MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.014 Electrodynamics

Problem Set 5 Solutions Available in Tutorials: March 18, 2002

Problem 5.1

a) For an arbitrary normal incidence wave with $\overline{E_i} = \hat{x}E_0 e^{-jkz}$

$\overline{E_t} = \hat{x}TE_0 e^{-jkz}$.

From air to glass,

$$\eta_{n1} = \frac{\eta_{t1}}{\eta_i} = \frac{\sqrt{\mu_0 / 2\varepsilon_0}}{\sqrt{\mu_0 / \varepsilon_0}} = \frac{1}{\sqrt{2}}.$$

The transmission coefficient for normal incidence from air to glass is therefore,

$$T_1 = \frac{2\eta_{n1}}{\eta_{n1} + 1} = \frac{2}{\sqrt{2} + 1}.$$

$$\therefore \overline{E_{t1}} = \hat{x}T_1 E_0 e^{-jkz}$$

Similarly, from glass to air

$$\eta_{n2} = \frac{\eta_{t2}}{\eta_1} = \frac{\sqrt{\mu_0 / \varepsilon_0}}{\sqrt{\mu_0 / 2\varepsilon_0}} = \sqrt{2} \text{ and}$$

$$T_2 = \frac{2\eta_{n2}}{\eta_{n2} + 1} = \frac{2\sqrt{2}}{\sqrt{2} + 1}.$$

The electric field of the sunlight entering the solar house is therefore,

$$\overline{E_{t2}} = \hat{x}T_2 T_1 E_0 e^{-jkz}$$

$$< \bar{S}_{t2} > = \frac{|E_{t1}|^2 |T_2|^2}{2} \text{Re}\left\{\frac{1}{\eta_{t2}^*}\right\}\hat{z}$$

$$= \frac{|T_1 E_0|^2 |T_2|^2}{2} \text{Re}\left\{\frac{1}{\eta_{t2}^*}\right\}\hat{z}$$

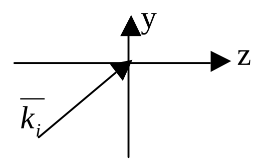
$$= \frac{32 E_0^2}{2\eta_0 (\sqrt{2} + 1)^4}\hat{z}$$

Since the incident power

$$< \bar{S}_i > = \frac{E_0^2}{2\eta_0},$$

$$\boxed{<\bar{S}_{t2}>/<\bar{S}_i> = \frac{32}{(\sqrt{2}+1)^4} * 100\% = 94.1\%}$$

b) From air to glass,



$$\Gamma = \frac{Z_n^{TE} - 1}{Z_n^{TE} + 1}, \text{ where}$$

$$Z_n^{TE} = \frac{\mu_t k_{iz}}{\mu_i k_{tz}} = \frac{k_{iz}}{k_{tz}} = \frac{\omega\sqrt{\mu_0\varepsilon_0}\cos\theta_i}{\omega\sqrt{2\mu_0\varepsilon_0 - \mu_0\varepsilon_0\sin^2\theta_i}} = \frac{0.5}{\sqrt{(2-0.75)}}$$

$$\boxed{\therefore \Gamma = \frac{0.4427-1}{0.4472+1} = -0.3819}$$

The y component of the wave does not encounter any boundaries. Therefore, we can conclude that power is conserved in the y direction. The ratio of incident and reflected power, therefore, depends on the z component.

$$\bar{E}_{iz} = \hat{x}E_0 e^{-jkz}$$

$$\bar{E}_{rz} = -\hat{x}\Gamma E_0 e^{-jkz}$$

$$\therefore <\bar{S}_{iz}> = \frac{E_0^2}{2\eta_0} \text{ and } <\bar{S}_{rz}> = \frac{E_0^2|\Gamma|^2}{2\eta_0}$$

$$\boxed{\therefore <\bar{S}_r>/<\bar{S}_i> = |\Gamma|^2 = 0.145 * 100\% = 14.5\%}$$

c) $$\boxed{\theta_B = \tan^{-1}\sqrt{\left(\frac{\varepsilon_t}{\varepsilon_i}\right)} = \tan^{-1}\left(\sqrt{2}\right) = 54.7^\circ}$$

Problem 5.2

a) Define our coordinate system as in Figure 9.36 of the text, making the radiation pattern (Figure 9.37) of a rectangular aperture applicable, for which we have:

$$G(k_x, k_y) = \frac{4pA}{l^2}\frac{\sin^2(k_x L_x/2)}{(k_x L_x/2)^2}\frac{\sin^2(k_y L_y/2)}{(k_y L_y/2)^2}.$$

The first nulls in the x and y directions for the two scenarios are calculated as

$$a_{x,null} = \frac{l}{L_x} \text{ and } a_{y,null} = \frac{l}{L_y}, \text{ where } \boxed{\lambda = \frac{c}{f} = \frac{3\times10^8}{10\times10^9} = 3\text{cm}}$$

If L_x = 15cm, then $a_{x,null}$ = 0.2 and $a_{y,null}$ = 0.033

If L_x = 90cm, then $a_{x,null}$ = 0.033 and $a_{y,null}$ = 0.2

Since we want the radar beam to be more directed (narrower) on the horizon ($\hat{y}$ direction), we choose the configuration that gives us the closest null in the y direction. **[Therefore, we want the short axis to be oriented vertically (L_x = 15cm).]**

When we scan the horizon, a narrower beam will help us pinpoint the location of the objects more accurately. Also, because of the curvature of the earth, a less directed beam in the x direction will help seeing objects that are far away.

b) The beam axis is z. The first nulls of the radiation patter are at $\boxed{\alpha_{x,null} = 0.2\text{rad}}$ and $\boxed{\alpha_{y,null} = 0.033\text{rad}}$. Because θ is small, the angle at which these nulls occur can be approximated by $a_{x,null}$ and $a_{y,null}$.

c) Assume we have an arbitrary transmitter/receiver radiating towards the ship's retro-reflector plate. At the transmitter/receiver we have:
Ptr: power transmitted; G: gain of transmitter/receiver; A: effective area of the transmitter/receiver.
The power received at the transmitter/receiver reflected back from the retro-reflector plate is given by the radar range equation,
$\boxed{P_{rec} = P_{tr}\left(\frac{G\lambda}{4\pi r^2}\right)^2 \frac{\sigma_s}{4\pi}}$, where σ_s is the scattering cross section of the plate. We want to find what this scattering cross section is. We also want it to be independent of the distance r, Ptr and Prec of this arbitrary transmitter/receiver. In order to do this let us consider the problem from the reflector's point of view. For the reflector plate we have:
Arp: effective area of the reflector plate; Grp: gain of the reflector plate. The power received by the reflector plate is: $\boxed{P_{rec}^{rp} = \frac{P_{tr}G}{4\pi r^2} A_{rp}}$, and the power reradiated back to the transmitter/receiver is: $P_{tr}^{rp} = P_{rec}^{rp}$. The power received by the transmitter/receiver is: $\boxed{P_{rec} = \frac{P_{tr}G}{(4\pi r^2)^2} A_{rp} G_{rp} A}$.
Now setting the two P_{rec} expressions equal to one another and using the relationship $\frac{A}{G} = \frac{A_{rp}}{G_{rp}} = \frac{\lambda^2}{4\pi}$, we have:

$$\boxed{P_{tr}\left(\frac{G\lambda}{4\pi r^2}\right)^2 \frac{\sigma_s}{4\pi} = \frac{P_{tr}G}{(4\pi r^2)^2} A_{rp} G_{rp} A \Rightarrow \sigma_s = \frac{4\pi}{\lambda^2} A_{rp}^2 = \frac{4\pi}{(0.03)^2}(40\text{cm})^2 = 2.23\times 10^3\,\text{m}^2}$$

d) Sally's radar is the transmitter and receiver.
$P_{rec} \geq 10^{-16}\,\text{W}$, $P_{tr} = 10\text{W}$, $\sigma_s = 1\text{m}^2$

$G_{tr/rec} = \frac{4\pi A}{\lambda^2} \frac{\sin^2(k_x L_x/2)}{(k_x L_x/2)^2} \frac{\sin^2(k_y L_y/2)}{(k_y L_y/2)^2}$ where A = (0.15m)(0.9m)=0.135 m^2.

Along the z axis, maximum $G_{tr/rec} = \frac{4\pi(0.135)}{(0.03)^2} = 1.885\text{x}10^3$

$$r^4 \le \frac{P_{tr}}{P_{rec}}\left(\frac{G_{tr/rec}\lambda}{4\pi}\right)^2 \frac{\sigma_s}{4\pi} = 161.1x10^{15}\,m$$
$$\therefore r \le 20.036x10^3\,m$$

Problem 5.3

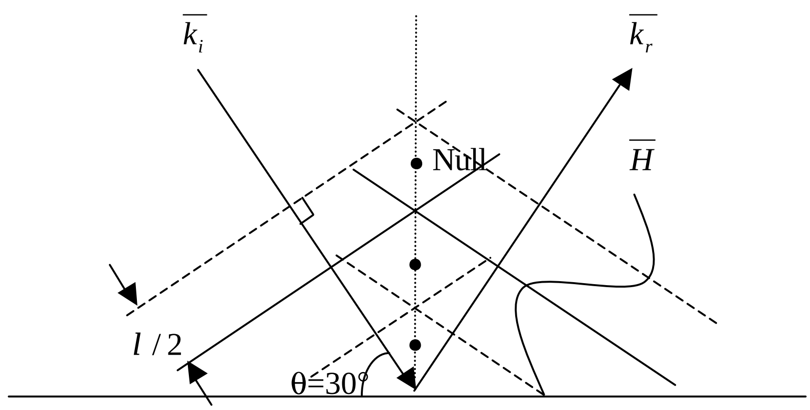
a) The loop needs to be oriented so that the normal vector of the surface of the loop is perpendicular to the receiver-sun axis. In other words, the surface normal vector of the loop needs to be perpendicular to the $\overline{k}$ of the incoming wave, and parallel to the $\overline{H}$ field of the incoming wave.

b)
$$10^{-15} = \frac{|\overline{E}|^2}{2h_0} = \frac{h_0 |\overline{H}|^2}{2}$$
$$|\overline{H}| = \sqrt{\frac{2\times10^{-15}}{h_0}}$$
$$\overline{H}(z,t) = \hat{y}\sqrt{\frac{2\times10^{-15}}{h_0}}\cos(wt - kz)$$
Where the direction of propagation (z) and the direction of the magnetic field (y) are chosen arbitrarily.
$$RMS(f(x)) = \sqrt{ave\{(f(x))^2\}}$$
$$\therefore RMS(\overline{H}(z,t)) = \sqrt{\frac{10^{-15}}{h_0}}$$

c)
$$\nabla\times\overline{E} = -\frac{\partial B}{\partial t}$$
$$\therefore \iint_s \nabla\times\overline{E}\cdot d\overline{s} = -\frac{\partial}{\partial t}\iint_s \overline{B}\cdot d\overline{s}$$
$$\therefore \oint_c \overline{E}\cdot d\overline{l} = -\frac{\partial}{\partial t}\iint_s \overline{B}\cdot d\overline{s}$$
$$\therefore V = \frac{\partial}{\partial t}\iint_s m_0\sqrt{\frac{2\times10^{-15}}{h_0}}\cos(wt - kz)\cdot d\overline{s}$$
$$\therefore V = -wm_0\left(100p(0.5)^2\right)\sqrt{\frac{2\times10^{-15}}{h_0}}\sin(wt - kz)$$
$$\therefore RMS(V) = wm_0\left(100p(0.5)^2\right)\sqrt{\frac{10^{-15}}{h_0}} = 188.5\times10^6 m_0\left(100p(0.5)^2\right)\sqrt{\frac{10^{-15}}{h_0}} = \boxed{3.0\times10^{-5}V}$$

d) Since the terrestrial surface is modeled as a perfectly conducting mirror, the incident wave is perfectly reflected. As a result, a standing wave pattern is created above the surface (Figure 4.6 on pg. 131 of the text). Since we want to maximize the magnetic flux going through the surface of the loop, we need to place the loop where the standing wave envelope of the magnetic field has a maxima. Therefore, we can place the loop at a distance that is a multiple of $\boxed{\frac{l}{2} = 5m \text{ from the surface.}}$ Furthermore, since the incident radiation is perpendicular to the terrestrial surface, the magnetic field is polarized parallel to the terrestrial surface. Therefore, we orient the loop such that the surface normal vector of the loop is parallel to the terrestrial surface.

e)



The maximas of the magnetic field standing wave pattern now occur at multiples of $\boxed{x = \frac{l/2}{\cos 60^\circ} = 10m}$. Therefore, two places where we can place the loop are 10m and 20m away from the surface. Since this is a TE wave, the total magnetic field is still polarized parallel to the terrestrial surface in the plane of incidence. Therefore, we orient the loop such that the surface normal vector of the loop is parallel to the terrestrial surface and in the plane of incidence.

f) The maximas of the magnetic field standing wave pattern occur at the same points as the TE wave. So the loop should be placed at the same distance away from the ground as the TE wave. However, since the magnetic field of a TM wave is normal to the plane of incidence, the loop should be oriented such that the surface normal vector of the loop is also normal to the plane of incidence.

g)

$$P_{rad} = \frac{p}{12} h_0 (ka)^4 \, |I|^2 \; N^2$$

$$R_r = \frac{2P_r}{|I|^2} = \frac{p}{6} h_0 (ka)^4 N^2 = \boxed{\frac{p}{6} h_0 \left(\frac{pd}{l}\right)^4 N^2}$$