Fourier Series and Fourier Transforms

EECS2 (6.082), MIT
Fall 2006

Lectures 2 and 3

Fourier Series

From your differential equations course, 18.03, you know Fourier's expression representing a T-periodic time function $x(t)$ as an infinite sum of sines and cosines at the fundamental frequency and its harmonics, plus a constant term equal to the average value of the time function over a period:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \quad (1)$$

where

$$\begin{aligned} \omega_0 &= \frac{2\pi}{T} \\ a_0 &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt \end{aligned} \quad (2)$$

and for integer $n > 0$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0 t)\,dt \quad (3)$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(n\omega_0 t)\,dt \quad (4)$$

Some comments on symmetries

1. The integrals could have been taken over any interval of length T, but the symmetric interval $[-\frac{T}{2}, \frac{T}{2}]$ is most convenient for us.

2. Note from the defining expressions that if $x(t)$ is an *even* function of t, i.e., if

$$x(-t) = x(t) \ ,$$

then $b_n = 0$ for all $n > 0$, so only *cosine* terms appear in the Fourier series.

Similarly, if $x(t)$ is an *odd* function of t, i.e., if

$$x(-t) = -x(t) \ ,$$

then $a_n = 0$ for all $n \geq 0$, so only *sine* terms appear in the Fourier series.

Recall some useful trigonometric identities

$$\begin{aligned}\cos(-p) &= \cos p\\ \sin(-p) &= -\sin p\\ \cos(p+q) &= \cos p\cos q - \sin p\sin q\\ \sin(p+q) &= \sin p\cos q + \cos p\sin q\end{aligned}$$

The latter two are most easily derived from Euler's identity:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

where $j = \sqrt{-1}$.

From the cosine-sum and sine-sum identities above, various other identities can be derived, e.g.,

$$\begin{aligned}2\cos p\cos q &= \cos(p+q) + \cos(p-q)\\ 2\sin p\sin q &= \cos(p-q) - \cos(p+q)\\ 2\sin p\cos q &= \sin(p+q) + \sin(p-q)\end{aligned}$$

Example 1 (sanity check)

Suppose $x(t) = K \cos \omega_0 t$, with $\omega_0 > 0$. Then

$$a_1 = \frac{2}{T} \int_{-T/2}^{T/2} K \cos^2(\omega_0 t)\, dt = K$$

while all other a_n and all b_n are 0. So the Fourier series for $x(t)$ is simply $K \cos \omega_0 t$, as it should be!

Similarly, the Fourier series for $x(t) = K \sin(\omega_0 t)$ is just this expression itself.

Example 2 (magnitude and phase)

Suppose $x(t) = a\cos(\omega_0 t) + b\sin(\omega_0 t)$, with $\omega_0 > 0$. This evidently is periodic with period $T = 2\pi/\omega_0$, and its Fourier series will be the same expression again. But there's an alternative representation that yields a bit more insight:

Rewrite $x(t)$ as

$$x(t) = \sqrt{a^2+b^2}\;\left(\tfrac{a}{\sqrt{a^2+b^2}}\cos(\omega_0 t) + \tfrac{b}{\sqrt{a^2+b^2}}\sin(\omega_0 t)\right)$$

and let

$$\theta = \arctan\left(\frac{b}{a}\right)$$

so

$$\frac{a}{\sqrt{a^2+b^2}} = \cos\theta\,, \qquad \frac{b}{\sqrt{a^2+b^2}} = \sin\theta$$

...→

Example 2 (continued)

(Construct for yourself a right-angled triangle that displays the preceding relations among a, b, and θ.)

Our previous expression for $x(t)$ then becomes

$$\begin{aligned} x(t) &= \sqrt{a^2+b^2}\Big(\cos\theta\cos(\omega_0 t)+\sin\theta\sin(\omega_0 t)\Big) \\ &= c\cos(\omega_0 t-\theta) \end{aligned}$$

where

$$c=\sqrt{a^2+b^2}$$

So adding a cosine and a sine of the same frequency, but possibly different amplitudes, yields (perhaps surprisingly) a pure sinusoid again, with magnitude and phase as specified above. We refer to the cosine and sine as having added "in quadrature", because the two are displaced from each other by 90 degrees ($\pi/2$ radians).

Magnitude/phase form of Fourier series

The transformation carried out on the $x(t)$ in the previous example can be equally well applied to a typical term of the Fourier series in (1), to obtain

$$a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$
$$= \sqrt{a_n^2 + b_n^2}\Big(\tfrac{a_n}{\sqrt{a_n^2+b_n^2}} \cos(n\omega_0 t) +$$
$$\tfrac{b_n}{\sqrt{a_n^2+b_n^2}} \sin(n\omega_0 t)\Big)$$

Letting

$$\theta_n = \arctan\left(\frac{b_n}{a_n}\right)$$

and

$$c_n = \sqrt{a_n^2 + b_n^2}$$

for $n \geq 0$ (with $c_0 = a_0$ and $\theta_0 = 0$), we get the following alternate form of (1) for a T-periodic function $x(t)$:

$$x(t) = \sum_{n=0}^{\infty} c_n \cos(n\omega_0 t - \theta_n) \qquad (5)$$

A two-sided Fourier series

It is convenient for many purposes to rewrite the Fourier series in yet another form, allowing both positive and negative multiples of the fundamental frequency. To obtain such a two-sided representation, note that

$$\begin{aligned} a_n \cos n\omega_0 t &= \frac{a_n}{2} \cos n\omega_0 t + \frac{a_n}{2} \cos n(-\omega_0) t \\ b_n \sin n\omega_0 t &= \frac{b_n}{2} \sin n\omega_0 t - \frac{b_n}{2} \sin n(-\omega_0) t \end{aligned}$$

Now for *all* integer n (negative, zero, and positive) define

$$A_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cos n\omega_0 t \, dt \tag{6}$$

$$B_n = -\frac{1}{T} \int_{-T/2}^{T/2} x(t) \sin n\omega_0 t \, dt \tag{7}$$

(The minus sign preceding the integral in the definition of B_n serves to make our notation consistent with the notation that is widely used in engineering.) ...$\rightarrow$

Two-sided Fourier series (continued)

From the preceding definitions, we conclude that $A_0 = a_0$, and for $n > 0$

$$\begin{aligned} A_n &= \frac{a_n}{2} = A_{-n} \\ B_n &= -\frac{b_n}{2} = -B_{-n} \end{aligned}$$

With these definitions, the Fourier series for a T-periodic function $x(t)$ can be written in the form

$$x(t) = \sum_{n=-\infty}^{\infty} A_n \cos n\omega_0 t - B_n \sin n\omega_0 t \qquad (8)$$

Note that the summation now runs symmetrically over all integer n (negative, zero, and positive), corresponding to terms at all integer multiples of the fundamental frequency.

Magnitude/phase form of two-sided series

By defining

$$\theta_n = \text{arctan}\left(\frac{B_n}{A_n}\right)$$

and

$$X_n = \sqrt{A_n^2 + B_n^2}$$

we can write

$$A_n = X_n \cos\theta_n \ , \qquad B_n = X_n \sin\theta_n$$

Using the now familiar procedure, we can use the preceding relations to rewrite the two-sided series in magnitude/phase form as

$$x(t) = \sum_{n=-\infty}^{\infty} X_n \cos(n\omega_0 t + \theta_n)$$

A step further

Let's go a small step beyond where you left off in 18.03:

If $x(t) = a_n \cos(n\omega_0 t)$, then the average value of $x^2(t)$ over a period — the "mean square" value of $x(t)$ — is

$$\begin{aligned} \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt &= \frac{1}{T}\int_{-T/2}^{T/2} a_n^2 \cos^2(n\omega_0 t)\,dt \\ &= \frac{a_n^2}{2} \end{aligned}$$

Extending this kind of calculation in a straightforward way, and invoking the standard trigonometric identities listed earlier, produces the various equivalent expressions shown on the next slide for the mean square value of a general periodic $x(t)$.

...⟶

Parseval's theorem

The mean square value of a T-periodic signal $x(t)$ is given in terms of its Fourier series coefficients by the following expressions:

$$
\begin{aligned}
& \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt \\
= \;& a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \\
= \;& c_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} c_n^2 \\
= \;& \sum_{n=-\infty}^{\infty}(A_n^2 + B_n^2) \\
= \;& \sum_{n=-\infty}^{\infty} X_n^2
\end{aligned}
$$

Some consequences of Parseval's

1. The mean square value of $x(t)$ is finite (for all cases of interest to us), so the infinite sums on the previous slide are all finite, which must mean that the Fourier coefficients all decay to 0 as $|n| \uparrow \infty$ — in particular, $|X_n| \downarrow 0$.

2. It is often useful or necessary to approximate $x(t)$ by a finite number of terms from its Fourier series, for instance by

$$x_N(t) = \sum_{n=-N}^{N} X_n \cos(n\omega_0 t + \theta_n)$$

Note that the highest frequency in this approximating signal is $N\omega_0$. The mean square *error* in this case is

$$\frac{1}{T}\int_{-T/2}^{T/2}\Big(x(t) - x_N(t)\Big)^2\, dt = \sum_{|n|>N} X_n^2$$

Complex notation for compact expressions

Recall

$$x(t) = \sum_{n=-\infty}^{\infty} A_n \cos n\omega_0 t - B_n \sin n\omega_0 t$$

where A_n and B_n are specified by the integrals in (6), (7). Defining

$$\widehat{X}_n = A_n + jB_n = X_n e^{j\theta_n}$$

so

$$\widehat{X}_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jn\omega_0 t}\, dt \tag{9}$$

allows us to rewrite the expression for $x(t)$ as

$$x(t) = \sum_{n=-\infty}^{\infty} \widehat{X}_n e^{jn\omega_0 t} \tag{10}$$

(The reason the imaginary part of the sum drops out is that $A_n = A_{-n}$ and $B_n = -B_{-n}$.)

Equations (9) and (10) comprise the complex form of the Fourier series representation for a T-periodic signal.

Non-periodic signals: From Fourier series to Fourier transforms

We are often interested in *non*-periodic signals, for instance an $x(t)$ of finite duration, or one that decays to 0 as $|t| \uparrow \infty$. The signals of interest to us typically satisfy

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty \quad \text{or} \quad \int_{-\infty}^{\infty} |x(t)|^2\, dt < \infty$$

We can consider such signals to have an *infinite period*, and can obtain a Fourier representation by taking the limit

$$T \uparrow \infty\ , \quad n\omega_0 = n\frac{2\pi}{T} \to \omega\ ,$$

$$\omega_0 \to d\omega\ , \quad T\widehat{X}_n \to \widehat{X}(\omega)$$

with summations suitably replaced by integrals.

The resulting $\widehat{X}(\omega)$ is termed the Fourier transform of $x(t)$.

...$\longrightarrow$

The Fourier transform

The resulting expressions replace (10) and (9) respectively by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{X}(\omega) e^{j\omega t} \, d\omega \qquad (11)$$

and

$$\widehat{X}(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt \qquad (12)$$

The first of this pair of equations, (11), is the Fourier synthesis equation, showing how a general time function may be expressed as a weighted combination of exponentials of all frequencies ω; the Fourier transform $\widehat{X}(\omega)$ determines the weighting.

The second of this pair of equations, (12), is the Fourier analysis equation, showing how to compute the Fourier transform from the signal.

Some additional observations

Remember that $\widehat{X}(\omega)$ is in general a *complex* number at each ω, even though $x(t)$ is real — reflecting the earlier definition $\widehat{X}_n = A_n + jB_n$. We shall denote the real and imaginary parts of $\widehat{X}(\omega)$ by $A(\omega)$ and $B(\omega)$ respectively, so

$$\widehat{X}(\omega) = A(\omega) + jB(\omega)$$

We shall denote the magnitude and angle of $\widehat{X}(\omega)$ by $X(\omega)$ and $\theta_X(\omega)$ respectively, so

$$|\widehat{X}(\omega)| = X(\omega) \qquad \angle\widehat{X}(\omega) = \theta_X(\omega)$$

Thus

$$\widehat{X}(\omega) = X(\omega)e^{j\theta_X(\omega)}$$