

LECTURE 13

Introduction to Filtering

This lecture introduces the idea of **filtering**, which we will use at the receiver to “pick out” signals in the frequency band of interest. Filters are widely applicable and used to solve many different problems. There are many different specialized techniques in place for filter design, and one can easily devote an entire course (or more) to this topic! We will focus on the essential ideas and describe **low-pass** and **high-pass** filters. The key concept in our discussion is the notion of **composition**, which is a way to take simple filters and combine them in series and parallel to realize more complex filters. Understanding filters in the *frequency domain* will turn out to be a convenient and useful approach.

■ 13.1 Complex Numbers and Frequency Response

One of the key points of the previous lecture was that if the input to an LTI system is an eternal complex exponential, $x[n] = e^{j\Omega n}$, $-\infty < n < \infty$, and if the LTI system has an L -length unit sample response, then the output, $y[n]$, is given by

$$y[n] = H(e^{j\Omega})e^{j\Omega n} \quad (13.1)$$

where

$$H(e^{j\Omega}) \equiv \sum_{m=0}^L h[m]e^{-j\Omega m}. \quad (13.2)$$

We referred to $H(e^{j\Omega})$, for Ω over the range $-\pi \leq \Omega \leq \pi$, as the frequency response of the LTI system.

For a given value of Ω , $H(e^{j\Omega})$ is a complex number, and below it will be convenient to use the polar form for a complex number. As a reminder, a complex number z can be written in Cartesian form, $z = x + jy$, where $x = \text{Re}(z)$ and $y = \text{Im}(z)$, or in polar form,

$$z = |z|e^{j\theta},$$

where $|z| = \sqrt{x^2 + y^2}$ is the magnitude of the complex number, θ is the angle, $x = \text{Re}(z) = |z|\cos\theta$, and $y = \text{Im}(z) = |z|\sin\theta$ (see Figure 13-1 for a summary diagram of the polar form).

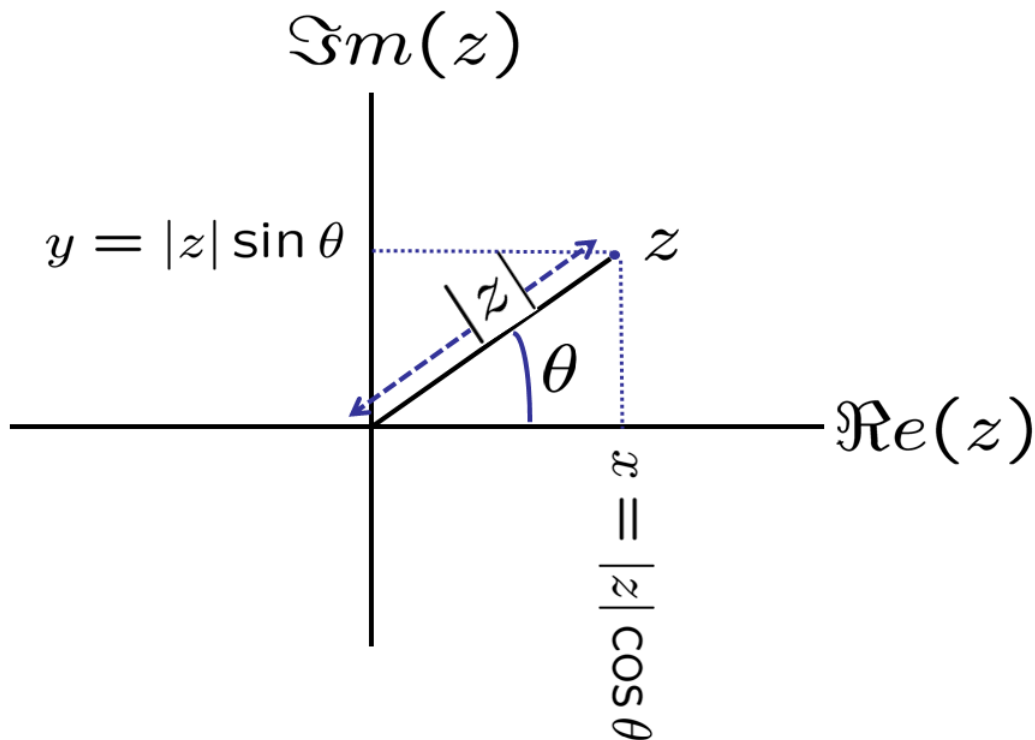


Figure 13-1: The polar form for a complex number $z = x + jy$.

Some operations with complex numbers are made simpler using the polar form, and some become more complicated. For example, the complex conjugate of z is $z^* = x - jy$ in Cartesian form, or $z^* = |z|e^{-j\theta}$ in polar form. Perhaps more compelling is the fact that the polar form for the product of two complex numbers is

$$z_1 z_2 = |z_1| |z_2| e^{j(\theta_1 + \theta_2)}.$$

The same identity is, in words, that magnitude of the product of two complex numbers is the product of the magnitudes, and the angle of the product is the sum of the angles. Please take special note of the fact that the polar representation of the sum of two complex numbers has no simple relation to the individual polar representations. For example, **the magnitude of the sum of two complex numbers is not necessarily the sum of the magnitudes** (though it can be). All one can say is

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

If the unit sample response for an LTI system is real, then

$$H(e^{j\Omega}) = H^*(e^{-j\Omega}),$$

and is referred to as the conjugate symmetry property. Another way of stating the conjugate symmetry property is: if $h[n]$ is real for all n , then $\text{Re}(H(e^{j\Omega})) = \text{Re}(H(e^{-j\Omega}))$, and $\text{Im}(H(e^{j\Omega})) = -\text{Im}(H(e^{-j\Omega}))$. Applying the conjugate symmetry property to the polar form

$$H(e^{j\Omega}) = |H(e^{j\Omega})|e^{j\theta},$$

implies that

$$H(e^{-j\Omega}) = H^*(e^{j\Omega}) = |H(e^{j\Omega})|e^{-j\theta}. \quad (13.3)$$

The polar form and the conjugate symmetry property can be used to improve our insights in to the case of using a co-sinusoidal waveform as an input of an LTI system. If

$$x[n] = \cos \Omega n = \frac{1}{2}e^{j\Omega n} + \frac{1}{2}e^{-j\Omega n}$$

then we can use the LTI system frequency response and superposition to derive the output

$$y[n] = \frac{1}{2}H(e^{j\Omega})e^{j\Omega n} + \frac{1}{2}H(e^{-j\Omega})e^{-j\Omega n}. \quad (13.4)$$

Substituting the polar form, $H(e^{j\Omega}) = |H(e^{j\Omega})|e^{j\theta}$, and exploiting conjugate symmetry,

$$y[n] = |H(e^{j\Omega})| \left(\frac{1}{2}e^{j(\Omega n + \theta)} + \frac{1}{2}e^{-j(\Omega n + \theta)} \right) = |H(e^{j\Omega})| \cos(\Omega n + \theta). \quad (13.5)$$

That is, if a co-sinusoidal waveform is applied to the input of an LTI system, the output is an amplitude-scaled and phase-shifted co-sinusoidal waveform of the same frequency.

As is clear from Equation 13.5, if $|H(e^{j\Omega})|$ is nearly zero for some frequencies, then sinusoidal and co-sinusoidal inputs at these frequencies will be “blocked” by the LTI system. And if $|H(e^{j\Omega})|$ is close to one for some frequencies, then sinusoidal and co-sinusoidal inputs at those frequencies will pass through the LTI system and emerge with nearly the same amplitude. Most of the time we will be unconcerned about the fact that the phase of the input sines and cosines may be altered, but there will be circumstances when phase must be considered.

■ 13.2 Designing Filters

Filters are usually specified in terms of frequency response (e.g. pass everything lower than a given frequency, or pass everything higher than a given frequency). The problem is that in order to apply a filter to an input signal, we need to convolve the filter input with the filter’s unit sample response to compute the output. That is, the filter’s output, $y[n]$, is computed from

$$y[n] = \sum_{m=0}^L h[m]x[n-m],$$

so we need a strategy for computing a filter’s unit sample response given a frequency response specification.

One general method for designing a filter is as follows. We mention it here to give you an idea of the technique. Suppose we want a filter whose frequency response at speci-

fied frequencies, $\Omega_1, \Omega_2, \dots, \Omega_K$ are given, $H(e^{j\Omega_1}), H(e^{j\Omega_2}), \dots, H(e^{j\Omega_K})$. We want to determine an L -length unit sample response for an LTI system so that the system's frequency response matches our specifications. That is, we want to determine the $L + 1$ unknown values $h[0], h[1], \dots, h[L]$.

We can unravel Equation 13.2 into a system of equations that we can solve for $h[0], h[1], \dots, h[L]$, as shown in Equation 13.6. Note that in the system in Equation 13.6, there are separate equations to enforce the specifications for $H(e^{j\Omega_K})$ and for $H(e^{-j\Omega_K})$, and the need to enforce the frequency response at both the positive and negative frequencies may seem superfluous. A perhaps plausible explanation for why both are needed is that if one wishes to design a filter that blocks $x[n] = \cos \Omega n$, then one must design the filter so that both $H(e^{j\Omega})$ and $H(e^{-j\Omega})$ are zero. As appealing as the cosine input explanation may be, the reasoning is not complete. If the unit sample response is real, conjugate symmetry implies that if $H(e^{j\Omega})$ is zero, then $H(e^{-j\Omega})$ is zero also. So, why the need to enforce both? The answer is that either the frequency response specification must enforce conjugate symmetry, so that the solution to the system of equations is real, or Equation 13.6 must be appended with constraints to force the computed unit sample response to be real.

In addition to the positive and negative frequency issue, there are additional subtleties associated with how the angle of the frequency response is specified in Equation 13.6, but these issues extend beyond the scope of 6.02. The key idea to remember is that filter design is a nearly algorithmic process, you specify the desired frequency response, run a program, and out pops a unit sample response that you can convolve with your input to filter it.

$$\begin{bmatrix} 1 & e^{j\Omega_1} & e^{j\Omega_1 2} & \dots & e^{j\Omega_1(L-1)} & e^{j\Omega_1 L} \\ 1 & e^{-j\Omega_1} & e^{-j\Omega_1 2} & \dots & e^{-j\Omega_1(L-1)} & e^{-j\Omega_1 L} \\ 1 & e^{j\Omega_2} & e^{j\Omega_2 2} & \dots & e^{j\Omega_2(L-1)} & e^{j\Omega_2 L} \\ 1 & e^{-j\Omega_2} & e^{-j\Omega_2 2} & \dots & e^{-j\Omega_2(L-1)} & e^{-j\Omega_2 L} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & e^{j\Omega_K} & e^{j\Omega_K 2} & \dots & e^{j\Omega_K(L-1)} & e^{j\Omega_K L} \\ 1 & e^{-j\Omega_K} & e^{-j\Omega_K 2} & \dots & e^{-j\Omega_K(L-1)} & e^{-j\Omega_K L} \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \\ \vdots \\ h[L-1] \\ h[L] \end{bmatrix} = \begin{bmatrix} H(e^{-j\Omega_1}) \\ H(e^{j\Omega_1}) \\ H(e^{-j\Omega_2}) \\ H(e^{j\Omega_2}) \\ \vdots \\ H(e^{-j\Omega_K}) \\ H(e^{j\Omega_K}) \end{bmatrix} \quad (13.6)$$

■ 13.2.1 Low-pass Filters

As mentioned above, we often specify filters in terms of the magnitude of the frequency response. For example, an extremely useful filter, one we can use directly and also use to construct other filters, is the **low-pass filter** (LPF). As its name indicates, the LPF takes an input signal, blocks all frequencies above a cut-off frequency Ω_c , but all frequencies less Ω_c are passed through without changing their amplitudes. A plot of the frequency-response magnitude for an ideal LPF is shown in Figure 13-2.

Generating an ideal LPF is impractical, as such a filter would have an infinitely long unit sample response, but non-ideal LPFs that approximate the ideal case are easy to generate. The unit sample and frequency response of an example approximate ideal LPF are shown in Figure 13-3. For this non-ideal LPF, input signals with frequencies less than Ω_{pass} pass through, and input signals with frequencies larger than Ω_{stop} are blocked. Input signals with frequencies between the pass and stop frequencies, in the transition region of the filter, are partially blocked. It is possible to trade-off narrowing the transition region against

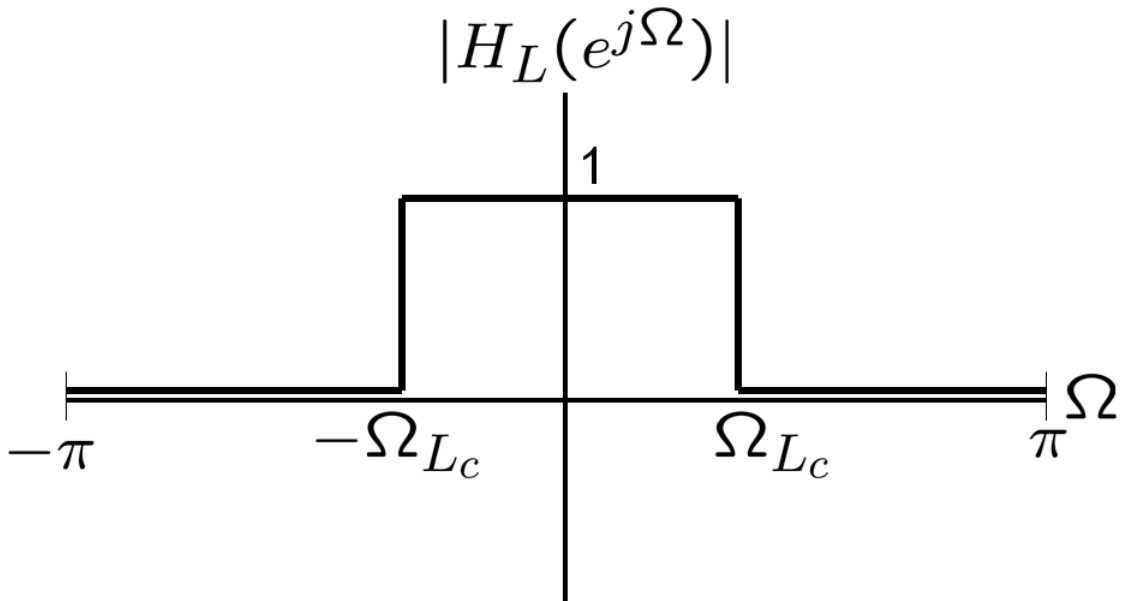


Figure 13-2: An ideal low-pass filter (LPF).

lengthening the non-ideal LPF's unit sample response.

One way to construct an LPF is to pick a set of Ω values all greater than Ω_{stop} and set $H(e^{j\Omega})$ to zero in Equation 13.6, then pick a mirror set of $-\Omega$ values, all more negative than $-\Omega_{\text{stop}}$, and set $H(e^{-j\Omega})$ to zero in Equation 13.6, and then pick values of Ω between $-\Omega_{\text{pass}}$ and Ω_{pass} and set $H(e^{j\Omega})$ to have magnitude one (though the angle must be chosen more carefully) in Equation 13.6. Finally, solve the resulting system of equations for the $L + 1$ values of the filter's unit sample response.

Although it is possible to resort to setting up a system of equations to compute the unit sample response whenever a filter is needed, a more instructive and powerful approach is that of **composition**, in which more complex filters are generated by combining simpler filters. The rest of this lecture is devoted to composition.

■ 13.3 Filter Composition

Suppose we have two filters with frequency responses H_1 and H_2 , respectively, and we pass a signal X through them **in series**, one after the other. What does the output, Y , look like, in terms of H_1 and H_2 (and X)?

Figure 13-4 shows the series composition of filters. Clearly, the intermediate signal is

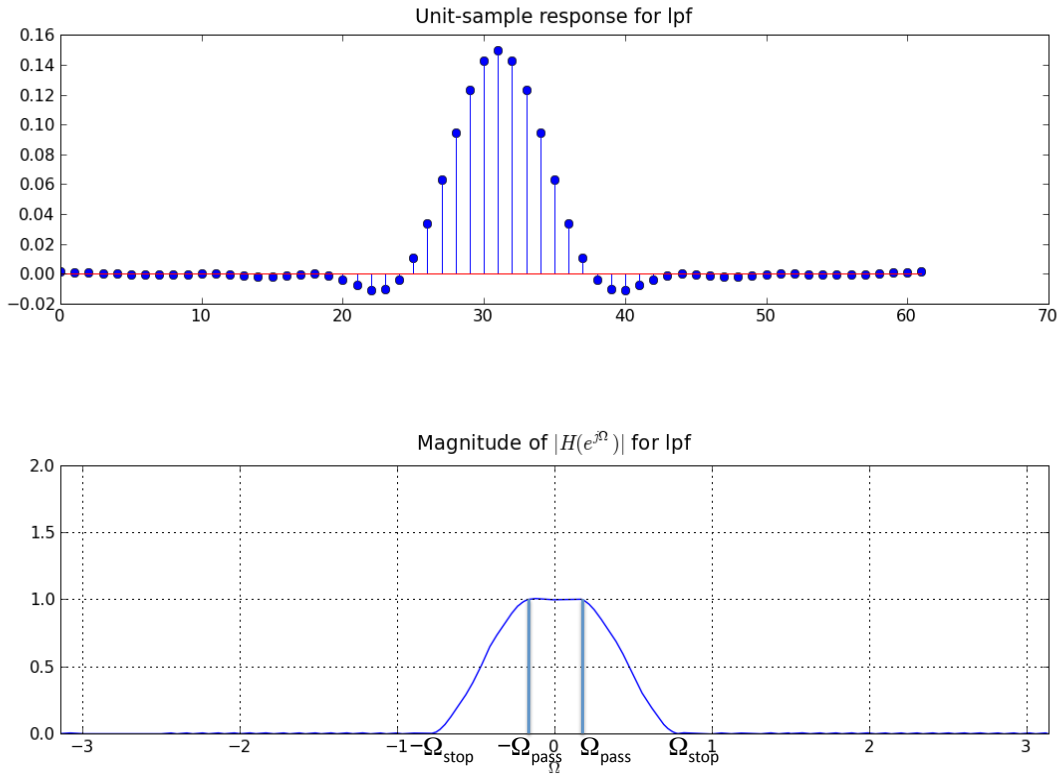


Figure 13-3: A non-ideal low-pass filter (LPF).

$W = H_1 * X$, and the output signal is $Y = H_2 * W$, where $*$ is the convolution operator. Therefore, for series composition, we have $Y = H_1 * H_2 * X$, which means that the H for a series composition of filters is $H = H_1 * H_2$. Or, in terms of sums, and assuming both H_1 and H_2 have L -length unit sample responses,

$$\begin{aligned} h[n] &= \sum_{m=0}^n h_1[m]h_2[n-m], n \leq L, \\ h[n] &= \sum_{m=0}^L h_1[m]h_2[n-m], n > L, \end{aligned} \quad (13.7)$$

and

$$y[n] = \sum_{m=0}^{2L} h[m]x[n-m].$$

The frequency response of a serially composed filter system is given by

$$H(e^{j\Omega}) = H_1(e^{j\Omega})H_2(e^{j\Omega}), \quad (13.8)$$

and therefore, using the product property of complex numbers, the magnitude of the fre-

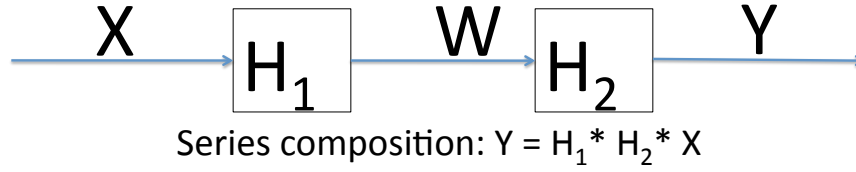


Figure 13-4: Series composition of filters.

quency response is

$$|H(e^{j\Omega})| = |H_1(e^{j\Omega})||H_2(e^{j\Omega})|. \quad (13.9)$$

So, the magnitude of the resulting frequency response is equal to the product of the frequency response magnitudes for the two filters that are serially composed. It is easy to see that these results extend to any number of filters composed in series.

■ 13.3.1 High-pass Filters and Parallel Composition

Suppose we want a filter that will take an input signal and block all frequencies below a cut-off frequency Ω_c , but all frequencies greater than Ω_c are passed through without changing their amplitudes. The magnitude of the frequency response for such an ideal **high-pass filter** (HPF) is shown in Figure 13-5.

Series composition is useful for a number of purposes, but can we construct a high-pass filter by serially composing LPFs? Obviously not, but we can apply a different kind of composition, **parallel composition**, as shown in Figure 13-6. For the case of parallel composition, the output signal $Y = H_1 * X + H_2 * X = (H_1 + H_2) * X$. Or, in terms of sums and assuming both H_1 and H_2 have L -length unit sample responses,

$$h[n] = h_1[n] + h_2[n]$$

and

$$y[n] = \sum_{m=0}^L h[m]x[n-m].$$

It is tempting to try to design a high-pass filter by creating an **all-pass filter**, one for which $|H_1(e^{j\Omega})| = 1$ for all Ω , and then adding the negative of a low-pass filter to the all-pass filter. To make an all-pass filter, consider using an LTI system whose unit sample response is the unit sample, as in $h_1[n] = \delta[n]$. Then

$$H_1(e^{j\Omega}) \equiv \sum_{m=0}^L h_1[m]e^{-j\Omega m} = 1.$$

Note that if the unit sample response is the delayed unit sample, $h_1[n] = \delta[n-L]$, the resulting system is still all-pass. The frequency response is not identically one, but is given

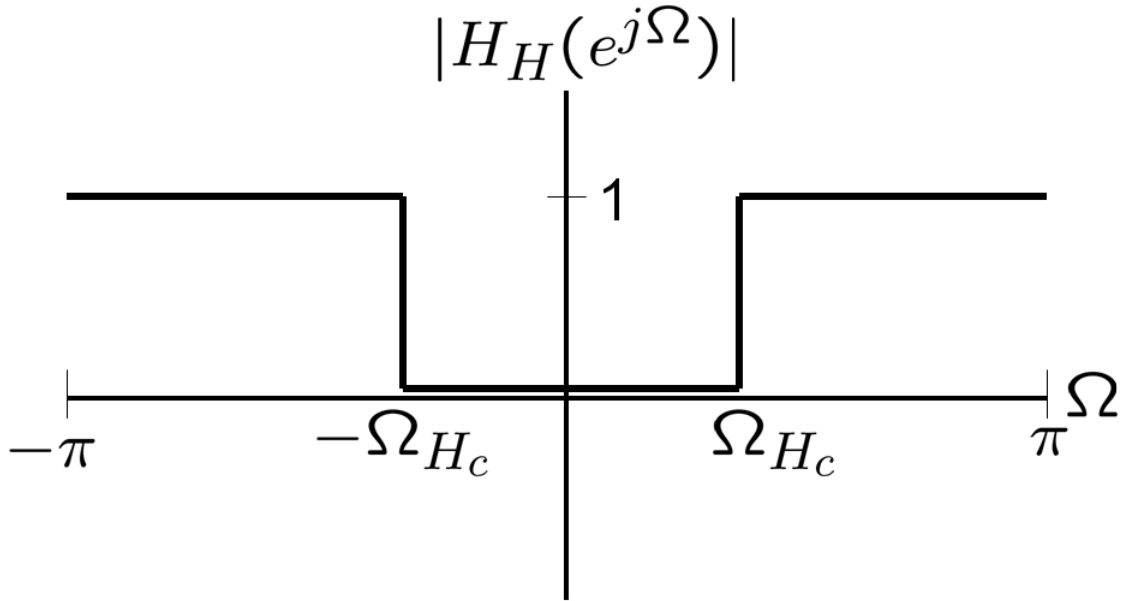


Figure 13-5: An ideal high-pass filter (HPF).

by

$$H_1(e^{j\Omega}) \equiv \sum_{m=0}^L h_1[m]e^{-j\Omega m} = e^{-j\Omega L},$$

and still has unit magnitude for all Ω ,

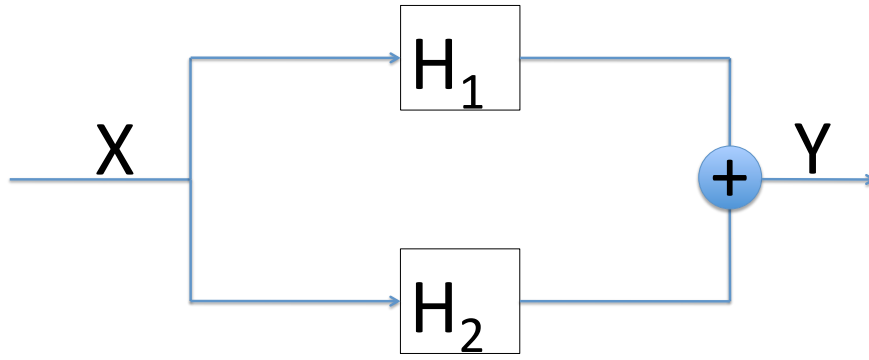
$$|H_1(e^{j\Omega})| = |e^{-j\Omega L}| = 1.$$

The problem with using the unit sample or the delayed unit sample as an all-pass filter, and then subtracting the low-pass filter, is that the magnitude of the frequency response for the parallel combination is given by

$$|H(e^{j\Omega})| = |H_1(e^{j\Omega}) + -H_2(e^{j\Omega})| \quad (13.10)$$

which is **NOT** equal to $|H_1(e^{j\Omega})| - |H_2(e^{j\Omega})|$.

As can be seen by inspecting the plot of the low-pass filter's unit sample response in Figure 13-3, the larger values in the unit sample response are clustered near sample 31, suggesting that there is a 31 sample delay associated with applying the LPF. Therefore, it seems reasonable to try to generate a high-pass filter using the parallel combination of the negative of the LPF and an all-pass filter whose unit sample response is a 31-sample de-



Parallel composition: $Y = (H_1 + H_2) * X$

Figure 13-6: Parallel composition of filters.

layed unit sample. The so-generated HPF's unit sample response and frequency response are plotted in Figure 13-7, and as is clear from the plot, the strategy was successful, if poorly justified.

The above heuristic approach to designing the parallel composition HPF is, well, unsatisfying. Did we just get lucky or are there generalizable principles? The answer is that if a filter, H_2 , has a unit sample response that is symmetric about one sample, then parallel composition with an appropriately delayed unit sample can produce filters with easily analyzed frequency responses. To see this, we can rewrite the frequency response definition, Equation 13.2, with a shift of the index m to the symmetry point, in this case, $\frac{L}{2}$. The resulting formula for the frequency response is

$$H_2(e^{j\Omega}) = \sum_{m=-\frac{L}{2}}^{\frac{L}{2}} h_2[\frac{L}{2} + m] e^{-j\Omega(\frac{L}{2} + m)} = \left(\sum_{m=-\frac{L}{2}}^{\frac{L}{2}} h_2[\frac{L}{2} + m] e^{-j\Omega m} \right) e^{-j\Omega \frac{L}{2}}. \quad (13.11)$$

Since $h_2[\frac{L}{2} + m] = h_2[\frac{L}{2} - m]$ as $\frac{L}{2}$ was the symmetry point, the sum

$$H_r(e^{j\Omega}) \equiv \left(\sum_{m=-\frac{L}{2}}^{\frac{L}{2}} h_2[\frac{L}{2} + m] e^{-j\Omega m} \right)$$

must be real, and we can rewrite the frequency response as

$$H_2(e^{j\Omega}) = H_r(e^{j\Omega}) e^{-j\Omega \frac{L}{2}}.$$

If the unit sample response for H_1 is $\delta[n - \frac{L}{2}]$, then

$$|H(e^{j\Omega})| = |H_1(e^{j\Omega}) - H_2(e^{j\Omega})| = |1 - H_r(e^{j\Omega})| e^{-j\Omega \frac{L}{2}} = |1 - H_r(e^{j\Omega})|. \quad (13.12)$$

In the case of our low-pass filter, $H_r(e^{j\Omega})$ is a real number and is equal to one at low fre-

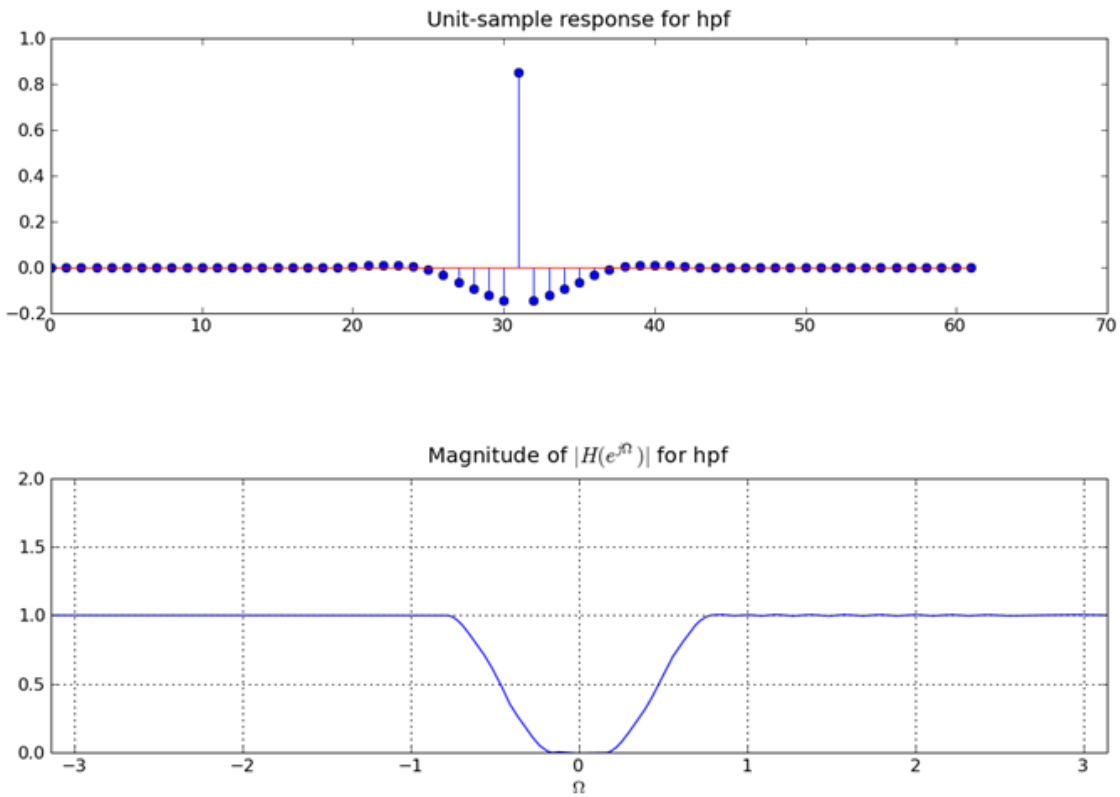


Figure 13-7: A high-pass filter (HPF) generated using parallel composition.

quencies and zero at high frequencies. Therefore $|H(e^{j\Omega})|$ will be zero at low frequencies and one at high frequencies, a high-pass filter as desired.