MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.014 Electrodynamics

Problem Set 10 Solutions Available in Tutorials: April 22, 2002

Problem 10.1

a) The definitions of the voltages and currents on the line as functions of position are: $V(z) = V_+e^{-jkz} + V_-e^{jkz}$ and $I(z) = Y_0(V_+e^{-jkz} - V_-e^{jkz})$. At both ends of the line we have open ended terminations, hence reflection coefficients of one. Thus $I(z=0) = 0 \Rightarrow V_+ = V_-$ and from this it follows that $V(z) = 2V_+\cos(kz)$ and $I(z) = -j2Y_0V_+\sin(kz)$. But we also need to have

$$\boxed{I(z=\ell) = 0 \Rightarrow \sin(k\ell) = 0 \Rightarrow k\ell = n\pi \rightarrow \omega_n = \frac{n\pi c}{\ell} = 3n\pi\times 10^8, \forall n \in [0,1,\ldots)}$$

b) The energy stored in the resonator is twice the electric energy stored in the resonator. Thus
$w_T = 2\int_0^{\ell=1}\langle w_C\rangle dz = \frac{1}{2}C\,|2V_+|^2\int_0^1 \cos^2(k(=n\pi)z)dz = 1\times 10^{-6}\ldots(1)$ and we know that $c^{-1} = \sqrt{LC}\ldots(2)$ and $\sqrt{\frac{L}{C}} = 100\ldots(3)$. From (2) and (3) we obtain $\boxed{C = \frac{100}{3}\,\text{pF}}$ and substituting this into (1) we obtain $\boxed{C\,|V_+|^2 = 1\times 10^{-6} \Rightarrow V_+ = 173.2\text{V}}$. The peak voltage magnitude value is attained at the ends of the lines and is $\boxed{V_{peak} = 2V_+ = 346.4\text{V}}$.

c) Place the resistor at $z = \ell$ (could be placed at 0 as well). We are dealing with the case discussed in Section 8.5 in the book, when a conductance is switched in at $z = a = \ell$. We see that $G = 1\times 10^{-4} << Y_0 = 1\times 10^{-2}$ and we can thus assume that the voltage and current profiles are not significantly perturbed by adding the conductance. The conductance also does not change the resonant frequencies. The power dissipated in the resonator circuit is
$P \approx \frac{1}{2}G\,|V(z=\ell)|^2 = 2G\,|V_+|^2\cos^2(n\pi)$ and when we look at the first nonzero resonant frequency (n=1) we calculate $\boxed{P = 2\times 10^{-4}(173.2)^2 = 6\text{W}}$ from which we calculate $\boxed{Q \approx \frac{\omega w_T}{P} = \frac{300\pi}{6} = 50\pi = 157.1}$

Problem 10.2

a) Using the guidance condition, $k_x d = mp$ and the dispersion relation $k^2 = k_x^2 + k_z^2$ we can derive the cut-off frequency w_m, above which $k_z > 0$.

$$w_m = \frac{mp}{d\sqrt{me}}.$$

$$TM_1 : f_{co} = \frac{w_1}{2p} = \boxed{0.75\times10^{10}\,Hz}$$

$$TM_2 : f_{co} = \frac{w_2}{2p} = \boxed{1.5\times10^{10}\,Hz}$$

$$TE_2 : f_{co} = \frac{w_2}{2p} = \boxed{1.5\times10^{10}\,Hz}$$

b) $$l_g = \frac{2p}{k_z} = \frac{2p}{\sqrt{w^2 me - (mp/d)^2}} = \boxed{1.15cm}$$

c) $$v_g = \left(\frac{\partial k_z}{\partial w}\right)^{-1} = v\sqrt{1-\left(\frac{mpv}{wd}\right)^2} = \boxed{1.3\times10^8\,m/s}$$