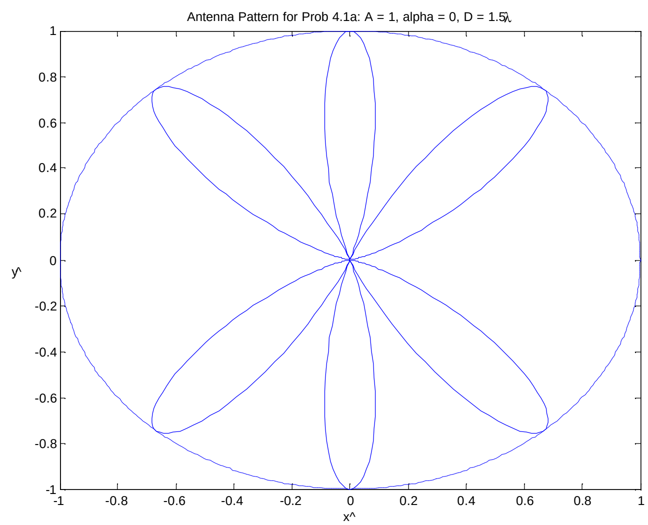
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.014 Electrodynamics

Problem Set 4 Solutions | Available in Tutorials: March 11, 2002

Problem 4.1

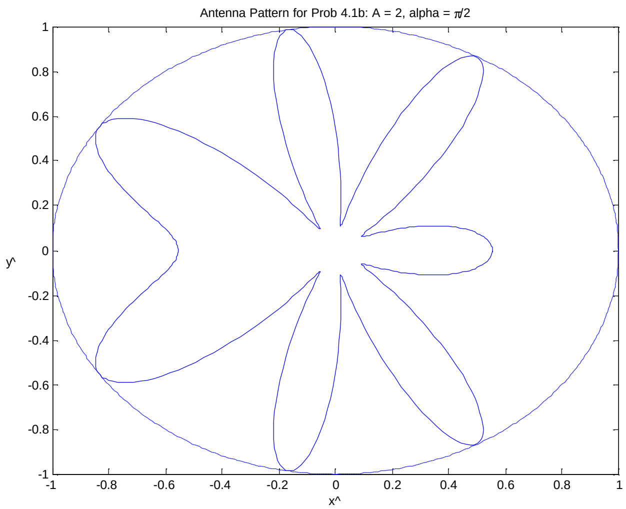
a) Following the text in section 9.4. We would look at the element factor and array factor for the 2 dipole case. The gain of the antenna array is computed using Eq. (9.4.9) and substituting (9.4.10) into (9.2.17) and the antenna pattern in the x-y plane, we obtain the radiation pattern formula given in Equations (2.4.8).
For our case we have A = 1, $\alpha = 0$, and D = 1.5λ, from which the following radiation pattern is obtained $\boxed{p(\theta) = 0.5 + 0.5\cos(3\pi\cos(\theta))}$ with its associated x-y plane sketch. Note that we have equal excitation implying that we can get nulls. (θ is the angle measured from the x axis to the y axis.)



The angles of min: Setting $\boxed{p(\theta) = 0 \Rightarrow \theta = a\cos\left(\frac{1+2n}{3}\right), \forall n = \{-1,0,1\}}$ integers. $\theta_{null} = \{0^\circ; \pm 70.53^\circ; \pm 109.47^\circ; 180^\circ\}$

The angles of max: Setting $\boxed{p(\theta) = 1 \Rightarrow \theta = a\cos\left(\frac{2n}{3}\right), \forall n = \{-1,0,1\}}$ integers.
$\theta_{max} = \{\pm 48.19^\circ; \pm 90^\circ; \pm 131.81^\circ\}$

b) $A = 2, \alpha = \pi/2, D = 1.5\lambda$. We do not have equal excitation, hence we will not have any nulls. We obtain $\boxed{p(\theta) = \frac{5}{9} - \frac{4}{9}\sin(3\pi\cos(\theta))}$ where this radiation pattern is sketched below.



The angles of min's: $\boxed{p(\theta) = 1/9 \Rightarrow \theta = a\cos\left(\frac{1}{6} + \frac{2}{3}n\right), \forall n = \{-1,0,1\}}$

$\theta_{min} = \{\pm 33.56^\circ; \pm 80.41^\circ; \pm 120^\circ\}$

The angles of max's: $\boxed{p(\theta) = 1 \Rightarrow \theta = a\cos\left(-\frac{1}{6} + \frac{2n}{3}\right), \forall n = \{-1,0,1\}}$

$\theta_{max} = \{\pm 146.44^\circ; \pm 99.59^\circ; \pm 60^\circ\}$

Note that we have half gain at $\theta = \{0; 180^\circ\}$.

c) If L = 1000λ, then kL = 2000π. Assuming A = 1, and α = 0, we have

$\boxed{p(?) = \frac{1}{2} + \frac{1}{2}\cos[2000p\cos(90 - ?)] = \frac{1}{2} + \frac{1}{2}\cos[2000p\sin(?)]}$.

The angles of the min's: $\boxed{2000\pi\sin(\theta) = (2n-1)\pi}$ or $\boxed{\sin(\theta) = \frac{(2n-1)}{2000}}$.

For small angles, $\theta_{min} \approx \frac{(2n-1)}{2000}$, and $\boxed{\Delta\theta \approx \frac{1}{1000} = \frac{\lambda}{L}}$ between nulls.

The angles of the max's: $\boxed{2000\pi\sin(\theta) = 2n\pi}$ or $\boxed{\sin(\theta) = \frac{n}{1000}}$.

There will be 1000 maxima and 1000 nulls (or 1000 lobes) between $\theta = 0$ and $\theta = \frac{\pi}{2}$.

The first null occurs at $\boxed{\theta = \frac{\lambda}{L}}$ and the second null occurs at $\boxed{\theta = \frac{2\lambda}{L}}$.

Problem 4.2

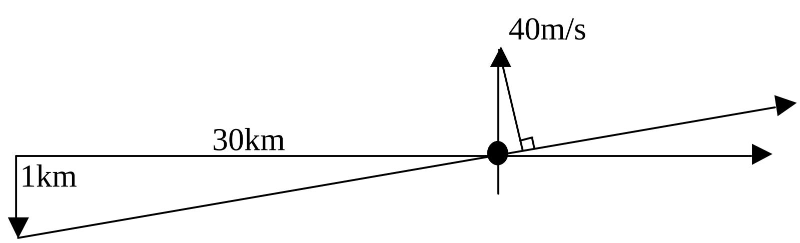
a) For two paths at Δf between nulls is $(t_2 - t_1)^{-1}$ Hz.
$t_1 = \frac{l_1}{c} = \frac{30\times10^3}{3\times10^8} = 1\times10^{-4}\text{s}$, $t_2 = \frac{l_2}{c} = \frac{31\times10^3}{3\times10^8} = 1.03\times10^{-4}\text{s}$
$\boxed{\Delta f = (t_2 - t_1)^{-1} = \pm 300\text{kHz}}$. Thus we have nulls at the nearest frequencies to 100MHz at: $\boxed{99.7\text{MHz};100.3\text{MHz}}$

b) $\lambda = c/f = 3\text{m}$. The distance between the two 'dipoles' is L=1km. Thus $kL = \frac{2\pi}{\lambda}L = \frac{2000\pi}{3}$. The antenna gain for this system is proportional to: $G(\phi) \propto (1+\cos(\frac{2000\pi}{3}\cos(\phi)+\alpha))$, from which it follows that the noise bursts will be obtained when

$$\boxed{\frac{2000\pi}{3}\cos(\phi_1)+\alpha = 0;\text{ and } \frac{2000\pi}{3}\cos(\phi_2)+\alpha = 2\pi \Rightarrow \cos(\phi_2)-\cos(\phi_1) \approx \Delta\phi \approx 0.003}$$

These noise bursts occur approximately
$\boxed{30\times10^3\tan(\Delta\phi) \approx 30\times10^3(0.003) = 90\text{m}}$ or $\boxed{30\lambda}$'s apart.

c)



The component of velocity in the direction of the propagating wave is

$$v = 40\cos q = 40\cos(88.07^\circ) = 1.3326$$

$$\boxed{f_D = f_0(1-\frac{v}{c}) = 100M(1-\frac{1.3326}{c}) = 0.444Hz}$$

Problem 4.3

a) Let $\overline{E}_2 = E_{2x}\hat{x} + E_{2y}\hat{y} + E_{2z}\hat{z}$, and using the boundary condition (BC) $\hat{n}\times(\overline{E}_1 - \overline{E}_2) = 0$ where $\hat{n}$ is directed from material 2 to 1 (i.e., $-\hat{z}$) we find,

$\boxed{(1-E_{2y})\hat{x}+E_{2x}\hat{y}=0 \Rightarrow E_{2y}=1, E_{2x}=0}$. Nothing can be stated about E_{2z} yet. We know that the boundary condition used, states that the tangential components of an E field are continuous across an interface (which we verified above).

Now look at the BC: $\hat{n}\bullet(\overline{D}_1-\overline{D}_2)=\rho_s$

For materials 1 and 2 σ=0. Thus the loss tangents of the materials are zero, hence the materials are lossless and at the boundary surface we have $\rho_s=0$.

Substituting $\overline{D}_i=\varepsilon_i\overline{E}_i, \forall i\in[1,2]$ into the BC we find:

$\boxed{-\hat{z}\bullet(\hat{y}(\varepsilon_1-\varepsilon_2)+\hat{z}(\varepsilon_1-\varepsilon_2E_{2z}))=0 \Rightarrow E_{2z}=\frac{\varepsilon_1}{\varepsilon_2}=0.5}$ thus $\boxed{\overline{E}_2=\hat{y}+0.5\hat{z}}$

b) If $\sigma_2=0$, material 2 is an ideal conductor. We know that the electric and magnetic field disappear in an ideal conductor (see page 123). For oblique wave incidence we have incident, reflected and transmitted waves. The transmitted wave is not going to exist.

For $\underline{\rho}_s(y)$**:** The reflected wave is given as: $\overline{E}_r = \hat{x}\Gamma E_0 e^{-jk_r.(y-z)}$. At the material boundary (z=0) and we look at the BC: $\hat{n}\times(\overline{E}_1-\overline{E}_2)=0$ we know that the tangential components of the electric field across the boundary should be continuous. Thus $E_0e^{-jk.(y)}+\Gamma E_0e^{-jk_r.(y)}=0$ holds for $\Gamma=-1$ and $k_r=k$. Because ME are linear $\overline{E}_1=\overline{E}_i+\overline{E}_r=\hat{x}E_0e^{-jky}(e^{-jkz}-e^{+jkz})$ and using $\overline{D}_1=\varepsilon_1\overline{E}_1$ in the BC: $\boxed{\hat{n}\bullet(\overline{D}_1)=\underline{\rho}_s(y)\Rightarrow \underline{\rho}_s(y)=0}$.

For $\underline{\overline{J}}_s(y)$**:** Assuming time harmonic form the general way to calculate an H field from an E field is: $\overline{H}=-\frac{\nabla\times\overline{E}}{j\omega\mu_0}=\frac{\overline{k}\times\overline{E}}{\omega\mu_0}$. Also $\overline{H}_1=\overline{H}_i+\overline{H}_r$, and calculating the incident and reflected H fields we find: $\overline{H}_i=\frac{(k\hat{y}+k\hat{z})\times(\hat{x}E_0e^{-jk.(y+z)})}{j\omega\mu_0}=\frac{E_0e^{-jk.(y+z)}}{\eta_1}(-\hat{z}+\hat{y})$ and

$\overline{H}_r=\frac{(k\hat{y}-k\hat{z})\times(-\hat{x}E_0e^{-jk.(y-z)})}{j\omega\mu_0}=\frac{E_0e^{-jk.(y-z)}}{\eta_1}(\hat{z}+\hat{y})$. Using the BC (and setting z=0): $\boxed{\hat{n}\times(\overline{H}_1)=\overline{J}_s(y)\Rightarrow \overline{J}_s(y)=\hat{x}\frac{2E_0}{\eta_1}e^{-jky}}$

Test (setting z = 0) BC: $\hat{n}\bullet(\mu_0\overline{H}_1)=-\hat{z}\bullet(\hat{y}\zeta+\hat{z}(E_0e^{-jky}-E_0e^{-jky}))=0$ which holds true.

Problem 4.4

a) Using Snell's law, we find: $\boxed{\sin(\theta_t) = \frac{k_i}{k_t} = \frac{\sqrt{\mu_0 \varepsilon_0}}{\sqrt{\mu_0 9\varepsilon_0}} \sin(\theta_i) = \frac{1}{3}\sin(\theta_i)}$

b) Using Snell's law: $\boxed{\frac{\sin(\theta_t)(=1)}{\sin(\theta_c)} = \frac{k_i}{k_t} = \frac{\sqrt{\mu_0 9\varepsilon_0}}{\sqrt{\mu_0 \varepsilon_0}} \Rightarrow \sin(\theta_c) = \frac{1}{3} \Rightarrow \theta_c = 19.47^\circ}$

c) Medium 2 is the incident medium. Medium 1 is the transmission medium. From the phase matching condition the projection of the phases onto the z axes should be matched for the incident, reflected and transmitted waves.

$k_{iz} = k_i \sin(\theta_i) = \frac{\omega\sqrt{9\mu_0\varepsilon_0}}{\sqrt{2}} \equiv k_{tz}$, from which we can calculate $k_{tx}^2 = k_t^2 - k_{tz}^2 = \omega^2\mu_0\varepsilon_0(1-9/2) = \omega^2\mu_0\varepsilon_0(-7/2)$. From this follows that $\boxed{k_{tx} = j\omega\sqrt{\mu_0\varepsilon_0 \frac{7}{2}} \Rightarrow \alpha = \omega\sqrt{\mu_0\varepsilon_0 \frac{7}{2}} = 2\pi\sqrt{\frac{7}{2}}\lambda^{-1}}$

Problem 4.5

a) The plasma frequency of the ionosphere is: $\omega_p = \sqrt{\frac{n_e q^2}{m\varepsilon_0}} = \sqrt{\frac{10^{12}(-1.6\times10^{-19})^2}{9.107\times10^{-31}(8.854\times10^{-12})}} = 5.638\times10^7[\mathrm{rad/s}]$, and also for the problem we have $\mu_t = \mu_i = \mu_0$ and $\varepsilon_1 = \varepsilon_0$.

The permitivity of the plasma is: $\varepsilon = \varepsilon_0\left(1-\frac{\omega_p^2}{\omega^2}\right)$. We have perfect reflection when $|\Gamma|^2 = 1$. We encounter 2 cases:

- $\Gamma = -1$, the reflected electric field is 180 degree out of phase with the incident electric field. This case is encountered when material 2 is highly conducting (see page 127 of text for a discussion),
- $\Gamma = 1$, the reflected electric field is in phase with the incident electric field. This case is encountered when material 2 has a very low value of for its permitivity (again page 127).

Seeing that for the plasma the permitivity can be made very small by playing around with the frequency. Moreover ε tends to zero when the frequency tens to the plasma frequency. Thus we are dealing with the second case, and the maximum frequency for which we will get perfect reflection ($\eta_n \to \infty \Rightarrow \varepsilon_t = 0$) is when $\boxed{\omega = \omega_p \Rightarrow f_{max} = 8.973\mathrm{MHz}}$

b) For oblique wave incidence when the grazing angle θ is greater than a critical angle θ_c, then we will get perfect reflection. Thus setting $\theta_c = 85^\circ$ and using Snell's law (where we realize that for perfect reflection $\theta_t = \pi/2$), thus we have:

$$\frac{1}{\sin(\theta_c)} = \frac{n_i}{n_t} = \frac{\sqrt{\mu_i \varepsilon_i}}{\sqrt{\mu_t \varepsilon_t}} = \left(\sqrt{1 - \frac{\omega_p^2}{\omega^2}}\right)^{-1}$$ from which we calculate

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{\omega_p^2}{1-\sin^2(\theta_c)}} = \frac{1}{2\pi} 6.469\times 10^8 = 102.96\mathrm{MHz}}.$$