MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.014 Electrodynamics

Problem Set 11 Solutions — Available in Tutorials: May 6, 2002

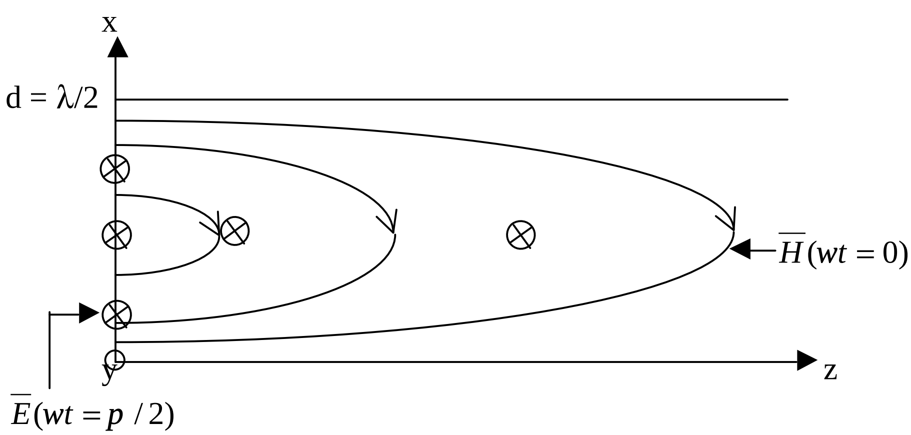
Problem 11.1

$$\overline{E}(x,y,z) = \hat{y}2\,jE_0 \sin(k_x x)e^{-az}$$

$$\therefore \overline{E}(x,y,z,t) = -\hat{y}2E_0 \sin(k_x x)e^{-az}\sin(wt)$$

$$\overline{H}(x,y,z) = \frac{\nabla\times\overline{E}}{-jwm} = \frac{j}{wm}\left[\hat{x}a\sin(k_x x) + \hat{z}k_x\cos(k_x x)\right]\cdot 2\,jE_0 e^{-az}$$

$$\therefore \overline{H}(x,y,z,t) = -\frac{2E_0}{wm}\left[\hat{x}a\sin(k_x x) + \hat{z}k_x\cos(k_x x)\right]e^{-az}\cos(wt)$$



$$\overline{E}(wt = 0) = \overline{H}(wt = p/2) = 0$$

Problem 11.2

a) TE$_{10}$ mode: m = 1; n = 0 from which it follows that $k_x = \frac{p}{a}; k_y = 0$. In section 7.4 the general solutions of the magnetic and electric fields for TE and TM modes are given. We thus have:

$E_x = 0$ (from equation 7.4.13); $E_z = 0$ (because we are dealing with a transverse electric field);

$$E_y(x,y,z) = -\frac{k_x}{k_0}E_0\sin(k_x x)e^{-jk_z z} \rightarrow E_y(x,y,z,t) = -\frac{k_x}{k_0}E_0\sin(k_x x)\cos(wt - k_z z)$$

(from equation 7.4.14);

$$H_x(x,y,z) = \frac{E_0 k_x k_z}{hk_0^2}\sin(k_x x)e^{-jk_z z} \rightarrow H_x(x,y,z,t) = \frac{E_0 k_x k_z}{hk_0^2}\sin(k_x x)\cos(wt - k_z z)$$

(from equation 7.4.15); $H_y = 0$ (from equation 7.4.15);

$$H_z(x,y,z) = -j\frac{E_0 k_x^2}{hk_0^2}\cos(k_x x)e^{-jk_z z} \rightarrow H_z(x,y,z,t) = \frac{E_0 k_x^2}{hk_0^2}\cos(k_x x)\sin(wt - k_z z)$$

(from equation 7.4.15). The surface current density on the guide walls is related to the magnetic field intensity by, $\bar{J}_s = \hat{n}\times\bar{H}$, where $\hat{n}$ is the outward normal from the wall surface and $\bar{H}$ is the magnetic field intensity at the wall. At t=0 and evaluating the surface current densities at the walls of the waveguide we find:

@ $x = 0$: $H_z(x=0,y,z,t=0) = -\frac{E_0 k_x^2}{hk_0^2}\sin(k_z z)$ and

$H_x(x=0,y,z,t=0) = 0$ from which it follows

$$\bar{J}_s(x=0) = -\hat{x}\times\hat{z}H_z(x=0,y,z,t=0) = -\hat{y}\frac{E_0 k_x^2}{hk_0^2}\sin(k_z z).$$

@ $x = a$: $H_z(x=a,y,z,t=0) = -\frac{E_0 k_x^2}{hk_0^2}(-1)\sin(k_z z)$ and

$H_x(x=a,y,z,t=0) = 0$ from which it follows

$$\bar{J}_s(x=a) = \hat{x}\times\hat{z}H_z(x=a,y,z,t=0) = -\hat{y}\frac{E_0 k_x^2}{hk_0^2}\sin(k_z z) = \bar{J}_s(x=0).$$

@ $y = 0$: $H_z(x,y=0,z,t=0) = -\frac{E_0 k_x^2}{hk_0^2}\cos(k_x x)\sin(k_z z)$ and

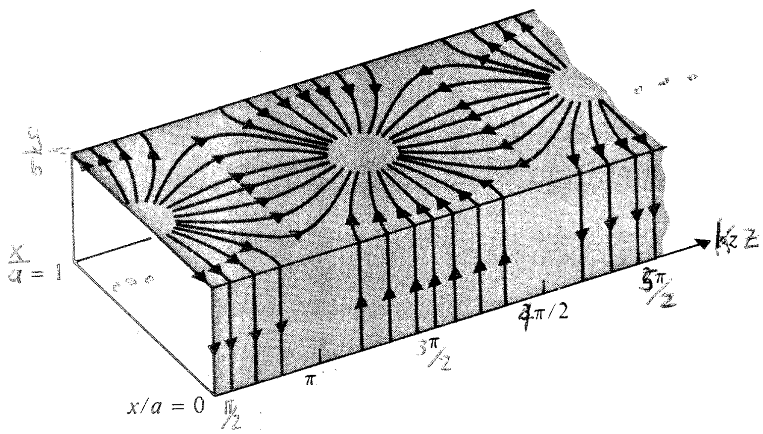
$H_x(x,y=0,z,t=0) = \frac{E_0 k_x k_z}{hk_0^2}\sin(k_x x)\cos(k_z z)$ from which it follows

$$\bar{J}_s(y=0) = -\hat{y}\times(\hat{x}H_x(x,y=0,z,t=0) + \hat{z}H_z(x=0,y,z,t=0))$$
$$= \hat{x}\frac{E_0 k_x^2}{hk_0^2}\cos(k_x x)\sin(k_z z) + \hat{z}\frac{E_0 k_x k_z}{hk_0^2}\sin(k_x x)\cos(k_z z).$$

@ $y = b$: $H_z(x,y=b,z,t=0) = -\frac{E_0 k_x^2}{hk_0^2}\cos(k_x x)\sin(k_z z)$ and

$H_x(x,y=b,z,t=0) = \frac{E_0 k_x k_z}{hk_0^2}\sin(k_x x)\cos(k_z z)$ from which it follows

$$\bar{J}_s(y=b) = \hat{y}\times(\hat{x}H_x(x,y=0,z,t=0) + \hat{z}H_z(x=0,y,z,t=0))$$
$$= -\hat{x}\frac{E_0 k_x^2}{hk_0^2}\cos(k_x x)\sin(k_z z) - \hat{z}\frac{E_0 k_x k_z}{hk_0^2}\sin(k_x x)\cos(k_z z) = -\bar{J}_s(y=0).$$



Surface currents on guide walls for TE_{10} mode in rectangular waveguide.

(Figure is adapted from "Field and Wave Electromagnetics", D.K. Cheng, p. 554, 2nd edition)

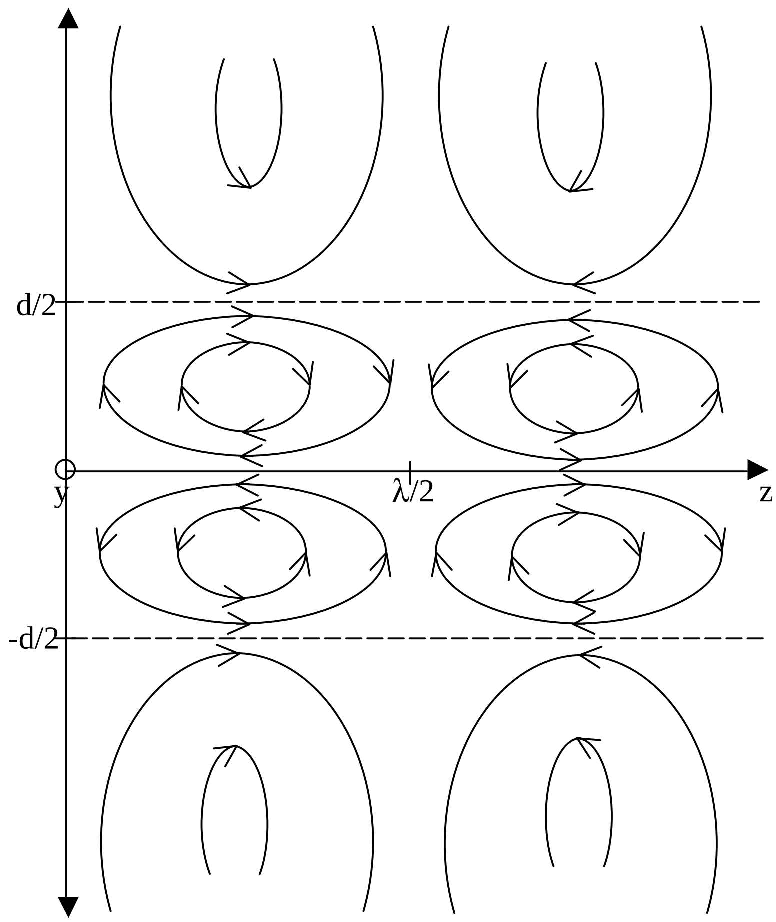
b) In order to break the current minimally thin slots can be placed:

- On the in the middle of the top or bottom surfaces (e.g. when $y = 0, b$ we need the slot to be placed at $x = \frac{a}{2}$). Note we are not specifying the length of the slot
- Vertical orientated thin slots on the side walls will also work (e.g. when $y = 0, a$).

Problem 11.3

a) TM_2

b)



c) $q = \tan^{-1}\left(\frac{k_z}{k_x}\right)$

$$q_c = \sin^{-1}\left(\sqrt{\frac{k_z^2 - a^2}{k_z^2 + k_x^2}}\right)$$

Problem 11.4

a) That the optical pulse can propagate on this line before distorting does not depend on the constants b_0, b_1. From the dispersion relationship given ($k \cong b_0 w_0 + b_1(w - w_0) + b_2(w - w_0)^2/2$) we can calculate the group and the phase velocity as follows: $\left.\frac{dk}{d\omega}\right|_{\omega_0} = \beta_1 = 1/v_g$ and $k/\omega_0 = \beta_0 = 1/v_p$ respectively. The dispersion relationship only pertains to the β_2 constant, and thus the distance that optical pulses can propagate on this line before distorting does not depend on β_0, β_1 .

b) Assume the velocity of the wave is the speed of light. Thus the width of the pulse train period is $\frac{c}{f} = 30cm$. The pulse train consists of a "1" and "0" that span half this width each. Thus half the width of a pulse is 0.075m. The group velocity as a function of frequency is given as follows:

$$v_g(\omega) = (\frac{dk}{d\omega})^{-1}\Big|_{\omega\neq\omega_0} = \frac{1}{\beta_1 + \beta_2(\omega - \omega_0)} = \frac{1}{v_g^{-1} + \beta_2(\omega - \omega_0)}$$

. We are told that distortion will happen when the highest velocity signals have moved ahead of the slowest signals by a distance of one-half the width of a single pulse, thus

$$| D_{wH} - D_{wL} |$$

$$= T\left|\frac{1}{v_g^{-1} + b_2 2p(3\times10^{14} + 10^{10} - 3\times10^{14})} - \frac{1}{v_g^{-1} + b_2 2p(3\times10^{14} - 10^{10} - 3\times10^{14})}\right|$$

$$= 0.075m$$

From the above we make T the subject. The distance traveled then is

$$\boxed{D = cT = \frac{3\times10^8(0.075)}{\left|\frac{1}{v_g^{-1} + b_2 2p(10^{10})} - \frac{1}{v_g^{-1} + b_2 2p(-10^{10})}\right|}}.$$