MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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6.014 Electrodynamics

Problem Set 1 Solutions — Available in Tutorials: February 14, 2002

Problem 1.1

a) Power = $v^2/R = 2^2/73 =$ $\boxed{55 \text{ mwatts}}$.

b) $P_r = G_t\, P_t/4\pi r^2 = 1.5 \times 0.055/4\pi 10^2 =$ $\boxed{6.57\times10^{-5} \text{ w/m}^2.}$

c) $A_e = G\lambda^2/4\pi = 1 \times (c/f)^2/4\pi = (3\times10^8/10^9)^2/4\pi =$ $\boxed{0.0072 \text{ m}^2}$ (if square, 8.8 cm on a side). Note that the gain over isotropic of an isotropic antenna is unity.

d) $P_{rec} = P_r\, A_e = 6.57\times10^{-5} \times 0.0072 =$ $\boxed{4.73\times10^{-7} \text{ watts}}$.

e) $R(bps) = P_{rec}/E_b \cong 4.73\times10^{-7}/4\times10^{-20} =$ $\boxed{11.8 \text{ Terabits/sec.}}$ In practice, the limiting data rate might be a few orders of magnitude less because of radio frequency interference. Even so, the bandwidth required for the resulting data rate would still exceed the RF transmission frequency employed (1 GHz). Usually communications signals occupy bandwidths and communicate data rates that are less than 20 percent of the central frequency ($\Rightarrow$ ~200 Mbps here). In practice we could reduce the transmitter power or the receiver sensitivity, and any remaining excess power would simply reduce further our low probability of error in the communicated bit stream.

Problem 1.2

a) $P_{tmax} = v^2/R_r = 4^2/73 = 0.219$ watts. If we send half "one's" at this power level and half "zero's" at no power, then the average transmitted power is $\boxed{0.109 \text{ watts.}}$

b) hf = photon energy, where $f = c/\lambda$. $P_{tmax} = 0.1\times0.219$ watts. Maximum photon flux $N = P_{tmax}/hf = P_{tmax}\lambda/hc = 0.0219\times0.6\times10^{-6}/(6.625\times10^{-34}\times3\times10^8) =$ $\boxed{6.62\times10^{16} \text{ photons/sec} = N.}$

c) If these are radiated isotropically, then we have $m = N/4\pi r^2$ photons $sec^{-1}\ m^{-2}$ at radius r, of which ηA_{det} are detected, where $\eta = 0.9$ and $A_{det} = 10^{-4}$ here. Therefore $n = 0.9\times6.62\times10^{16}\times10^{-4}/4\pi 10^2 =$ $\boxed{4.74\times10^9 \text{ detected photons/sec.}}$

d) If we assume 100 photons/detectable bit, then the maximum data rate is $\boxed{\sim 5\times10^7}$ bps.

Note that direct sunlight (~1 kw/m^2) on the detector would add P_{sun} = 0.1 watt. The number of detected solar photons would be $0.9\times0.1/hf = 0.09\ \lambda/hc = 0.09\times6\times10^{-7}/6.625\times10^{-34}\times3\times10^{8} \cong 2.7\times10^{17}$. This number of solar photons could be reduced by a factor of 1000 by a narrowband optical filter at the receiver, and would correspond to 3×10^{14} detected photons per second, all random. If we slice each second into 5×10^{7} pieces, one piece per bit, then 6×10^{6} background photons would be detected in each such brief interval, and the rms variation in photon count per "zero" interval would be $(6\times10^{6})^{0.5}$ = 2450 photons/sec, large compared to the detection threshold of 100 photons per bit that we assumed. By setting the detection threshold for "one" near 10,000 photons, excursions of ~four standard deviations would be required to produce a bit error. Conservative design would therefore increase the average number of detected photons required to represent a "one" to perhaps 20,000 or more, reducing the available data rate to ~250 kbps.

e) Since so much depends on manufacturing costs and unknown interference levels, it is difficult to be certain which mode, RF or optical, would be best for this application. However, at these data rates both RF and optical systems can be integrated on single chips, so manufacturing cost is less likely to control than is interference, and path blockage. In general, RF is more likely to fill a room through scattered papers, books, etc. than are optical signals, which are moreover vulnerable to direct sunlight if they operate in the visible. If we are guided by our answers to Problems 1.1 and 1.2, the preferred choice is simple—RF.

Problem 1.3

a) $P_{rec} = P_t G_t G_r \lambda^2/(4\pi r)^2$ Since we are using the same antennas in both directions, and λ and r are also the same in both directions, P_{rec} is simply a constant times P_t, where a one-watt transmitter (like a cell phone) is reasonable for the astronaut and 10 kW is reasonable for the ground transmitter. Thus this argument suggests that costs will be set by the needs of the downlink, which is disadvantaged by a factor of ~10^4. However the astronaut's isotropic antenna will pick up all N interfering transmitters on Earth; N could be millions if the same frequency band is used by individuals. For example, one billion people in view of the moon might produce 10 million simultaneous calls, of which one percent might be in our band (then N $\cong 10^5$ calls). If the interferers also radiate ~1 watt isotropically, then we need to have a ground station antenna that has gain in the direction of the astronaut that is $10(SNR)\times10^5$ greater, or 10^6. Since 10-kW transmitters and gains of 10^5 are cheap, we want to use such systems. They should suffice, however, because we have a 10^4 power advantage over a 1-watt interfering cell phone, plus a 10^5 antenna gain advantage, or 10^9 overall advantage, which is large compared to our assumed set of 10^6 1-watt isotropic interferers. Were this not the case, the uplink antenna gain might have to be increased substantially, making it the dominant cost element.

b) $P_{rec} = P_t G_t G_r \lambda^2/(4\pi r)^2$. We assume no two astronauts will talk to earth simultaneously in the same narrow band. If we use a 1-watt transmitter for the astronaut, r = 380,000 km, $G_t = 1$, $\boxed{G_r = 10^5}$, and $\lambda = 0.03$ m, then $P_{rec} = 1\times1\times10^5(0.03)^2/(4\pi 3.8\times10^8)^2 = 3.95\times10^{-18}$ watts, which should be compared to the received power requirement $RE_b = 2000\times4\times10^{-20} = 8\times10^{-17}$ watts. Since P_{rec} is ~20 times less than the required RE_b, either the ground station antenna gain must be increased by a factor of ~20; or the astronaut's transmitter must radiate >~20 watts, or some combined solution must be adopted. If we assume the antenna costs grow as $G^{1.2}$ and power costs increase linearly with power P_t, then the factor of 20 might be applied entirely to the astronauts' transmitter power (making it 20 watts), unless the baseline cost of that transmitter were larger than that of the antenna.

In practice the final optimized design might divide the "cost pain" among several elements. These element costs typically increase faster than linear with the critical parameters. For example, to optimize this system the ground station antenna gain might be increased a factor of 1.5, the astronaut's helmet antenna gain might be increased to 3.33 (requiring the astronaut to generally face the earth when phoning, the helmet power might be increased to 3 watts, the voice data rate (and quality) might be reduced (say to 1.5 kbps), the astronauts' battery weight might be increased a factor of 1.5, and the phone speaking time per charge might be reduced a factor of two.

Most engineering design proceeds approximately in this fashion. A nominal point design is analyzed, and then all system parameters, including performance, are incremented to produce a superior design using quantitative metrics that combine cost, performance, and risk.

Problem 1.4

a) Since $\omega = 10^9$, $f = \omega/2\pi \cong \boxed{15.9 \text{ MHz.}}$

b) The argument $(10^8 t + z)$ remains constant as time increases if z goes negative. The wave is therefore propagating in the negative z direction.

c) Setting the first time derivative of the argument $(10^8 t + z)$ to zero (the argument doesn't change for a fixed point on the moving waveform), we find $dz/dt = 10^8$, less than the speed of light, so $\lambda = v/f = 10^8/1.59\times10^7 \cong \boxed{6.3 \text{ m.}}$

d) $v = (\varepsilon\mu_o)^{-0.5} = 10^8$[m/s], whereas $(\varepsilon_o\mu_o)^{-0.5} \cong 3\times10^8$. Therefore $(\varepsilon/\varepsilon_o)^{0.5} = 3$, or $\varepsilon = 9\varepsilon_o = \boxed{7.97\times10^{-11} \text{ [F/m]}}$

e) We can use Faraday's law, $\partial \overline{H}/\partial t = -(\nabla\times \overline{E})/\mu_o$, and then integrate, or we can take advantage of the fact that we have only a single uniform plane wave, which implies

$\overline{\mathrm{H}}$ is in phase with $\overline{\mathrm{E}}$, at right angles, and $1/\eta_o$ times greater, where $\eta_o = (\mu_o/\varepsilon)^{0.5} = (1.257\times10^{-6}/7.97\times10^{-11})^{0.5} = 126$ ohms. Here, for fun, let's use Faraday's law.

$$\nabla\times\overline{\mathrm{E}} = \det\begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial x \\ E_x & E_y & E_z \\ \hat{x} & \hat{y} & \hat{z} \end{vmatrix} = \hat{y}\,\partial E_x/\partial z = -\hat{y}\sin(10^8 t + z)\text{, so}$$

$$\overline{\mathrm{H}} = \int \hat{y}\,[\sin(10^8 t + z)/\mu_o]dt = -\hat{y}\cos(10^8 t + z)/10^8\mu_o = \boxed{-\hat{y}\cos(10^8 t + z)/126\ [\mathrm{A/m}]}$$

Problem 1.5

a) (i) $\boxed{3\hat{x}+\hat{y}+2\hat{z}}$, (ii) $2 - 2 + 2 = \boxed{1}$, (iii) $\overline{\mathrm{A}}\times\overline{\mathrm{B}} = \det\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ \hat{x} & \hat{y} & \hat{z} \end{vmatrix}$

$= (-1-2)\hat{x} + (2-2)\hat{y} + (4+1)\hat{z} = \boxed{-3\hat{x}+5\hat{z}}$

(iv) $2^2+1^2+1^2 = 6$, so $|\overline{\mathrm{A}}| = 6^{0.5} = \boxed{2.45}$

(v) $\overline{\mathrm{A}}\bullet\overline{\mathrm{B}} = 1 = |\overline{\mathrm{A}}||\overline{\mathrm{B}}|\cos\theta \Rightarrow \theta = \cos^{-1}\left(1/[2.45\times2.45]\right) = \boxed{80.4°}$

b) $\boxed{\overline{\mathrm{a}}(\mathrm{t}) = (\hat{x}+\hat{y})\cos\omega t + (\hat{y}-\hat{z})\sin\omega t}$

The real parts ⇒ cos ωt, imaginaries ⇒ -sin ωt

c) $\boxed{\underline{\mathrm{B}} = -2j\hat{x}+\hat{y} = 2.24e^{j2.68}}$