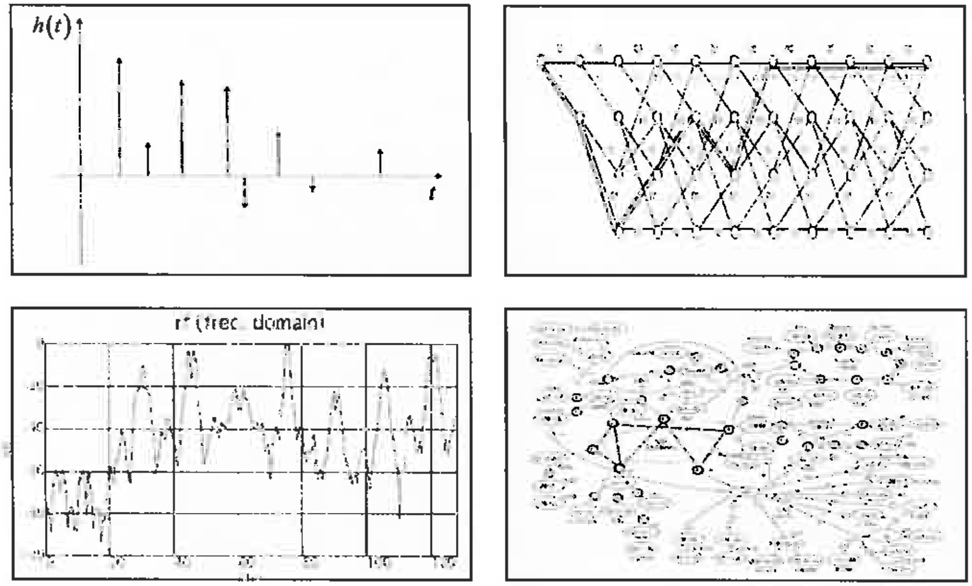


INTRODUCTION TO EECS II

DIGITAL
COMMUNICATION
SYSTEMS

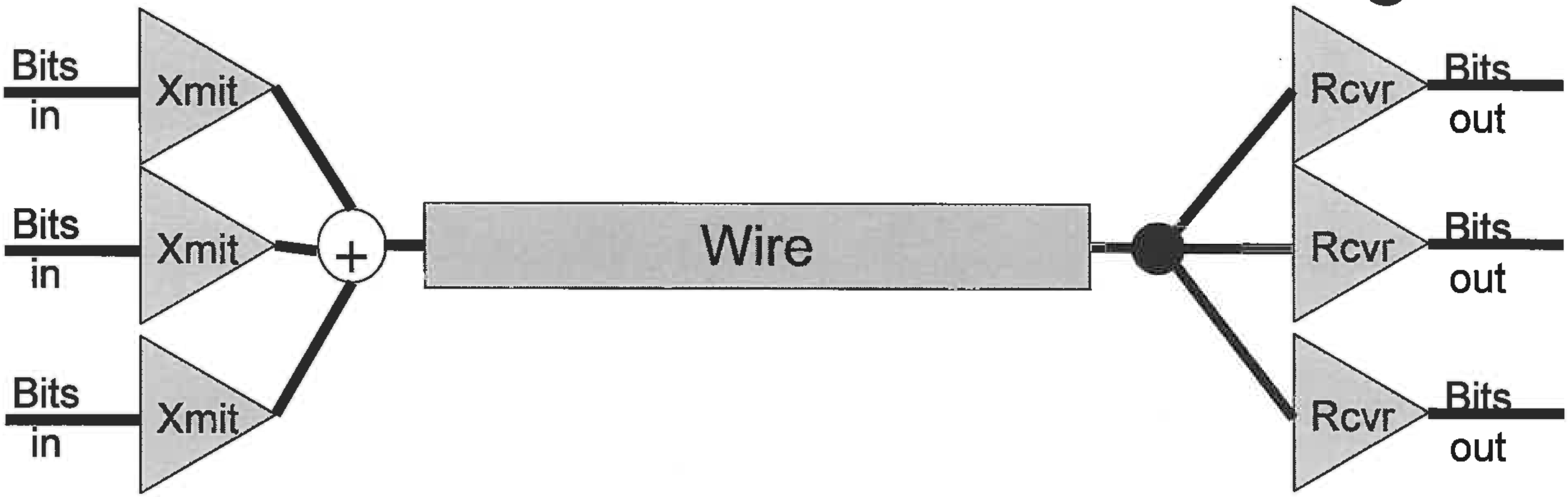
6.02 Fall 2010
Lecture #12

- Summary of progress
- The problem of sharing in frequency domain
- Sines → LTI → Sines by experiment
- Discrete-Time Complex Exponentials
- Frequency Response

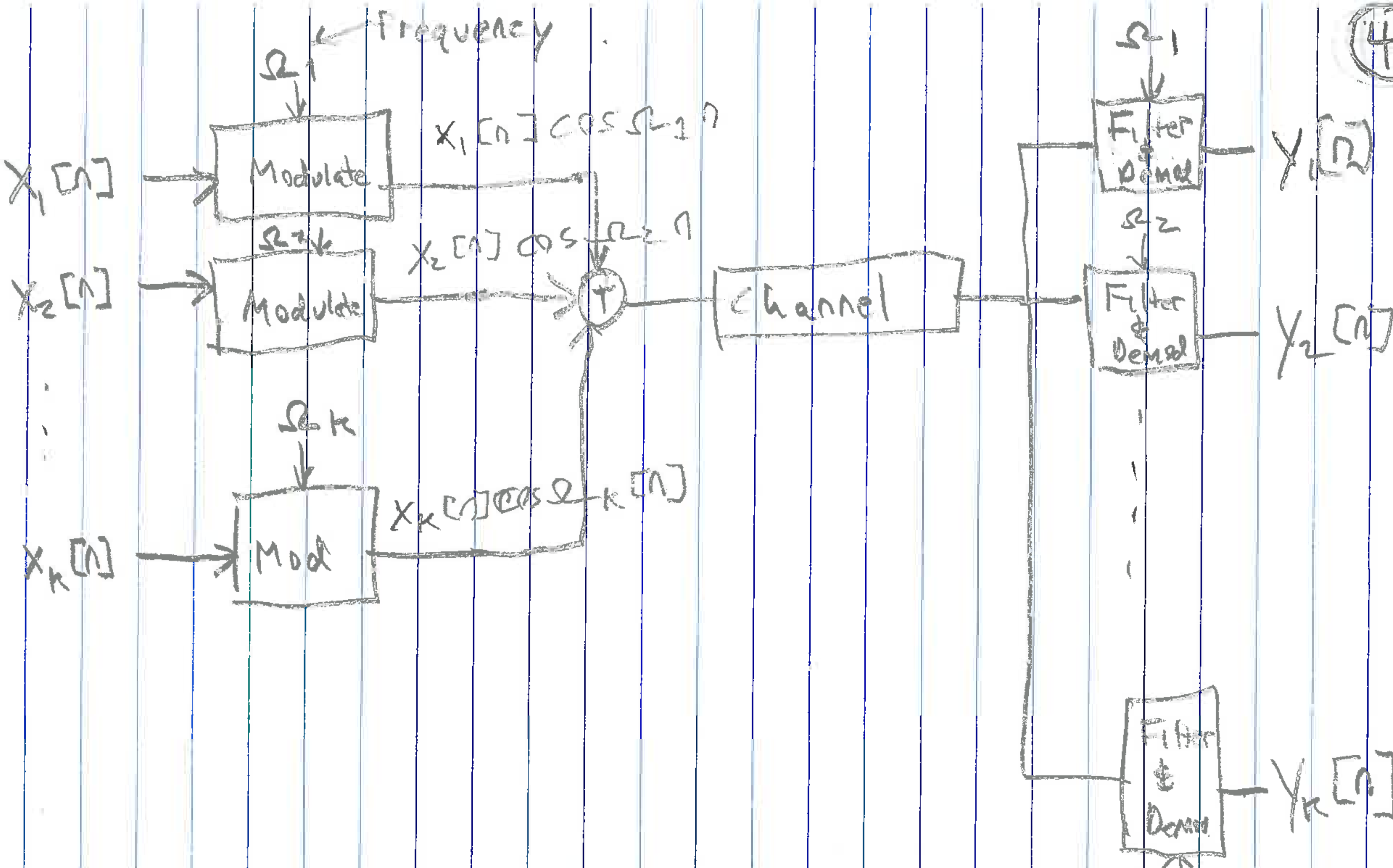
First Problem - Low BER on Wires

- Problems and Analysis Techniques:
  - Intersymbol Interference (ISI)
    - LTI Systems and Unit Sample Responses
    - Eye Diagrams
  - Noise
    - Probability Density and Cumulative Distribution Functions
    - Normal (Gaussian) random variables
- Solution Approaches
  - Techniques based on LTI models
    - Deconvolution
  - Error Detection and Correction Codes
    - Parity bits, Convolutional Code, Viterbi Algorithm

Second Problem - Resource Sharing



- Two Approaches
  - Time Sharing (e.g. TDMA)
    - Xmit-Rcvr pair gets a dedicated time slot or contend for slots
    - Used in both wired and wireless networks
  - Frequency Sharing (aka Frequency Division Multiplexing)
    - Each Xmit-Rcvr gets part of the spectrum (explained later)
    - Used in Broadcast TV, cell phones and WiFi.
  - Most WiFi systems use TDMA and FDM
- Next Two Weeks on issues in FDM

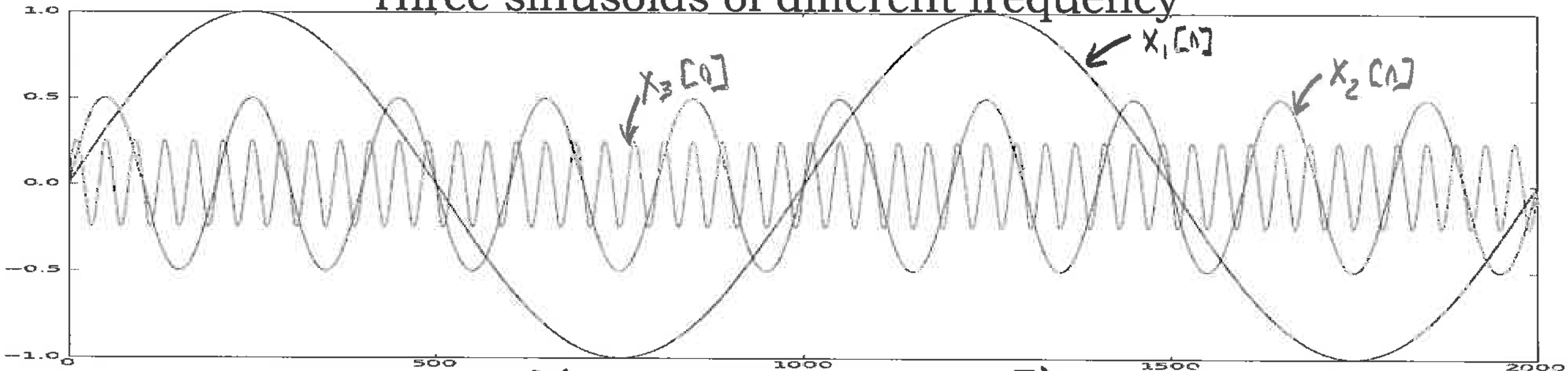


Larger Picture

5

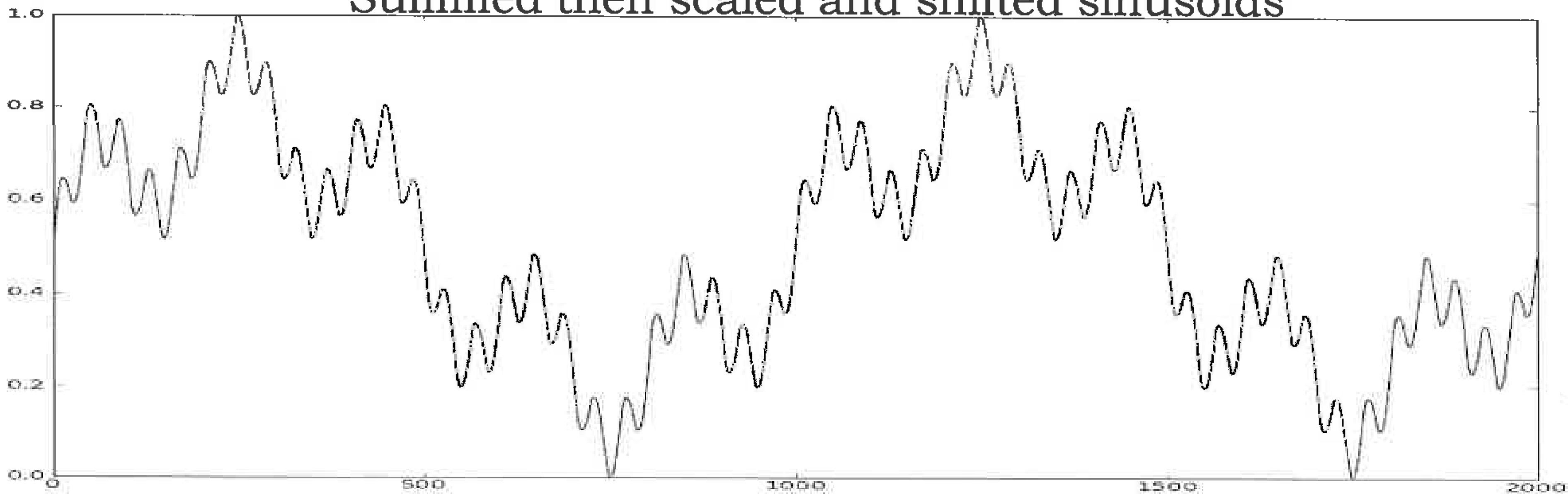
Sinusoids and LTI Systems

Three sinusoids of different frequency



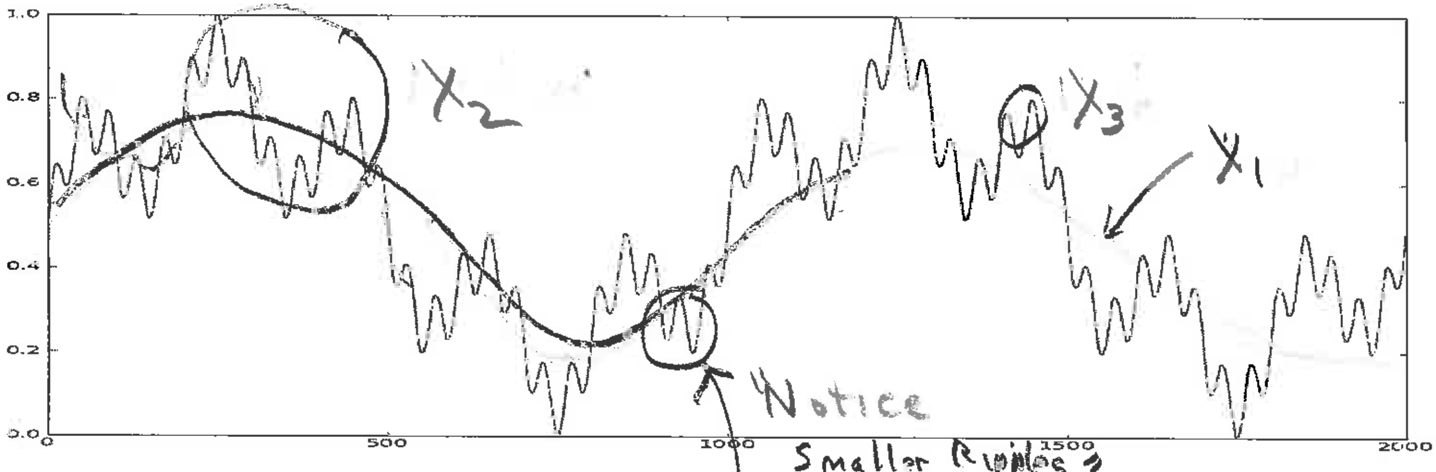
$$x_{3s}[n] = \frac{2}{7}\left(x_1[n] + x_2[n] + x_3[n] + \frac{7}{4}\right)$$

Summed then scaled and shifted sinusoids

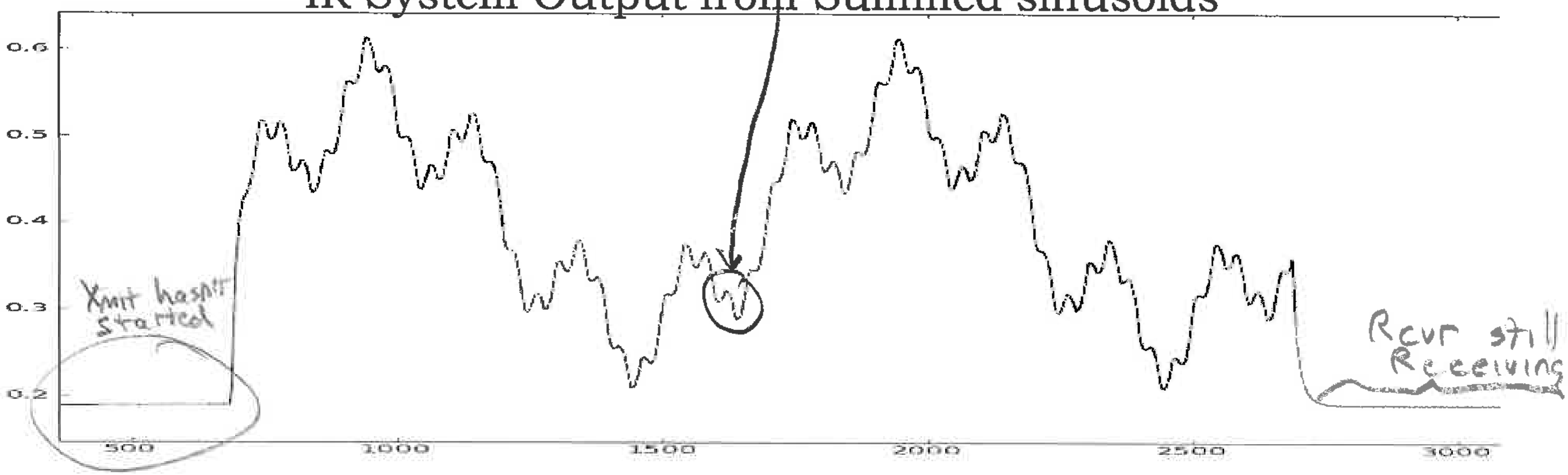


Sinusoids and LTI Systems

Summed then scaled and shifted sinusoids



IR System Output from Summed sinusoids



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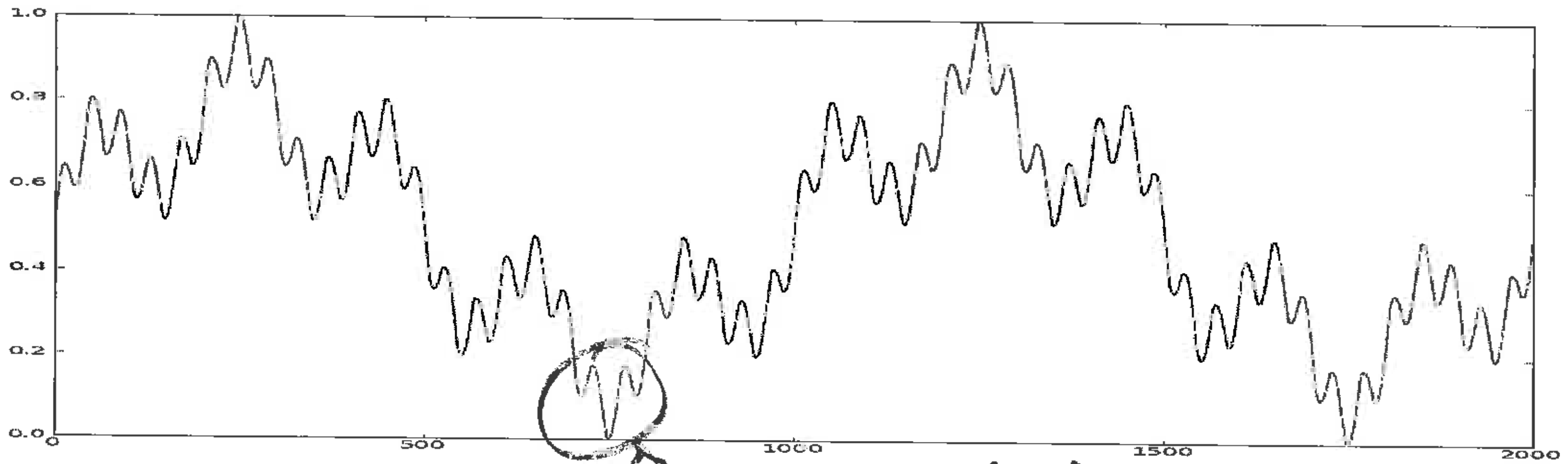
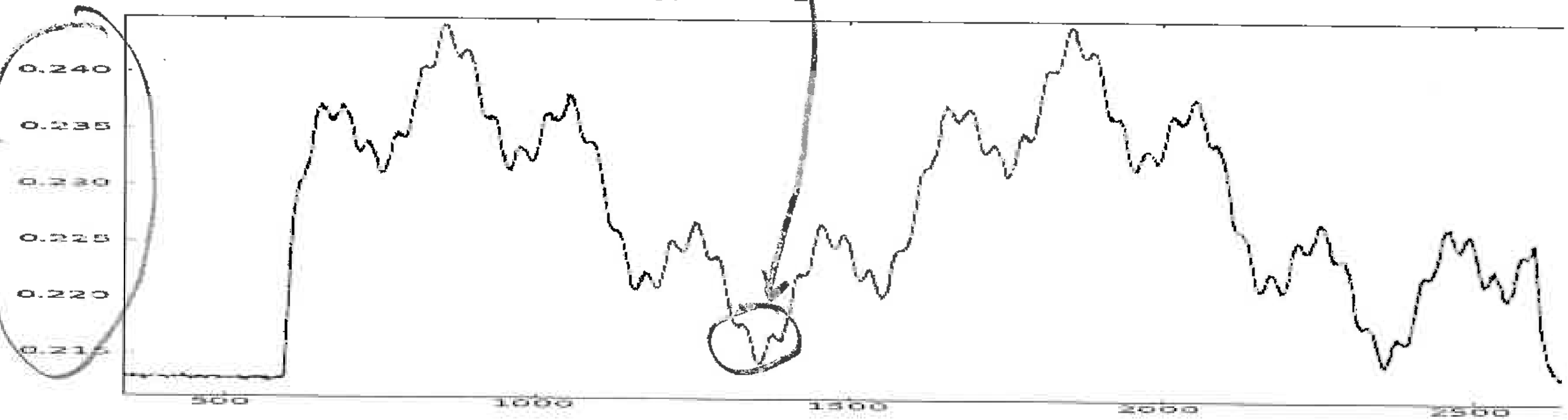
Sinusoids and LTI Systems

Summed then scaled and shifted sinusoids



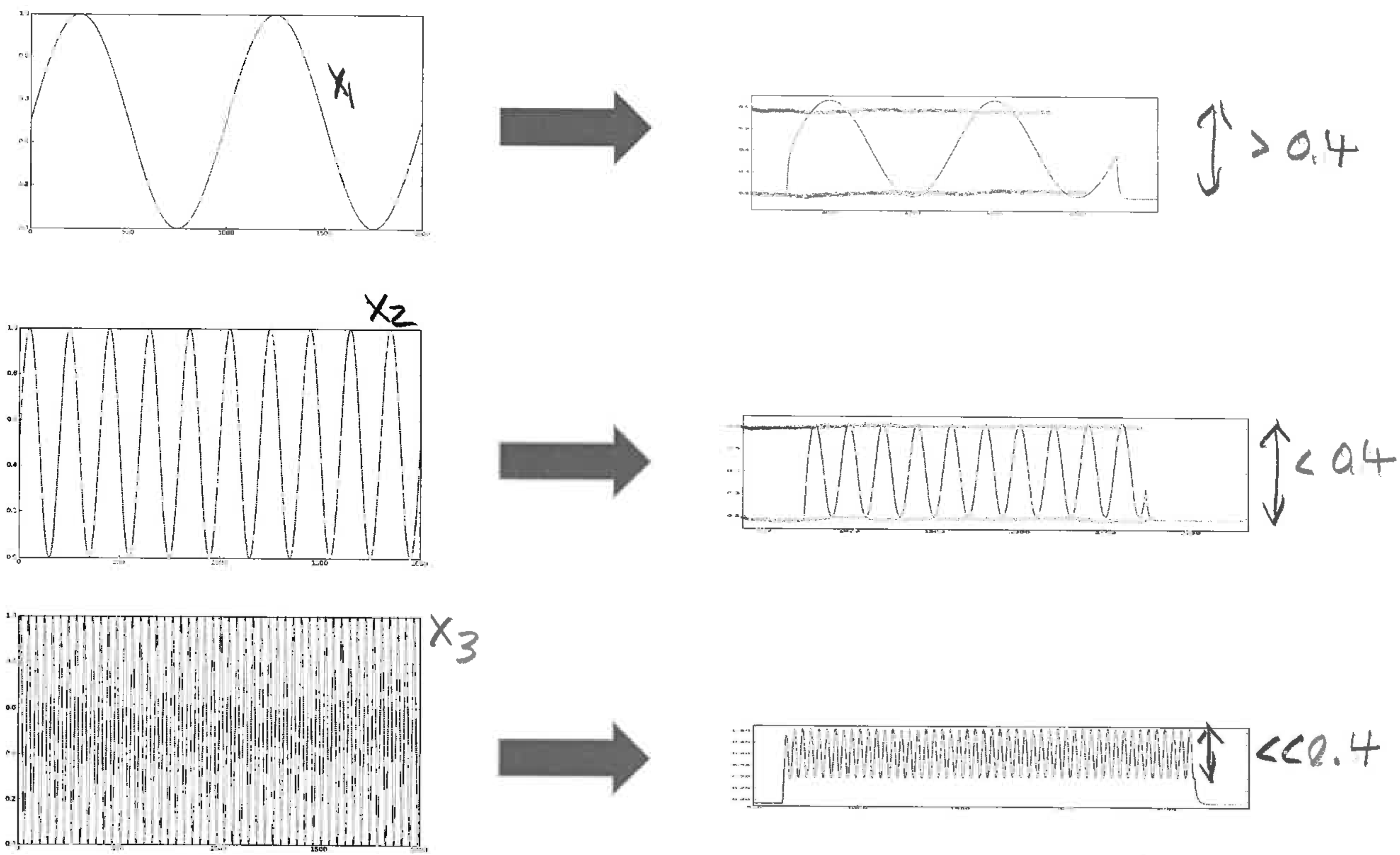
still see ripples from x_3

IR System (off ceiling) Output from Summed sinusoids



Note smaller Range

IR System Response to Different Sines



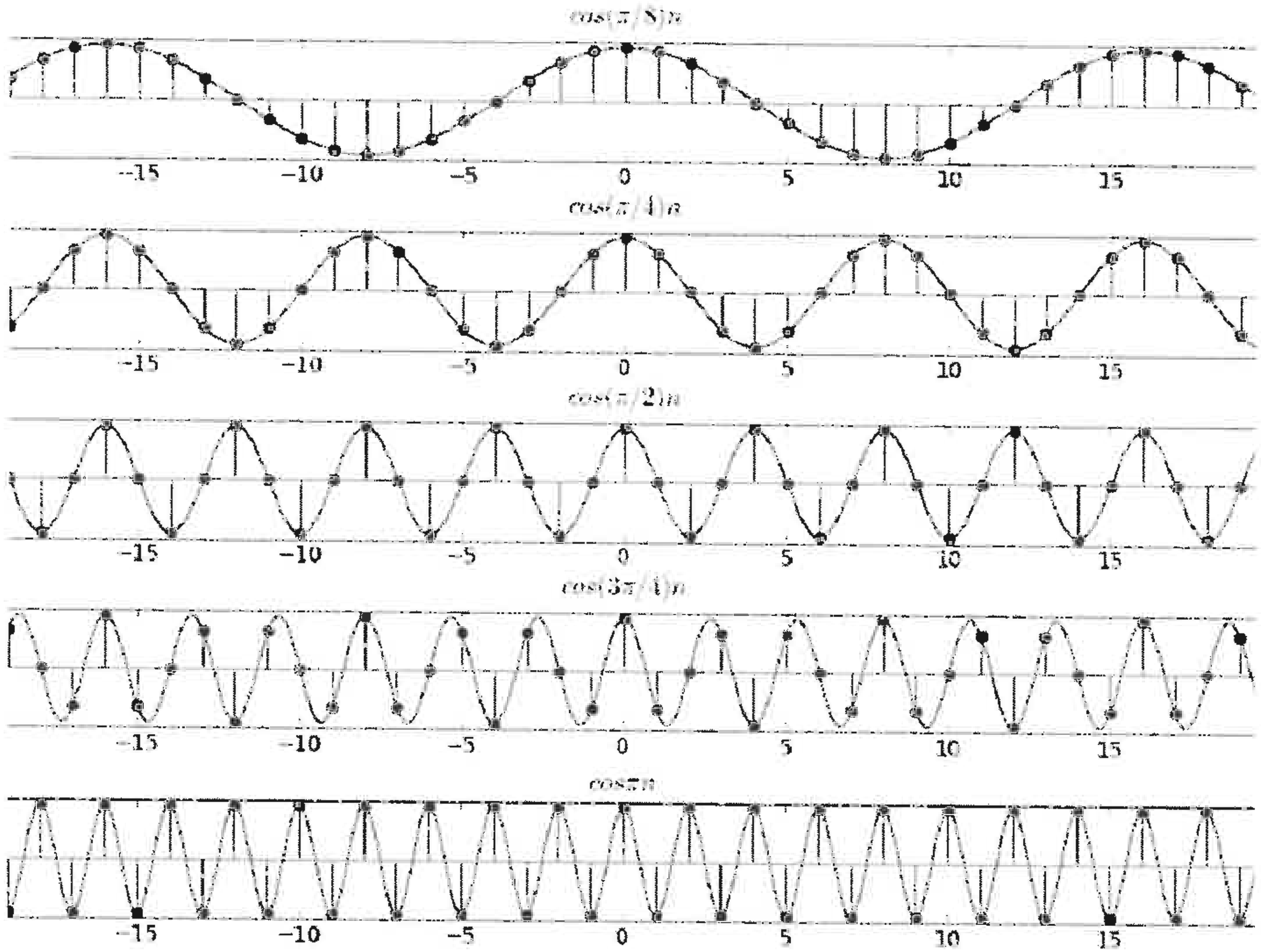
9

Sinusoids in LTI Systems

- Sinusoids → LTI Systems → Sinusoids
  - Different Frequencies do not seem to mix
    - Xmit: encode data on different frequencies (Modulation)
    - Rcvr: decode data from different frequences (Filtering)
  - Different Frequencies have different "Gain"
    - E.G. the IR Channel:
      - higher frequencies → lower "gain".
- Analyzing Sinusoids in LTI systems
  - Model as Eternal signals (-infinity < n < infinity)
  - Difficulty Computing response to $x[n] = \cos \Omega n$
  - Simplify response computation using complex exponentials

$$\cos \Omega n = \frac{e^{j\Omega n} + e^{-j\Omega n}}{2} \qquad \sin \Omega n = \frac{e^{j\Omega n} - e^{-j\Omega n}}{2j}$$

Discrete Time Cosines $-\infty < n < \infty$



Why is pi highest frequency?

Correspondence

Suppose $f_s = 4 \cdot 10^6$

$\Omega = \pi/8$

$$\cos \frac{1}{8}\pi n \Rightarrow \cos 2\pi \underbrace{\left(\frac{f_s}{16}\right)}_{250{,}000 \text{ cycle/sec}} t$$

or

$$\cos \pi \frac{f_s}{8} t$$

$$\cos \frac{\pi}{4} n \Rightarrow \cos 2\pi \underbrace{\left(\frac{f_s}{8}\right)}_{500{,}000} t \quad \frac{\text{cycles}}{\text{sec}}$$

$$\cos \frac{3\pi}{4} n \Rightarrow \cos 2\pi \underbrace{\left(\frac{3f_s}{8}\right)}_{1{,}500{,}000} t \quad \frac{\text{cycles}}{\text{sec}}$$

$$\cos \pi n \Rightarrow \cos 2\pi \underbrace{\left(f_s/2\right)}_{2{,}000{,}000} t \quad \frac{\text{cycles}}{\text{sec}}$$

Sampling at $4 \cdot 10^6$ — continuous waveform

LTI Systems for Eternal Sines and Cosines

- Convolution Changes a little

x[n] is eternal

$$y[n] = \sum_{m=0}^{\infty} h[m]x[n-m] = \sum_{m=0}^{\infty} h[m]\cos\Omega[n-m]$$

h[m] is still causal

- If unit sample response finite length (L)

$$y[n] = \sum_{m=0}^{L} h[m]\cos\Omega[n-m]$$

How do we simplify?

Frequency Response with Complex Exponentials

13

- From convolution

$$y[n] = \sum_{m=0}^{L} h[m] e^{j\Omega(n-m)}$$

Reorganizing

$$y[n] = \underbrace{\left(\sum_{m=0}^{L} h[m] e^{-j\Omega m}\right)}_{\text{A } \textit{complex} \text{ number if the sum converges}} e^{j\Omega n}$$

$$y[n] = H\left(e^{j\Omega}\right) e^{j\Omega n}$$

$$\boxed{H(e^{j\Omega}),\ -\pi < \Omega < \pi, \text{ is the frequency response}}$$

Most Interested in Magnitude

$$H(e^{j\Omega}) = a + bj \qquad j = \sqrt{-1}$$

$$\left\| H(e^{j\Omega}) \right\| = \sqrt{a^2 + b^2}$$

$$H(e^{j\Omega}) = \left\| H(e^{j\Omega}) \right\| \underbrace{e^{j\theta}}_{\text{Angle}} \quad -\pi \le \theta \le \pi$$

$$H(e^{-j\Omega}) = a - bj$$

$$\left\| H(e^{-j\Omega}) \right\| = \sqrt{a^2 + b^2} = \left\| H(e^{j\Omega}) \right\|$$

$$H(e^{-j\Omega}) = \left| H(e^{j\Omega}) \right| e^{-j\theta} \quad -\pi \le \theta \le \pi$$

$\boxed{\text{$|H(e^{j\Omega})|$ is an even function of Ω; $\measuredangle H(e^{j\Omega})$ is an odd function of Ω}}$

Going back to Cosines

- Suppose

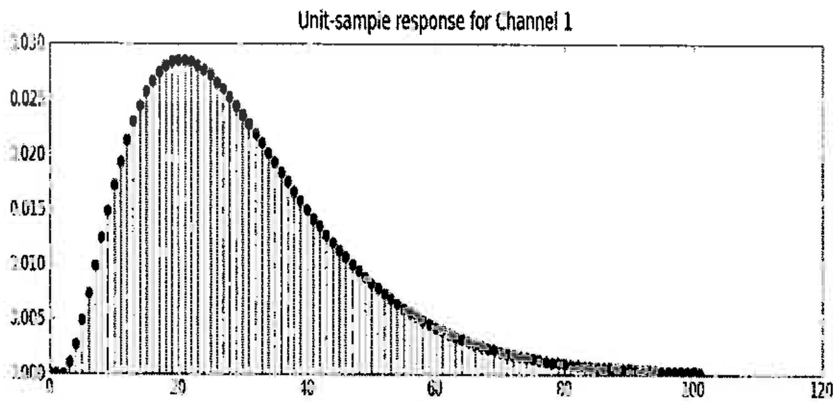
$$x[n] = \cos \Omega n = \tfrac{1}{2} e^{j\Omega n} + \tfrac{1}{2} e^{-j\Omega n}$$

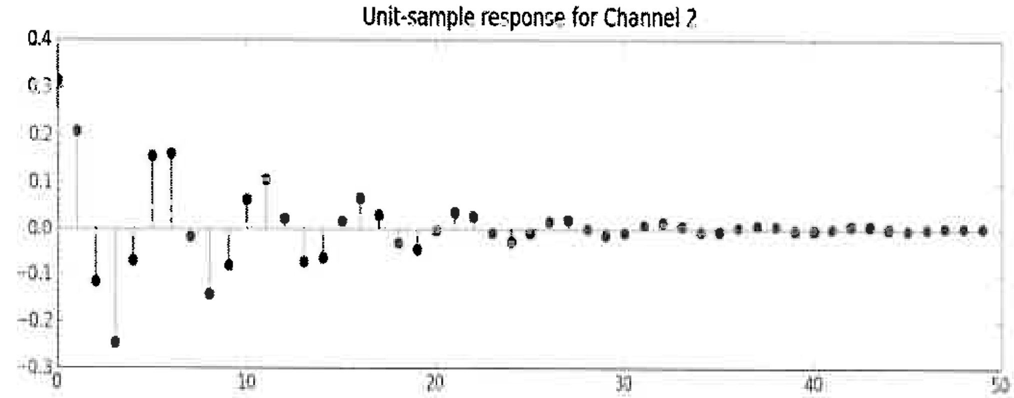
$$y[n] = \left(\textstyle\sum_{m=0}^{L} h[m] e^{-j\Omega m}\right) \frac{e^{j\Omega n}}{2} + \left(\textstyle\sum_{m=0}^{L} h[m] e^{j\Omega m}\right) \frac{e^{-j\Omega n}}{2}$$

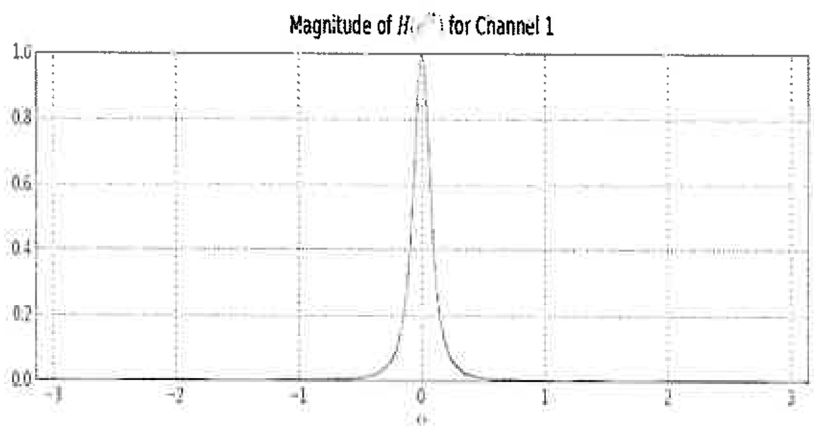
$$y[n] = \tfrac{1}{2} H\left(e^{j\Omega}\right) e^{j\Omega n} + \tfrac{1}{2} H\left(e^{-j\Omega}\right) e^{-j\Omega n}$$

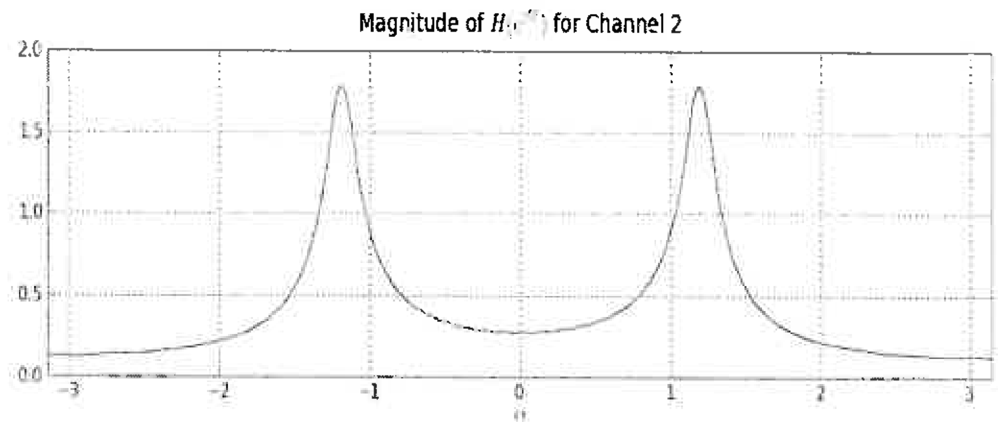
- Just using LTI superposition property!

Channel 1 and 2 Frequency Responses



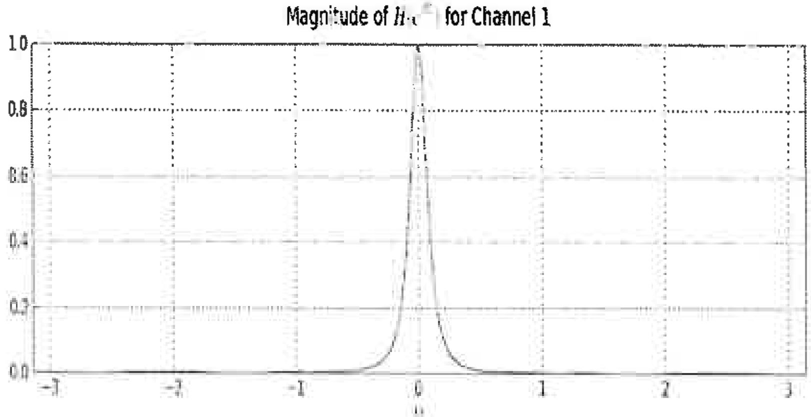


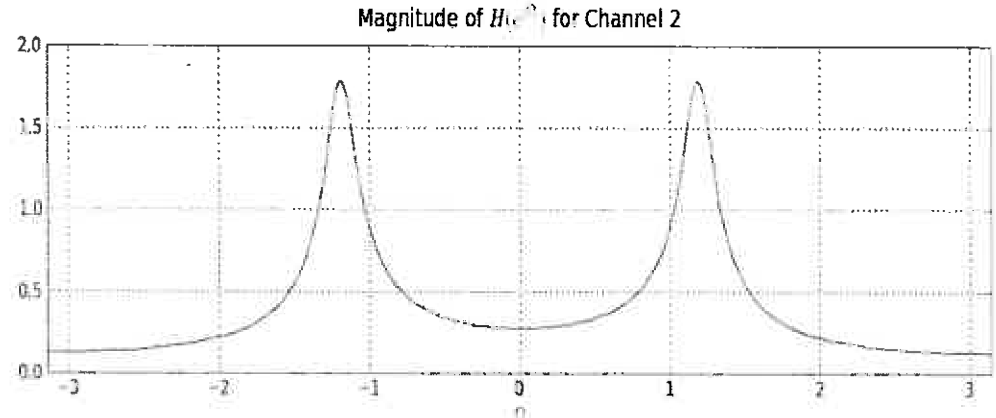




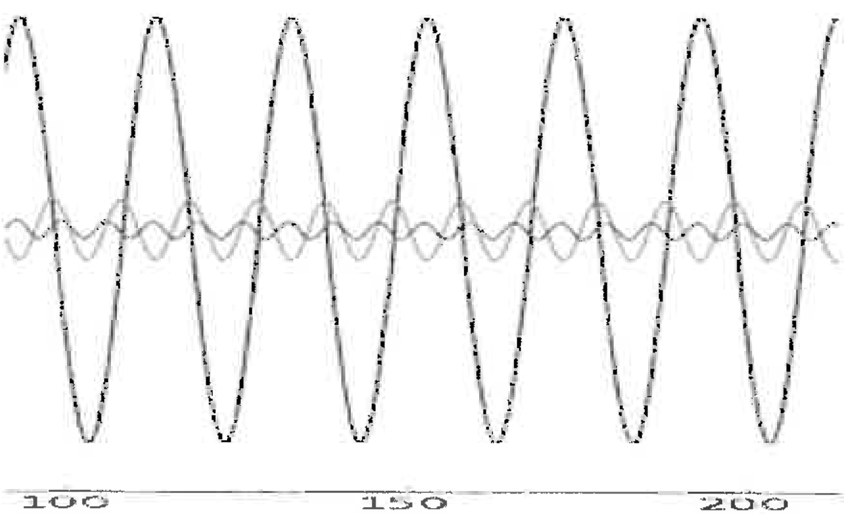
Channel 1 and 2 Frequency Responses

$$x[n] = \cos(\Omega n) \qquad \Omega = \left\{\frac{\pi}{10}, \frac{2\pi}{10}, \frac{3\pi}{10}\right\}$$

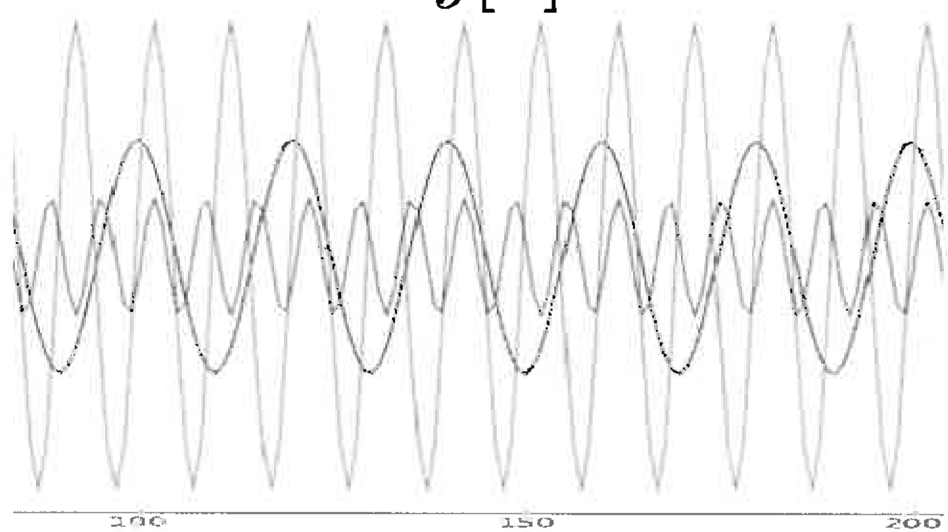




$y[n]$



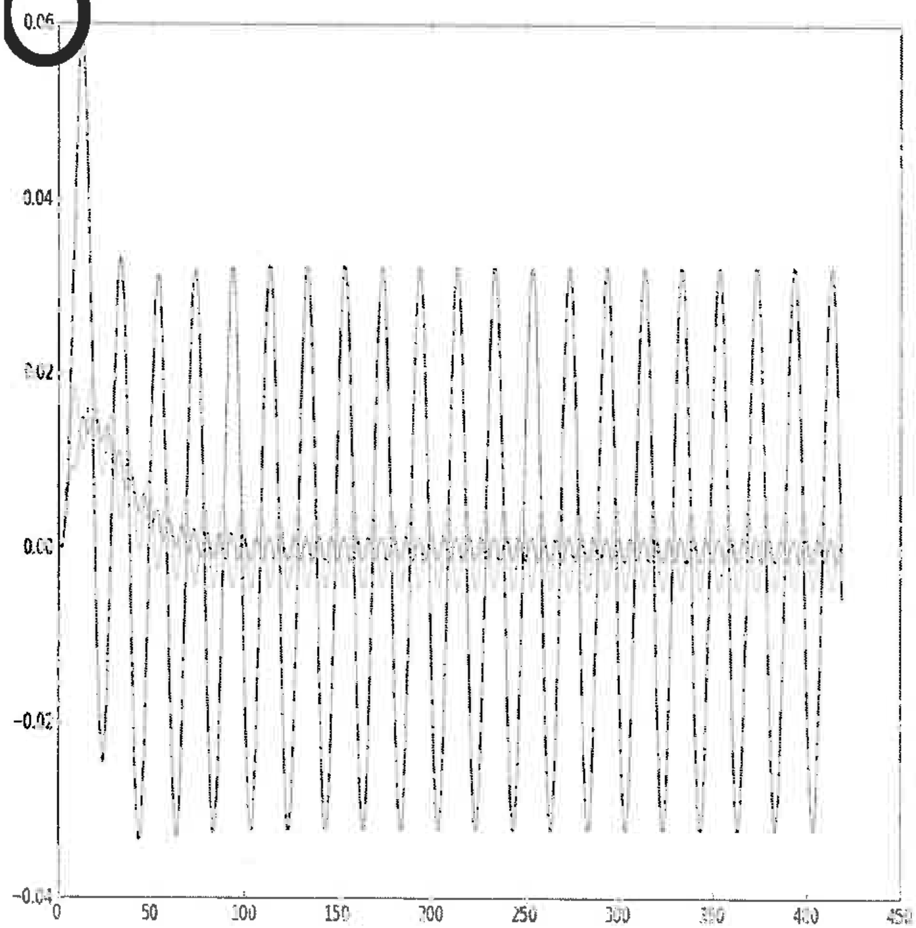
$y[n]$



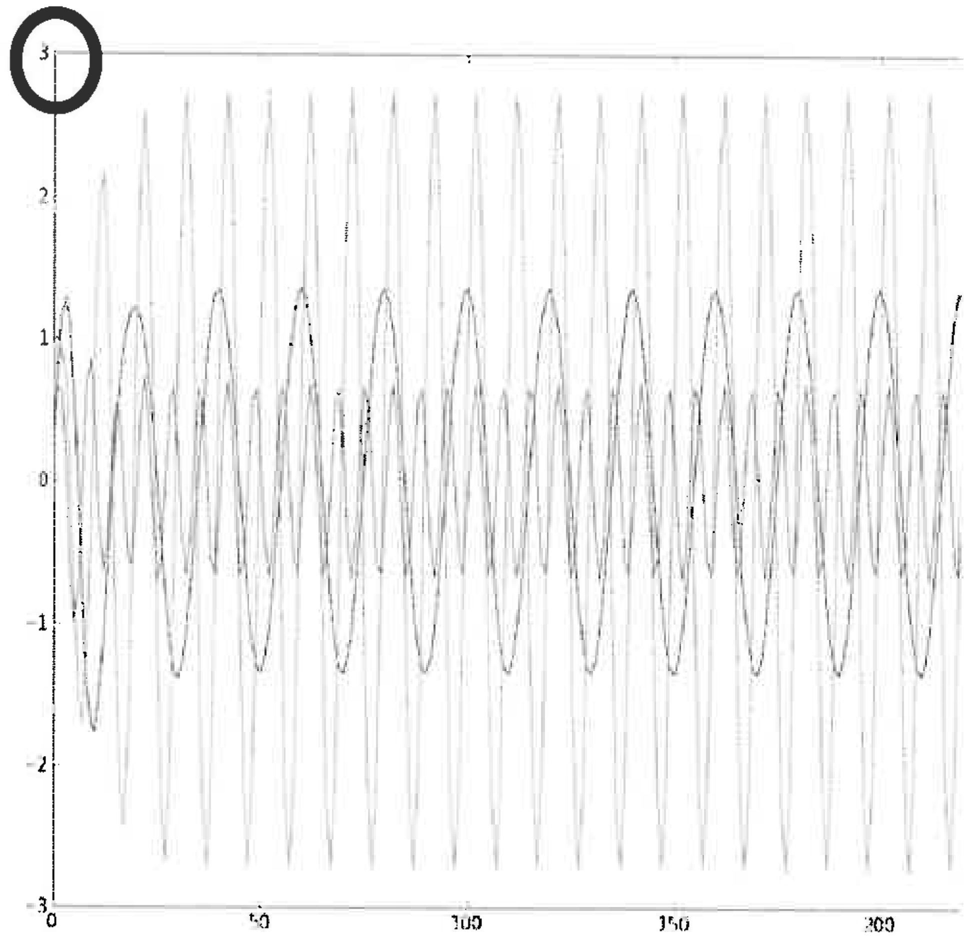
Non-eternal Example: Cosine starts at zero

$$x[n] = \cos(\Omega n)u[n] \;\; \Omega = \left\{\frac{\pi}{10}, \frac{2\pi}{10}, \frac{3\pi}{10}\right\}$$

Slow Channel



Fast Channel



Summary and Next Time

- Frequency Division Multiplexing
  - Eternal frequencies do not mix when input to LTI systems
  - Frequency division multiplexing

$$x[n] = A_1 e^{j\Omega_1 n} + \ldots + A_K e^{j\Omega_K n}$$

  - Frequency Response

$$H\left(e^{j\Omega}\right) \quad \Omega \in [-\pi, \pi]$$

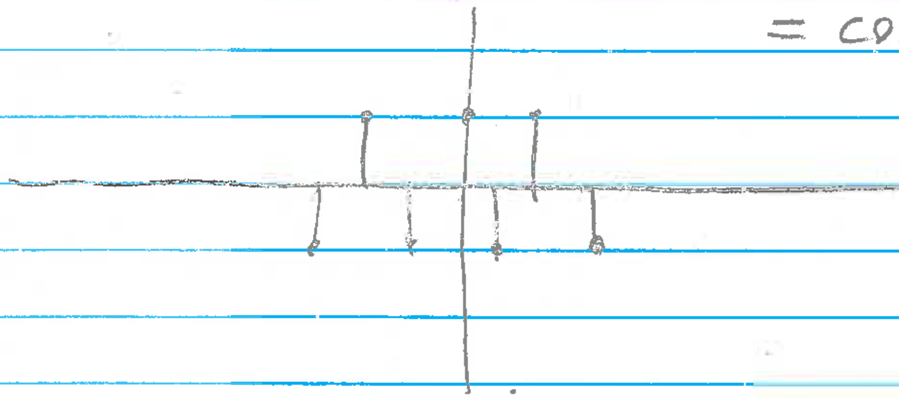
- Next Two Weeks
  - Need Filters at receiver to separate messages.
  - How do we analyze more complicated messages

$$x[n] = x_1[n] e^{j\Omega_1 n} + \ldots + x_K[n] e^{j\Omega_K n}$$

    - Do the modulated messages still stay separated?
  - Fourier Analysis and Spectrum
  - Modulation Schemes

Only consider $-\pi \le \Omega \le \pi$

Highest frequency $= e^{j\pi n} = \cos \pi n + j \sin \pi n$
$= \cos \pi n = (-1)^n$



$$e^{j(-\pi)n} = \cos(-\pi)n + j\sin(-\pi)n$$
$$= \cos \pi n = (-1)^n$$

Consider $\Omega = \pi + \Delta \quad \Delta > 0$ (outside $-\pi \to \pi$)

$$e^{j(\pi+\Delta)n} = \cos(\pi+\Delta)n + j\sin(\pi+\Delta)n$$

$$e^{j\pi n} e^{j\Delta n} = (-1)^n (\cos \Delta n + j \sin \Delta n)$$

$$e^{j(-\pi+\Delta)n} = e^{j(-\pi)n} e^{j\Delta n}$$
$$= (-1)^n (\cos \Delta n + j \sin \Delta n)$$

$$e^{j(\pi+\Delta)n} = e^{j(-\pi+\Delta)n}$$

$\underbrace{\Omega = \pi + \Delta}_{\text{outside } -\pi \to \pi}$ same as $\underbrace{\Omega = \Delta - \pi}_{\text{inside } -\pi \to \pi}$

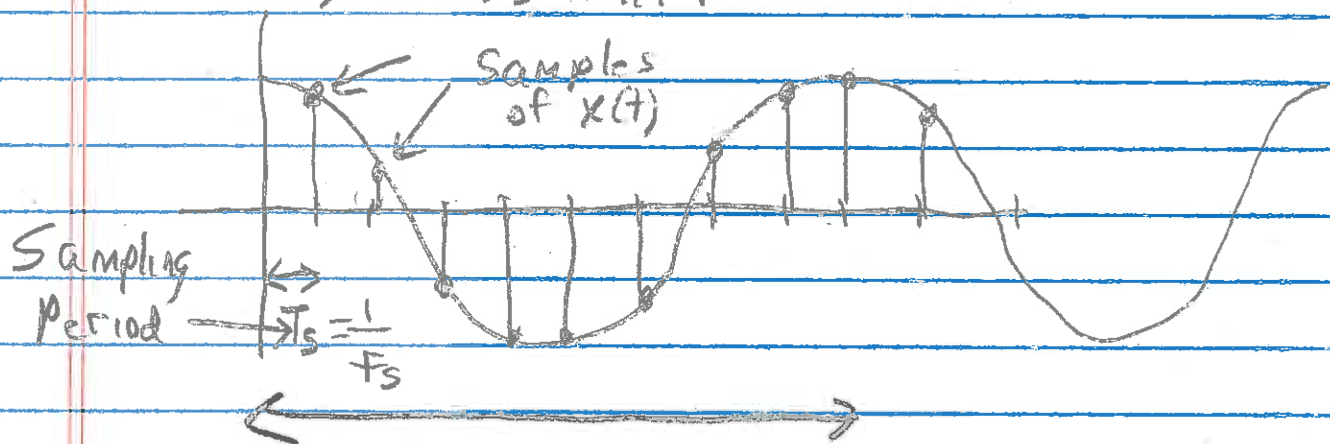
Connection To Sample Rate

Continuous time Signal

$$x(t) = \cos 2\pi f t$$

f ↑ frequency in cycles/second

$x(t) = \cos 2\pi f t$



$$T = \frac{1}{f} = \text{period of cosine}$$

Sampling Period $\equiv T_s$

Sampling frequency $\equiv f_s = 1/T_s$

$$x[n] = x(t)\Big|_{t=nT_s} = \cos \underbrace{2\pi f T_s}_{\Omega} n$$

Maximum Frequency

If $\Omega_{max} = \pi \Rightarrow 2\pi f_{max} T_s = \pi \Rightarrow f_{max} = \frac{1}{2T_s} = \frac{f_s}{2}$

$\Rightarrow -\pi \le \Omega \le \pi$ means $-\frac{f_s}{2} \le f \le f_s/2$

Example of Frequency Response

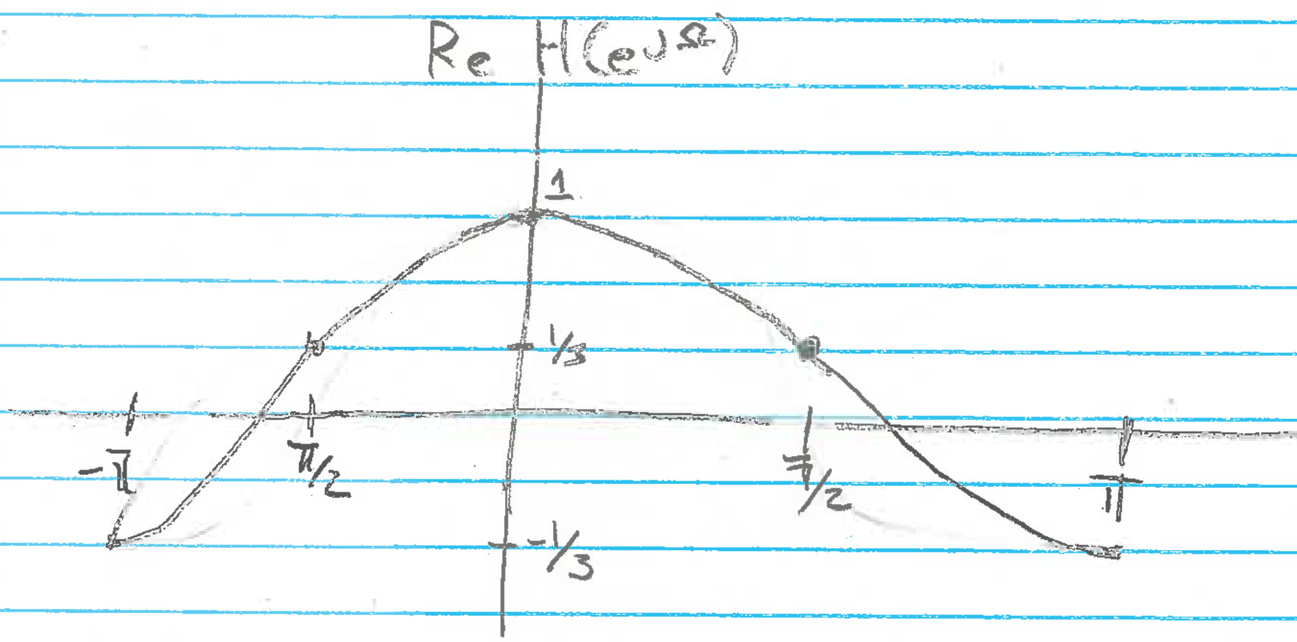
$$H(e^{j\Omega}) = \frac{1}{3}e^{j\Omega} + \frac{1}{3}e^{-j\Omega} + \frac{1}{3}$$

$$= \frac{1}{3}(\cos\Omega + j\sin\Omega)$$

$$+ \frac{1}{3}(\cos\Omega - j\sin\Omega)$$

$$+ \frac{1}{3}$$

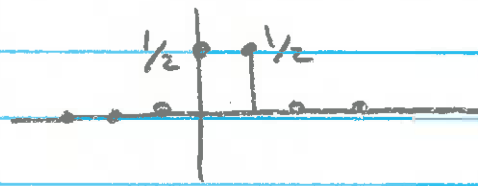
$$= \frac{2}{3}\cos\Omega + \frac{1}{3}$$



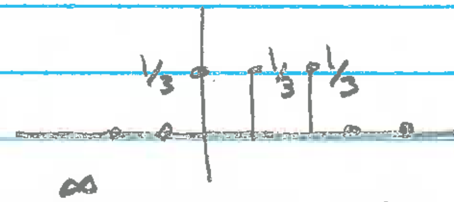
Example

Frequency Response in two cases

$h_1[n]$



$h_2[n]$

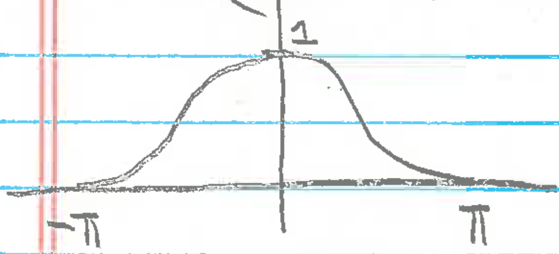


$$H(e^{j\Omega}) = \sum_{m=0}^{\infty} h[m]\, e^{-j\Omega m}$$

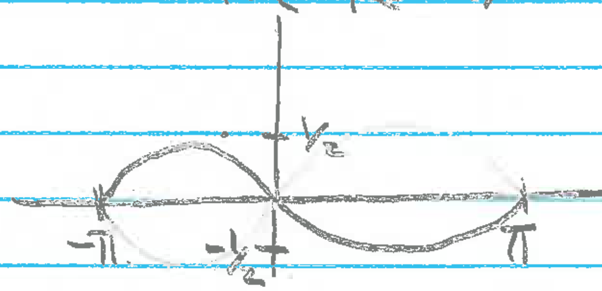
$$H_1(e^{j\Omega}) = h_1[0]\cdot e^{-j\Omega 0} + h_1[1]\, e^{-j\Omega \cdot 1}$$
$$= \tfrac{1}{2}(1 + \cos\Omega) - \tfrac{1}{2}j \sin\Omega$$

$$H_2(e^{j\Omega}) = \tfrac{1}{3} e^{-j\Omega 0} + \tfrac{1}{3} e^{-j\Omega 1} + \tfrac{1}{3} e^{-j\Omega \cdot 2}$$
$$= \tfrac{1}{3}(1 + \cos\Omega + \cos 2\Omega) - \tfrac{1}{3}j(\sin\Omega + \sin 2\Omega)$$

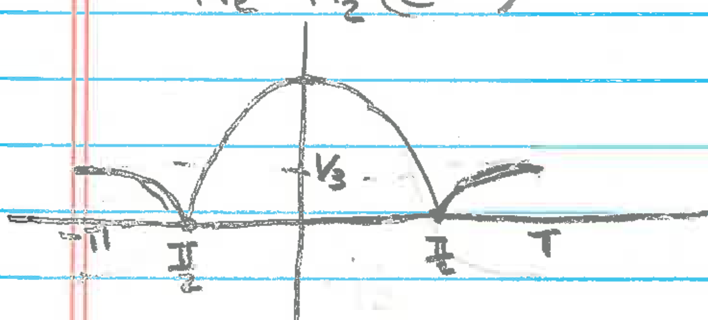
$Re(H_1(e^{j\Omega}))$



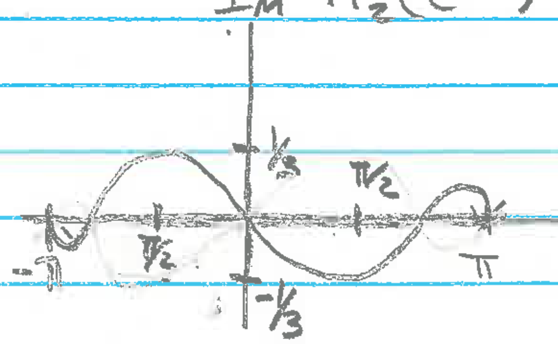
$Im(H_1(e^{j\Omega}))$



$Re\ H_2(e^{j\Omega})$



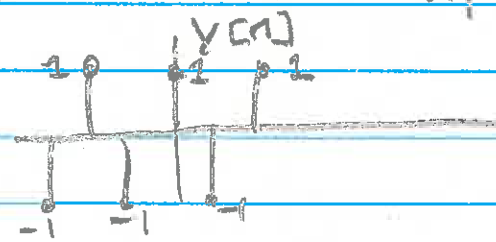
$Im\ H_2(e^{j\Omega})$

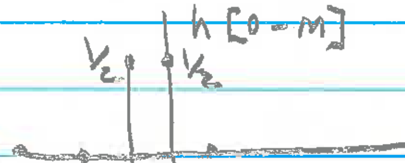


Note if $x[n] = (-1)^n = e^{j\pi n}$

$$y_1[n] = H_1(e^{j\pi})\, e^{j\pi n} = 0 \cdot e^{j\pi n} = 0$$

Makes sense convolve h_1 with y



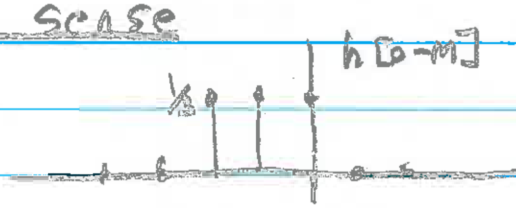


$$\Rightarrow y[0] = 1\cdot\tfrac{1}{2} + (-1)\tfrac{1}{2} = 0$$

$$y[n] = (-1)\tfrac{1}{2} + (1)\tfrac{1}{2} = 0$$

$$y_2[n] = H_1(e^{j\pi})\, e^{j\pi n} = \tfrac{1}{3} e^{j\pi n} = \tfrac{1}{3}(-1)$$

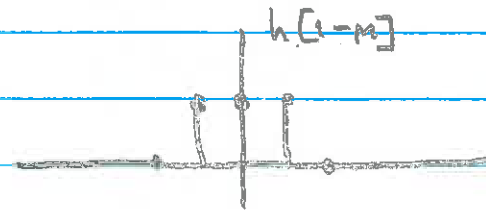
Makes sense



$y[0] = \frac{1}{3}$

$y[1] = -\frac{1}{3}$

$y[2] = \frac{1}{3}$



Suppose $x[n] = \cos \Omega n = \frac{1}{2}\left(e^{j\Omega n} + e^{-j\Omega n}\right)$

$$y[n] = \sum h[m] \cos \Omega(n-m)$$

$$y[n] = H(e^{j\Omega})\tfrac{1}{2}e^{j\Omega n} + H(e^{-j\Omega})\tfrac{1}{2}e^{-j\Omega n}$$

Consider $\Omega = \frac{\pi}{2}$ and system 2

$$H(e^{j\pi/2}) = 0 + -\tfrac{1}{3}j$$

$$H(e^{-j\pi/2}) = 0 + \tfrac{1}{3}j$$

$$y[n] = -\tfrac{1}{3}j\,\tfrac{1}{2}e^{j\frac{\pi}{2}n} - \tfrac{1}{3}j\,\tfrac{1}{2}e^{-j\frac{\pi}{2}n}$$

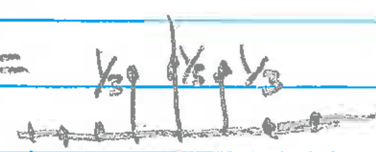
$$= -\tfrac{1}{6}j\left(\cos\tfrac{\pi}{2}n + j\sin\tfrac{\pi}{2}n - \left(\cos\tfrac{\pi}{2}n - j\sin\tfrac{\pi}{2}n\right)\right)$$

$$= -\tfrac{1}{6}j\left(2j\sin\tfrac{\pi}{2}n\right)$$

$$= \tfrac{1}{3}\sin\tfrac{\pi}{2}n$$

Same Amplitude Different phase

Suppose $h[n] =$



(Analyzed Before)

$$y[n] = \underbrace{H(e^{j\frac{\pi}{2}})}_{1/3}\tfrac{1}{2}e^{j\frac{\pi}{2}n} + \underbrace{H(e^{-j\frac{\pi}{2}})}_{1/3}\tfrac{1}{2}e^{-j\frac{\pi}{2}n}$$

$$= \tfrac{1}{6}\left(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}\right)$$

$$= \tfrac{1}{3}\cos \tfrac{\pi}{2} n$$

Consider Deconvolution
(Noise-Free Case)

$X \to \boxed{\text{Channel } H} \to Y \to \boxed{\text{Deconvolver (Ideal)}} \to W$

$$y[n] = \sum_{m=0}^{L} h[n-m]\, x[m]$$

$$\sum h[n-m]\, w[m] = y[n]$$

H_{chan}: $h[n],\ n \in [0, \ldots, L]$; $H(e^{j\Omega})\ \ \Omega \in [-\pi, \pi]$

H_{dec} ?

$$H_{dec}(e^{j\Omega}) = \frac{1}{H(e^{j\Omega})}$$

Made clearer Next Lecture

$$w[n] = H_{dec}(e^{j\Omega}) \cdot \left(H_{chan}(e^{j\Omega})\, e^{j\Omega n}\right) = \underbrace{H_{dec}(e^{j\Omega})\, H_{chan}(e^{j\Omega})}_{=1} e^{j\Omega n}$$