MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.014 Electrodynamics

Problem Set 8 Solutions | Available in Tutorials: | April 15, 2002

Problem 8.1

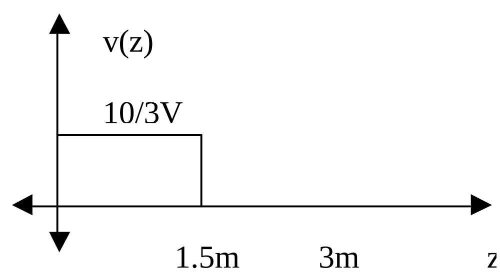
a) For t < 0 the transistor is a short circuit and the transmission line system is shorted out. For $t \geq 0$ the circuit we are investigating is in the standard transmission line problem format. Thus we have a DC voltage source of 10V; a source resistor of 200Ω; a RL load with $R = 100\Omega, L = 0.4\mu H$; a transmission line with a characteristic impedance of Z_0 and for this line the time it takes to traverse the line is

$$\nu = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{2} \Rightarrow t_{traverse} = \frac{\ell}{\nu} = \frac{3(2)}{c} = 2\times 10^{-8}\,s.$$

For $0 \leq t < \frac{\ell}{\nu}$ the waveform does not see the load impedance yet. Therefore the voltage is divided between the source impedance and the line as follows,

$V(z,t) = \frac{Z_0}{Z_0 + R_s} V_S u(t - \frac{z}{\nu}) = \frac{10}{3} u(t - \frac{z}{\nu})$, for $t \in [0, \frac{\ell}{\nu})$. From this it follows,

$$v(z) = V(z, t = 10^{-8}) = \begin{cases} \frac{10}{3}, \text{ for } z \in [0, \frac{\ell}{2}] \\ 0, \text{o/w} \end{cases}$$



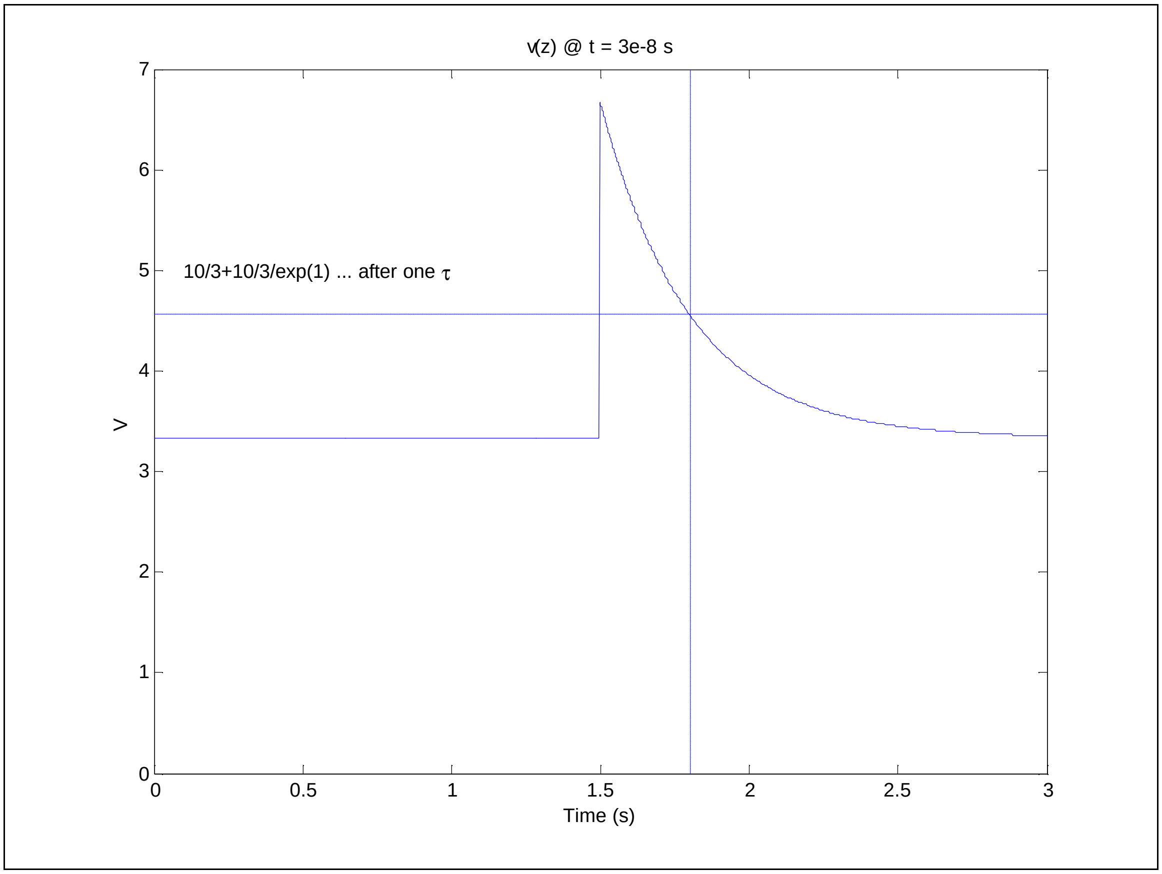
b) In this problem a Thevenin equivalent (which will be a lumped parameter circuit) is constructed of the transmission line system w/o the load. This equivalent circuit is then used to solve a lumped parameter linear circuit. This circuit is a straight forward first order RL circuit with a voltage source V_{th}, resistance $(Z_{th} + R_L)$ and inductance L_L. The time constant for this circuit is $\tau = \frac{L_L}{R_L + Z_{th}}$.

<u>Thevenin Equivalent Circuit:</u> We are required to look at the transmission line system with the load removed and we then consider the two cases where the subsequent system is open circuited and short circuited.

<u>Open Circuit</u>, now we have $V_L|_{oc} = V^+ + V^- = 2V^+ = V_{th}$ because of the positive unity voltage reflection coefficient at the load.

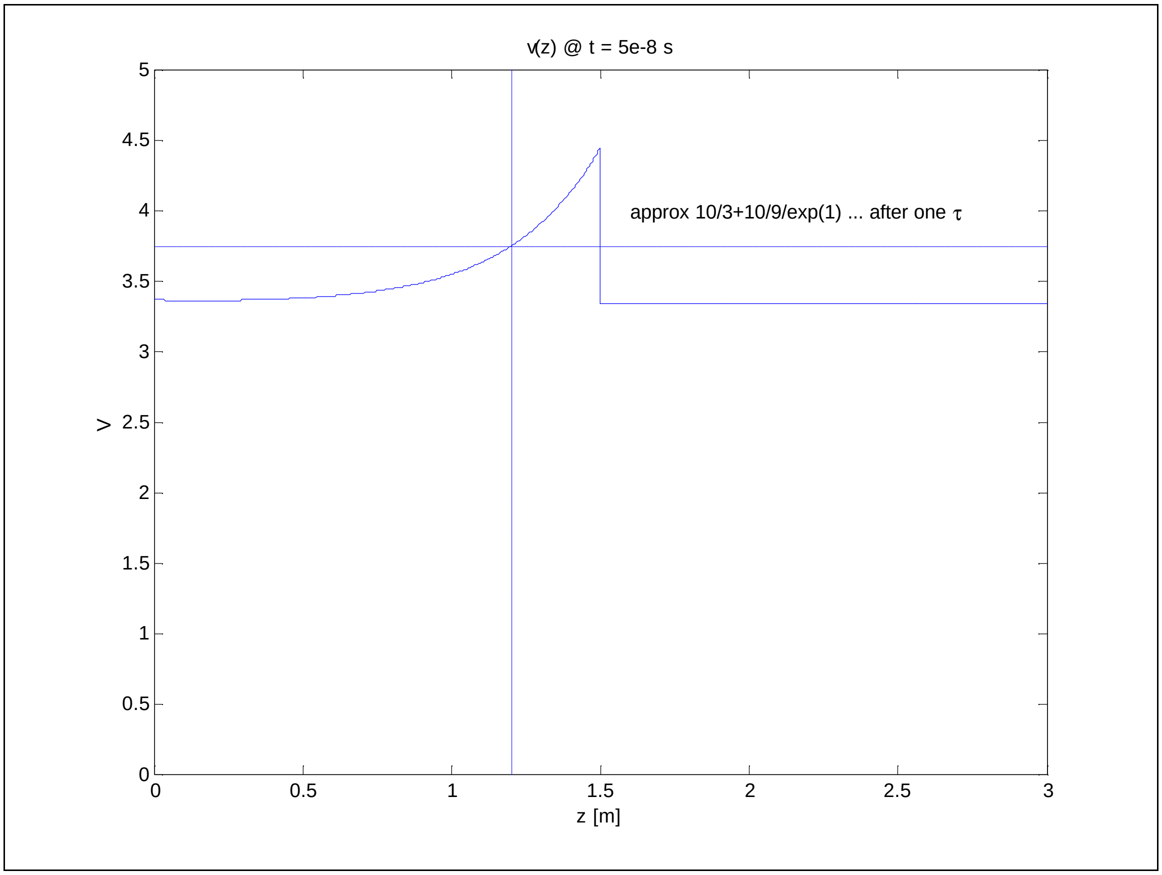
Close Circuit, now we have $I_L|_{sc} \equiv \frac{1}{Z_0}(V^+ - V^-) = \frac{1}{Z_0}(2V^+) = I_{th}$ because of the negative unity voltage reflection coefficient due to the short circuit. Now $\boxed{Z_{th} = \frac{V_{th}}{I_{th}} = Z_0}$, $V_{th} = \frac{20}{3}u(t - \frac{3}{\nu}) = \frac{20}{3}u(t - 2\times10^{-8})V$ and $\tau = 2\times10^{-9}s$.

The total voltage on the line at $t = 3\times10^{-8}s$ is given as (derived in problem 8.2a) $\boxed{V_{total}(t = 3\times10^{-8}, z) = \frac{10}{3} + \frac{10}{3}\exp\left(-\frac{(z-1.5)}{0.3}\right)u(z-1.5)[V]}$, and is visualized as follows …



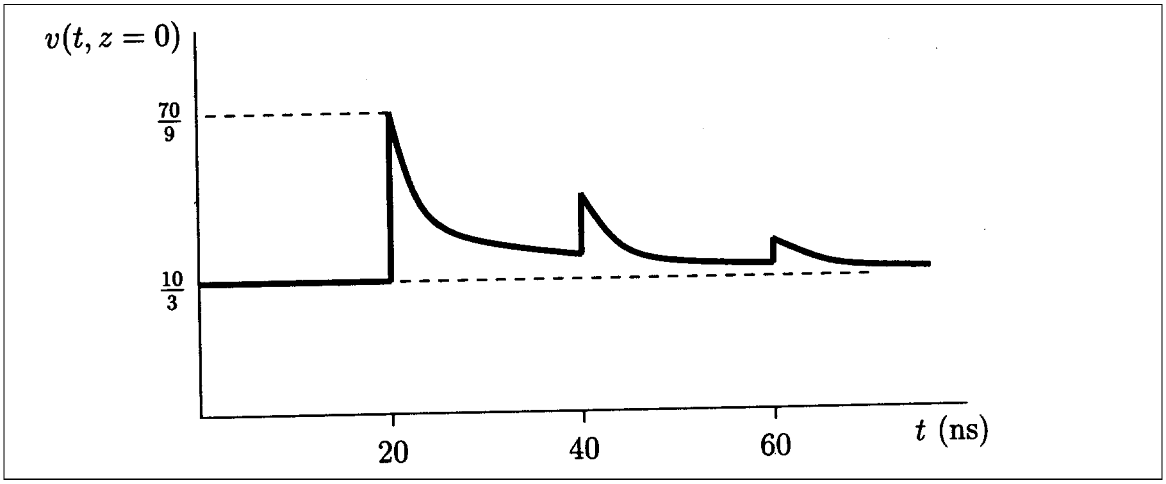
c) At the source side we do not have a matched conditions and the backward traveling wave will be reflected and then travel forward on the line. The reflection coefficient is $\Gamma_S = \frac{R_S - Z_0}{R_S + Z_0} = \frac{1}{3}$. Thus $V_2^+ = \frac{1}{3}V_1^-$, and $V_{total} = V_1^+ + V_1^- + V_2^+$ and at $t = 5\times10^{-8}s$ it can be shown that:

$\boxed{v(z) = \frac{10}{3}\left(1 + \exp\left(-\frac{(z+1.5)}{0.3}\right) + \frac{1}{3}\exp\left(-\frac{(1.5-z)}{0.3}\right)u(1.5-z)\right)[V]}$ where v(z) can be visualized as follows …



d) In order to find the asymptotic voltage across the transistor we know that the transients would have died away and that $V(t \to \infty) = \frac{R_L}{R_L + R_S} V_S = \frac{10}{3} V$. The maximum voltage across the transistor will be encountered at $t = \frac{2\ell}{v} = 4 \times 10^{-8} s$ when we have the summation of three voltage wavefronts (V_1^+, V_1^-, V_2^+) and thus we have $\boxed{V_{max}(t) = \frac{10}{3} + \frac{10}{3} + \frac{1}{3} \cdot \frac{10}{3} = \frac{70}{9} = 7.78V > 6.67V}$. We see that the maximum voltage across the transistor is greater than twice the equilibrium voltage on the line. The actual maximum voltage is a function of the DC voltage source, line characteristics, source impedance and load impedance. The worst scenario would have been if we had a reflection coefficient of 1 at the source end and then the voltage spike across the transistor would have been three times the steady state voltage on the line. Thus as a rule of thumb we can specify the breakdown voltage of the transistor to be three times the steady state voltage on the line. The voltage across the switching transistor as a function of time[1] …

[1] Thanks to bigjim@mit.edu for the neat figure …



Problem 8.2

a) For $0 < t < \frac{l}{v}$,

$$V_+(t,z) = \frac{Z_0}{Z_0 + R_s} V_s u(t - z/v).$$

When V_+ reaches the load at $t = l/v,$ we look at the thevinin equivalent network at the load to solve the problem. The governing equation for the current at $z = l$ is

$$L\frac{di(t)}{dt} + \frac{Z_0 + R_L}{L} i(t) = 2V_+(t, z = l)$$

$$\therefore i(t) = \frac{2V_+(t,z=l)}{Z_0 + R_L}\left(1 - e^{-(t-l/v)/t}\right) \text{ where } t = \frac{L}{Z_0 + R_L} = 2\text{ns}.$$

$$V_L(t) = L\frac{di(t)}{dt} + R_L i(t) = \frac{2V_+(t,z=l)}{Z_0 + R_L}\left(R_L - Z_0 e^{-(t-l/v)/t}\right)$$

Since $V_L(t) = V_+(t, z=l) + V_-(t, z=l),$

$$V_-(t,z=l) = V_L(t) - V_+(t,z=l) = \frac{Z_0}{Z_0 + R_s} V_s e^{-(t-l/v)/t} u(t - l/v)$$

$$\therefore V_-(t,z) = \frac{Z_0}{Z_0 + R_s} V_s e^{-(t+z/v-2l/v)/t} u(t + z/v - 2l/v)$$

Due to reflection at the source,

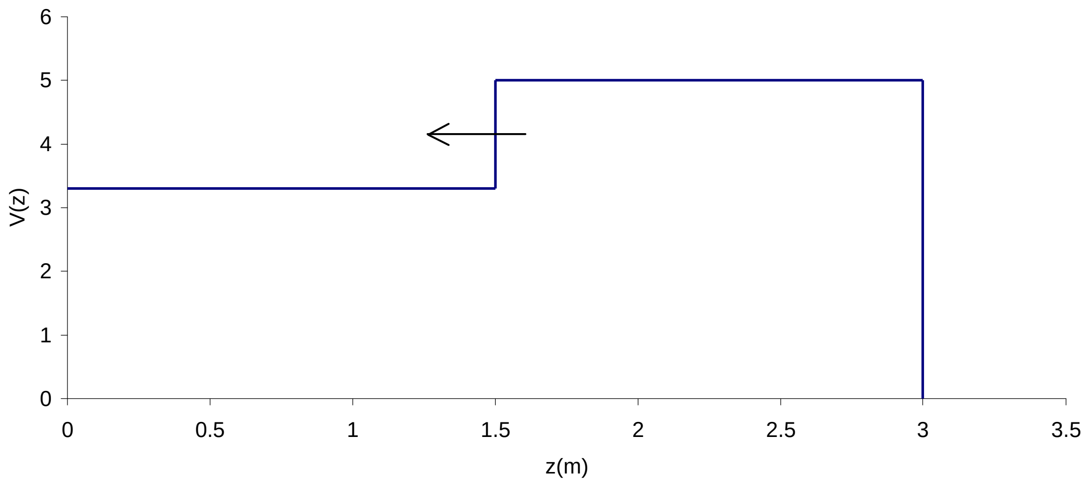
$$\therefore V_+^{(2)}(t,z) = \frac{R_s - Z_0}{R_s + Z_0} V_-(t,-z) = \frac{1}{3} V_-(t,-z) = \frac{Z_0}{3(Z_0 + R_s)} V_s e^{-(t+z/v-2l/v)/t} u(t + z/v - 2l/v)$$

$$\therefore V_{total}(t,z) = V_+ + V_- + V_+^{(2)}$$

$$= \frac{1}{3}V_s u(t - z/v) + \frac{1}{3}V_s e^{-(t+z/v-2l/v)/t} u(t + z/v - 2l/v) + \frac{1}{9}V_s e^{-(t-z/v-2l/v)/t} u(t - z/v - 2l/v)$$

$$\therefore V_{total}(t = 3\times10^{-8}, z) = \frac{1}{3}V_s u(t - z/v) + \frac{1}{3}V_s e^{-(t+z/v-2l/v)/t} u(t + z/v - 2l/v)$$

b)



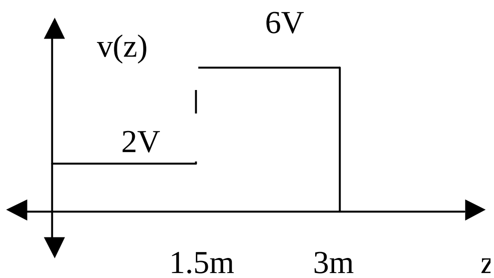
Problem 8.3

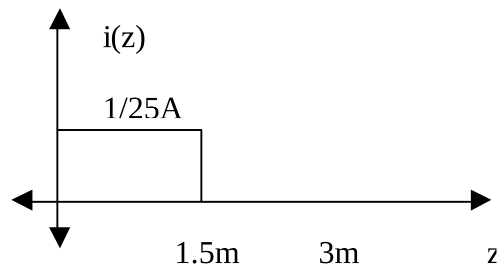
a) Prior to the switch opening all the transient waveforms on the line have died away and we are left with a standing waveform. This is motivated by the statement that the system is in equilibrium for $t < 0$. Thus it does not make sense to talk about forward and backward traveling waves.

We have $\boxed{V(z, t<0 \,|\, \text{equilibrium}) = V(z, \tau \to \infty) = \frac{R_L}{R_L + R_S} V_S = 2V}$ (where τ is another time axis and related to our t time axis such that when $\tau \to \infty \Rightarrow t = 0$).
The current waveform is also be a constant along the line and is

$\boxed{I(z, t<0 \,|\, \text{equilibrium}) = I(z, \tau \to \infty) = \frac{1}{R_L + R_s} V_S = \frac{1}{25} A}$.

b) When the circuit is broken at the load a new boundary condition is enforced at the load. For the open circuit we know that the current at $z = 3m$ must be 0 for all time $t \geq 0$. Thus a backward traveling current wave will be induced with an amplitude of $\boxed{I_{-}(z,t) = -\frac{1}{25} A, \forall t \in [0, 2\times10^{-8}) \text{ and } z \in [0,3]}$, yielding the resultant current at the point in space to be zero. The accompanying voltage wave is calculated as: $\boxed{V_{-}(z,t) = -I_{-}(z,t)Z_0 = 4V, \forall t \in [0, 2\times10^{-8}) \text{ and } z \in [0,3]}$. The total voltage $V(z, t = 10^{-8}) = v(z)$ and current $I(z, t = 10^{-8}) = i(z)$ waveforms thus look as follows:





Problem 8.4

a) $$P_{rec} = E_{rec} f_D = P_{tr}\left(\frac{Gl}{4pr^2}\right)^2 \frac{s_s}{4p} = P_{tr}\left(\frac{l}{4pr^2}\right)^2\left(\frac{4pA}{l^2}\right)^2 \frac{s_s}{4p} = P_{tr}\frac{A^2 s_s}{r^4 l^2 4p}$$

Since $f_D = f_0(1-v/c) \approx \frac{c}{l}$ $\quad (v << c)$

$$l = P_{tr}\frac{A^2 s_s}{4pr^4 E_{rec} c} = 0.16mm$$

b) $$r^4 = P_{tr}\frac{A^2 s_s}{4p\,l E_{rec} c}$$

$\therefore r = 6360m$