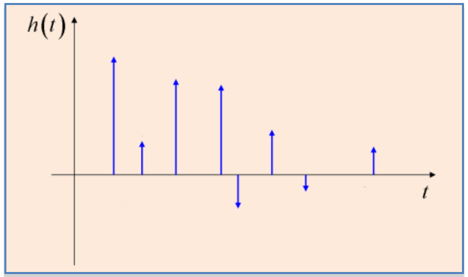
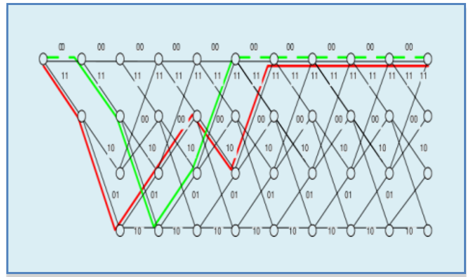
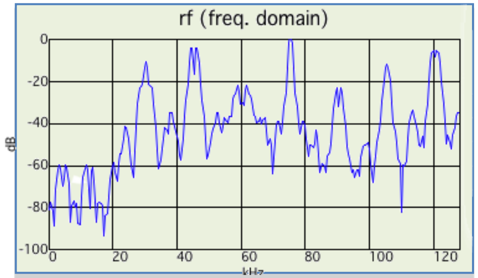
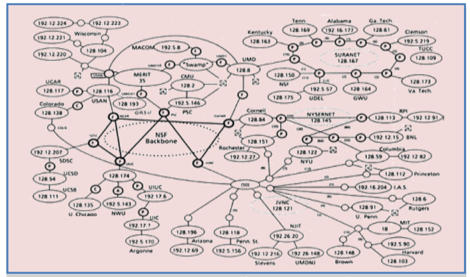








INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Fall 2010
Lecture #13

- Frequency Response and Filters.
- Equations for Filter Design.
- Effects of low-pass, high-pass and band-pass filters
- Series and Parallel Composition.

Basic Tools: DT Sinusoids and Complex Exponentials

- Continuous Time to Discrete Time Sinusoids

$$x(t) = \cos 2\pi f t$$

Sample Period $\Rightarrow$ $$x(nT_s) = \cos 2\pi f T_s n$$

Sample Rate $\Rightarrow$ $$x(n\frac{1}{f_s}) = \cos 2\pi \frac{f}{f_s} n$$

$\Rightarrow$ $$x[n] = \cos 2\pi \frac{f}{f_s} n = \cos \Omega n$$

- Complex Exponentials

$$cos\Omega n = \frac{1}{2} e^{j\Omega n} + \frac{1}{2} e^{-j\Omega n} \quad sin\Omega n = \frac{1}{2j} e^{j\Omega n} - \frac{1}{2j} e^{-j\Omega n}$$

Basic Tools: Frequency Response

- Frequency Response

$$x[n] = e^{j\Omega n} \qquad y[n] = \sum_{m=0}^{L} h[m]x[n-m]$$

$$y[n] = H\left(e^{j\Omega}\right)e^{j\Omega n} \qquad H\left(e^{j\Omega}\right) = \sum_{m=0}^{L} h[m]e^{-j\Omega m}$$

Complex Number

$H(e^{j\Omega})$, $-\pi < \Omega < \pi$, is the frequency response

$$cos\Omega n = \frac{1}{2}e^{j\Omega n} + \frac{1}{2}e^{-j\Omega n} \qquad sin\Omega n = \frac{1}{2j}e^{j\Omega n} - \frac{1}{2j}e^{-j\Omega n}$$

Designing Filter's Unit Sample Response

- Design a Unit Sample response with L+1 nonzero values with a given set of frequency responses.

$$\begin{bmatrix} 1 & e^{j\Omega_1} & e^{j\Omega_1 2} & \dots & e^{j\Omega_1 (L-1)} & e^{j\Omega_1 L} \\ 1 & e^{-j\Omega_1} & e^{-j\Omega_1 2} & \dots & e^{-j\Omega_1 (L-1)} & e^{-j\Omega_1 L} \\ 1 & e^{j\Omega_2} & e^{j\Omega_2 2} & \dots & e^{j\Omega_2 (L-1)} & e^{j\Omega_2 L} \\ 1 & e^{-j\Omega_2} & e^{-j\Omega_2 2} & \dots & e^{-j\Omega_2 (L-1)} & e^{-j\Omega_2 L} \\ \vdots & \vdots & & \vdots & \vdots & \\ 1 & e^{j\Omega_K} & e^{j\Omega_K 2} & \dots & e^{j\Omega_K (L-1)} & e^{j\Omega_K L} \\ 1 & e^{-j\Omega_K} & e^{-j\Omega_K 2} & \dots & e^{-j\Omega_K (L-1)} & e^{-j\Omega_K L} \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \\ \vdots \\ h[L-1] \\ h[L] \end{bmatrix} = \begin{bmatrix} H\left(e^{-j\Omega_1}\right)) \\ H\left(e^{j\Omega_1}\right)) \\ H\left(e^{-j\Omega_2}\right)) \\ H\left(e^{j\Omega_2}\right)) \\ \vdots \\ H\left(e^{-j\Omega_K}\right)) \\ H\left(e^{j\Omega_K}\right)) \end{bmatrix}$$

- 2K constraints (desired frequency responses)
- L+1 Unknowns (values of h to be designed)

#Eqns = # Unknowns → 2K = L+1

Low Pass Filter Background

$$x[n] = e^{j\Omega n} \quad \text{or} \quad x[n] = e^{-j\Omega n}$$

$$y[n] = H(e^{j\Omega})\, e^{j\Omega n}$$

or

$$H(e^{j\Omega})\, e^{-j\Omega n}$$

Why Negative Frequencies?

$$x[n] = \cos \Omega n = \frac{1}{2} e^{j\Omega n} + \frac{1}{2} e^{-j\Omega n}$$

$$y[n] = -\frac{1}{2} H(e^{j\Omega}) e^{j\Omega n} + \frac{1}{2} H(e^{-j\Omega}) e^{-j\Omega n}$$

Magnitude of $H(e^{j\Omega})$

If unit sample response finite length

$$H(e^{j\Omega}) = \sum_{m=0}^{L} h[m]\, e^{-j\Omega m}$$

if h is causal

$$H(e^{-j\Omega}) = \sum_{m=0}^{L} h[m]\, e^{j\Omega m}$$

$$e^{j\Omega m} = \cos \Omega m + j \sin \Omega m$$
$$e^{-j\Omega m} = \cos \Omega m - j \sin \Omega m$$

$$\Rightarrow \quad H(e^{j\Omega}) = \sum h[m] \cos \Omega m + j \sum h[m] \sin \Omega m$$
$$H(e^{-j\Omega}) = \sum h[m] \cos \Omega m - j \sum h[m] \sin \Omega m$$

$$|H(e^{j\Omega})| = \sqrt{\left(\sum h[m] \cos \Omega m\right)^2 + \left(\sum h[m] \sin \Omega m\right)^2}$$

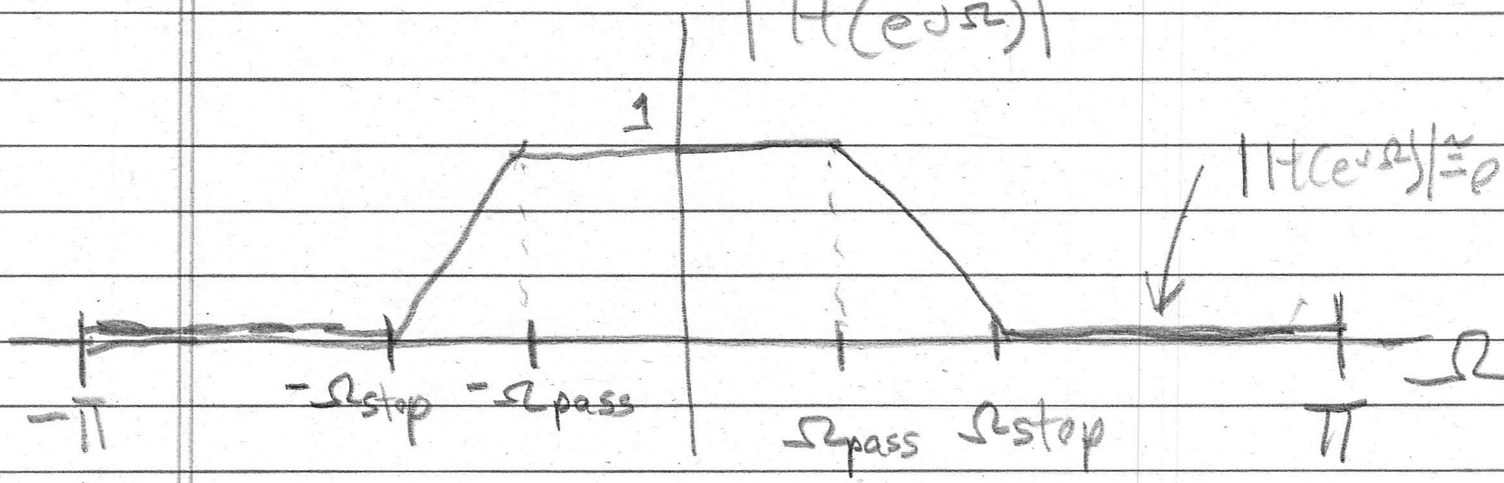
$$|H e^{-j\Omega}| = \text{same.}$$

$$\Rightarrow |H(e^{j\Omega})| = |H(e^{-j\Omega})|$$

Low Pass Filter

$$|H(e^{j\Omega})| = |H(e^{-j\Omega})| \cong 0 \quad \Omega > \Omega_{stop}$$

$$|H(e^{j\Omega})| = |H(e^{-j\Omega})| \simeq 1 \quad \Omega < \Omega_{pass}$$



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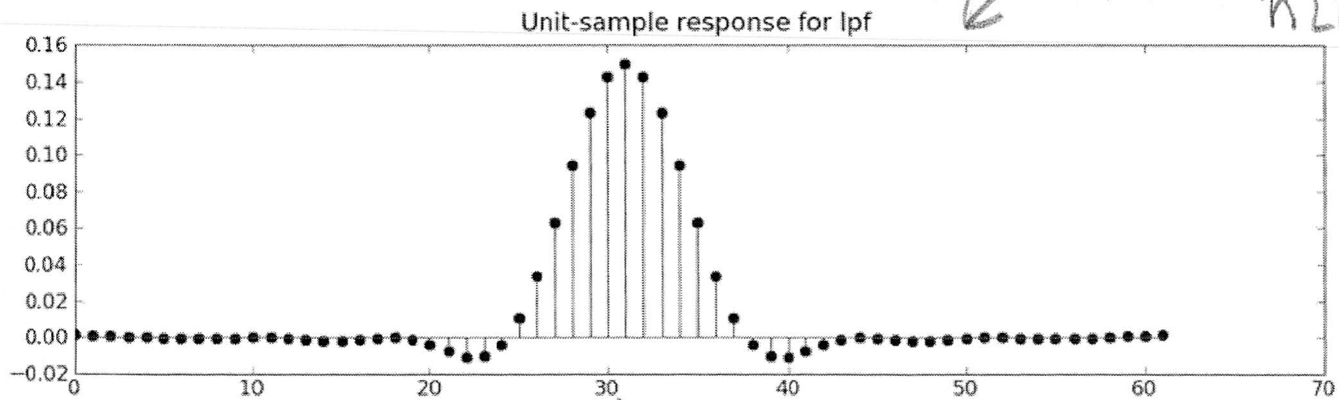
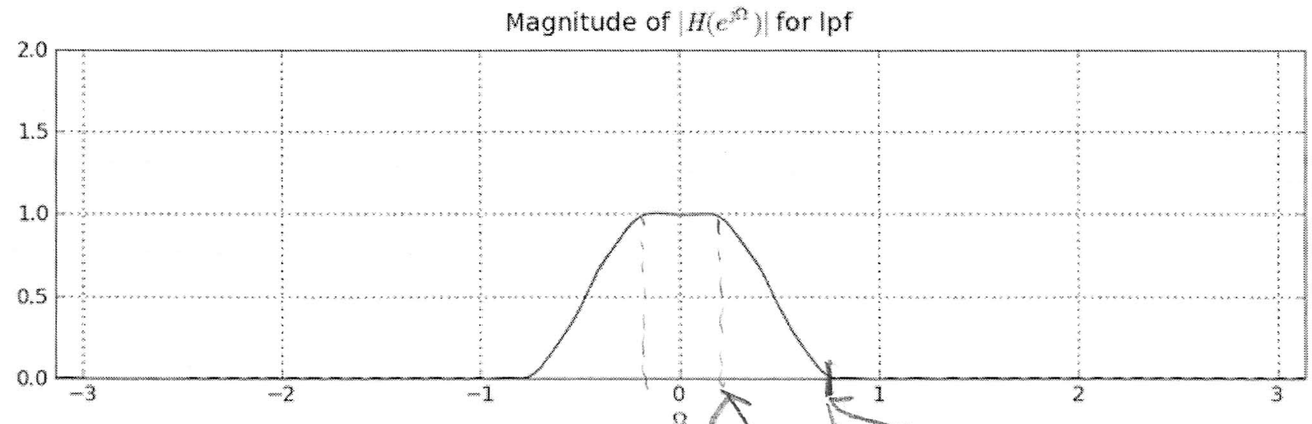
Low Pass Filter

Filter is causal
h[n] = 0 n < 0



Note 31 samples



Ω_{pass}

Ω_{stop}

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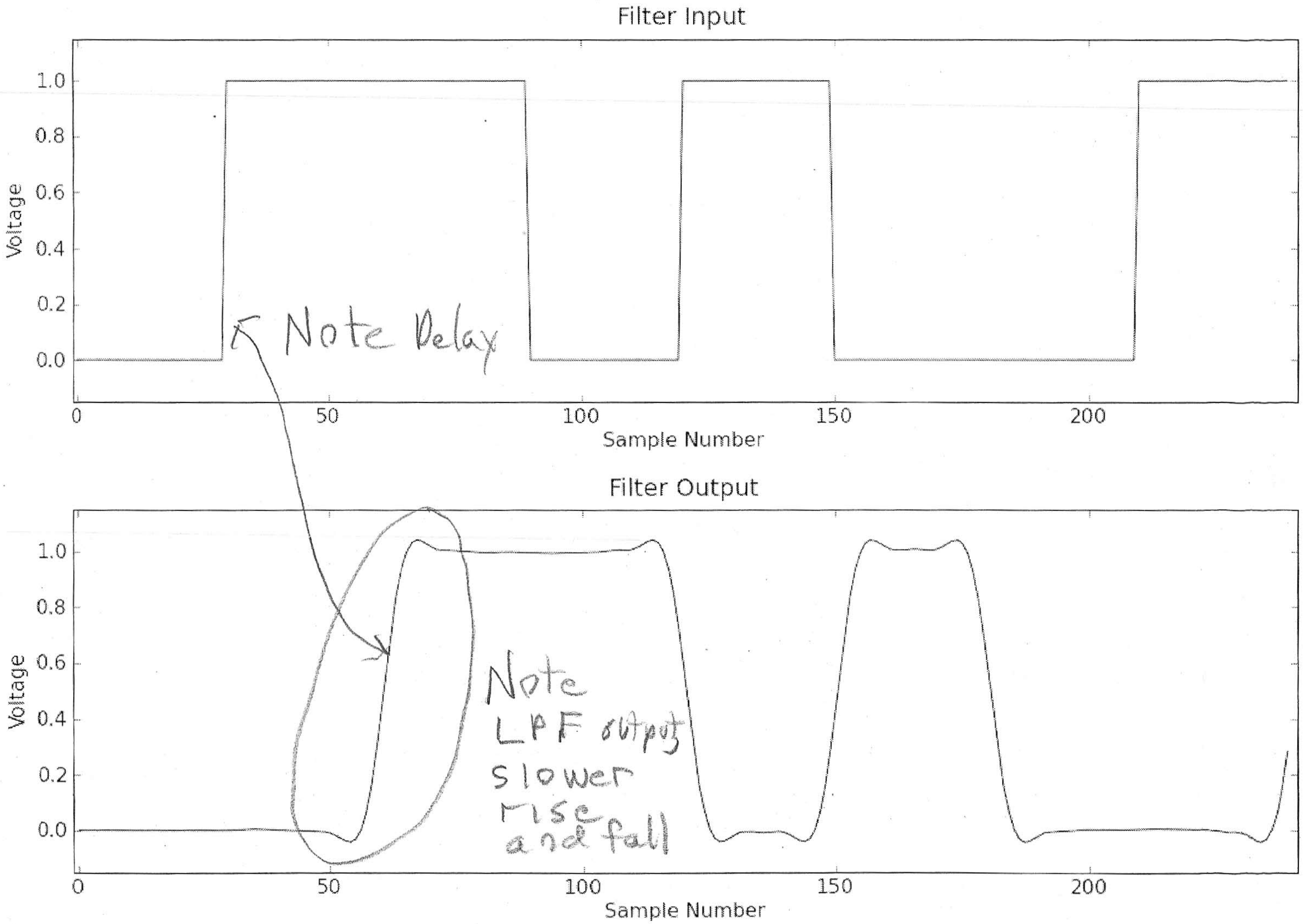
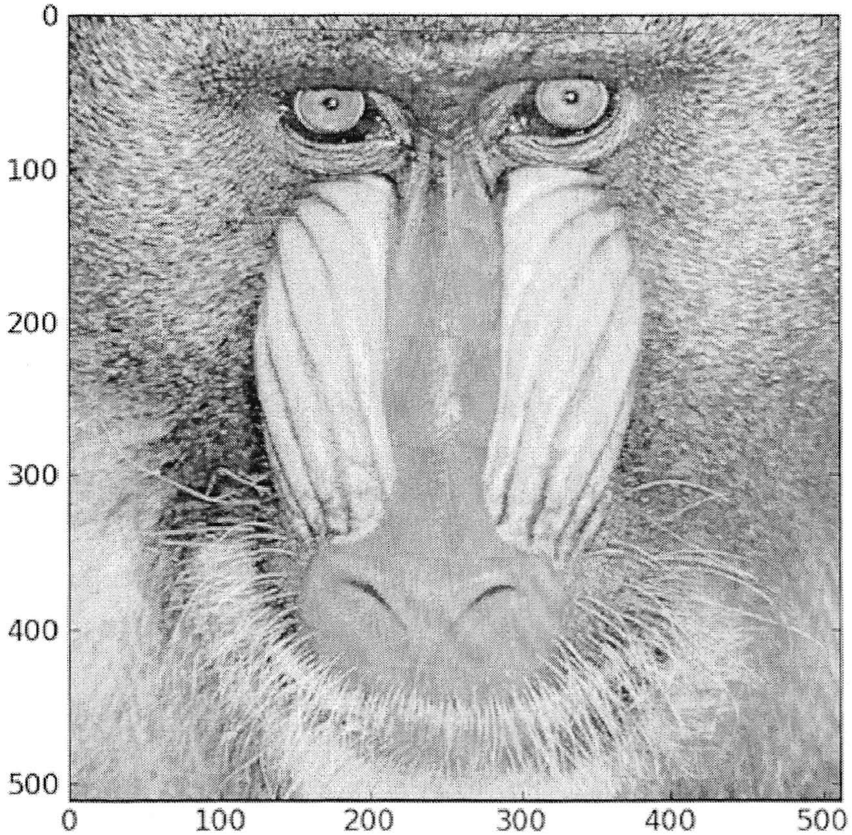
Low Pass Filter Applied to Bits



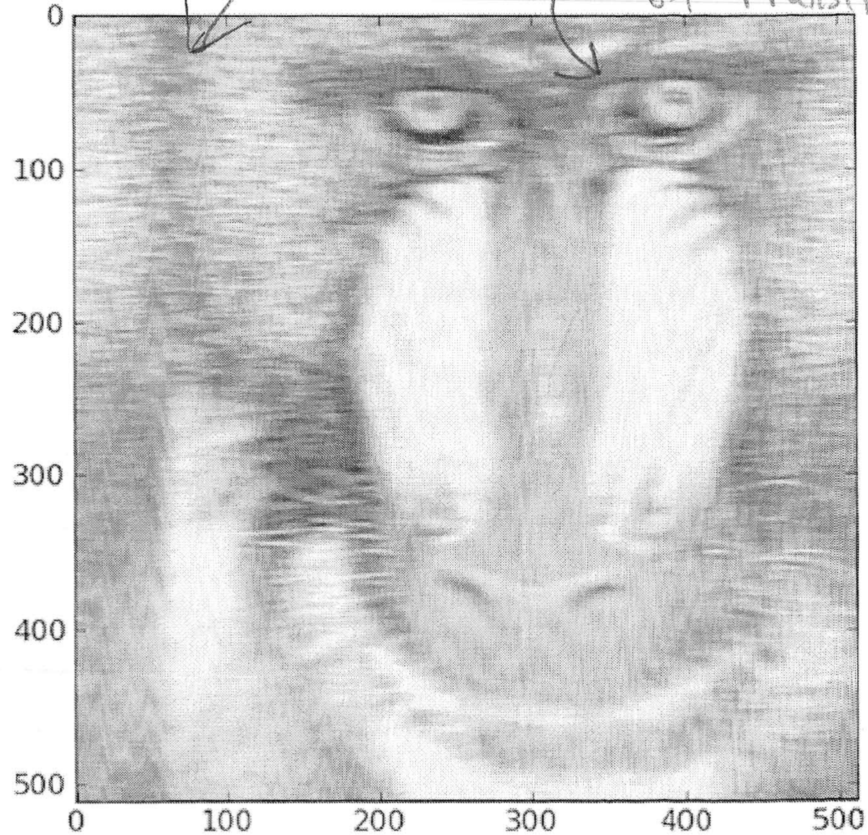
Low Pass Filter Applied to Image

Notice shift due to filter delay

Notice smearing of transitions



Input



Low-pass filtered output

Filters By Composition

Series Composition

$$X \rightarrow \boxed{H_1} \xrightarrow{W} \boxed{H_2} \rightarrow Y$$

$$W = H_1 * X$$

$$Y = H_2 * W$$

$$\Rightarrow Y = H_2 * (H_1 * X)$$

$$= (H_2 * H_1) * X$$

causal h_1

OR $$h[n] = \sum_{m=0}^{m=n} h_2[m]\, h_1[n-m]$$

causal h_2

$$y[n] = \sum_{m=0}^{n} h[m]\, x[n-m]$$

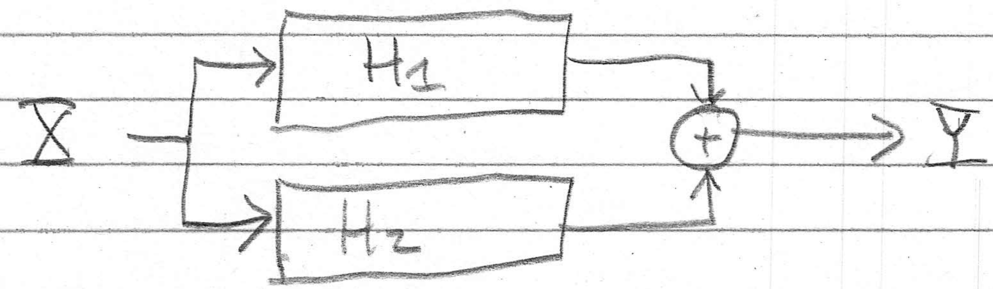
Frequency Response

$$x[n] = e^{j\Omega n} \Rightarrow w[n] = H_1(e^{j\Omega}) e^{j\Omega n}$$

$$y[n] = H(e^{j\Omega}) e^{j\Omega n} = H_2(e^{j\Omega}) H_1(e^{j\Omega}) e^{j\Omega n}$$

$$|H(e^{j\Omega})| = |H_2(e^{j\Omega})|\,|H_1(e^{j\Omega})|$$

Parallel Composition



$$Y = H_1 * X + H_2 * X$$

$$= (H_1 + H_2) * X = H * X$$

OR

$$y[n] = \sum_{m=0}^{n} h_1[m]\, x[n-m] + \sum_{m=0}^{n} h_2[m]\, x[n-m]$$

$$= \sum_{m=0}^{n} \left(h_1[m] + h_2[m]\right) x[n-m]$$

$$\Rightarrow \quad h[n] = h_1[n] + h_2[n]$$

Frequency Response

$$x[n] = e^{j\Omega n} \qquad y[n] = H_1(e^{j\Omega})\, e^{j\Omega n} + H_2(e^{j\Omega})\, e^{j\Omega n}$$

$$y[n] = \left(H_1(e^{j\Omega}) + H_2(e^{j\Omega})\right) e^{j\Omega n}$$

$$|H(e^{j\Omega})| = |H_1(e^{j\Omega}) + H_2(e^{j\Omega})|$$

Simple Example

$$H_1: \quad h_1[n] = \delta[n]$$

$$H_2: \quad h_2[n] = \delta[n-5]$$

$$H_1(e^{j\Omega}) = \sum_{m=0}^{L} h_1[m]e^{-j\Omega m} = e^{-j\Omega 0} = 1$$

$$H_2(e^{j\Omega}) = \sum_{m=0}^{L} h_2[m]e^{-j\Omega m} = e^{-j\Omega 5}$$

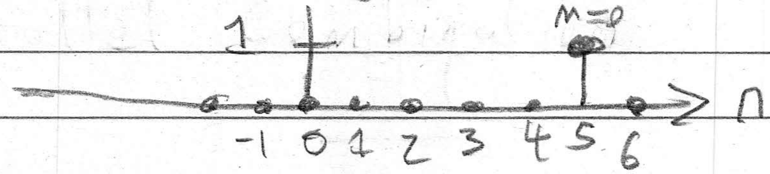
Series Composition

$$X \rightarrow \boxed{H_1} \rightarrow \boxed{H_2} \rightarrow Y$$

$$H(e^{j\Omega}) = H_1(e^{j\Omega})\,H_2(e^{j\Omega}) = 1 \cdot e^{-j\Omega 5}$$

$$|H(e^{j\Omega})| = 1$$

$$h[n] = \sum_{m=0}^{n} h_1[m]\,h_2[n-m] = \delta[n-5]$$



Parallel Composition

X → $\boxed{H_1}$ (+), $\boxed{H_2}$ (−) → ⊕ → Y $\qquad h[n] = \delta[n] - \delta[n-5]$

$$H(e^{j\Omega}) = H_1(e^{j\Omega}) - H_2(e^{j\Omega}) = 1 - e^{-j\Omega 5}$$

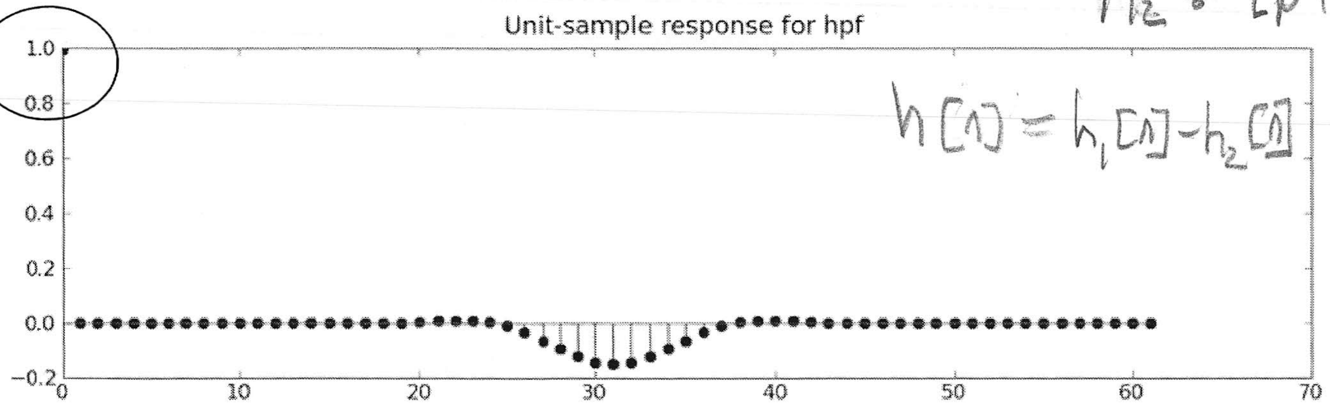
$$|H(e^{j\Omega})| = \sqrt{(1-\cos(5\Omega))^2 + (\sin(5\Omega))^2} = \sqrt{2-2\cos 5\Omega}$$

Note: $|H(e^{j0})| = 0$. Constant input ⇒ zero output

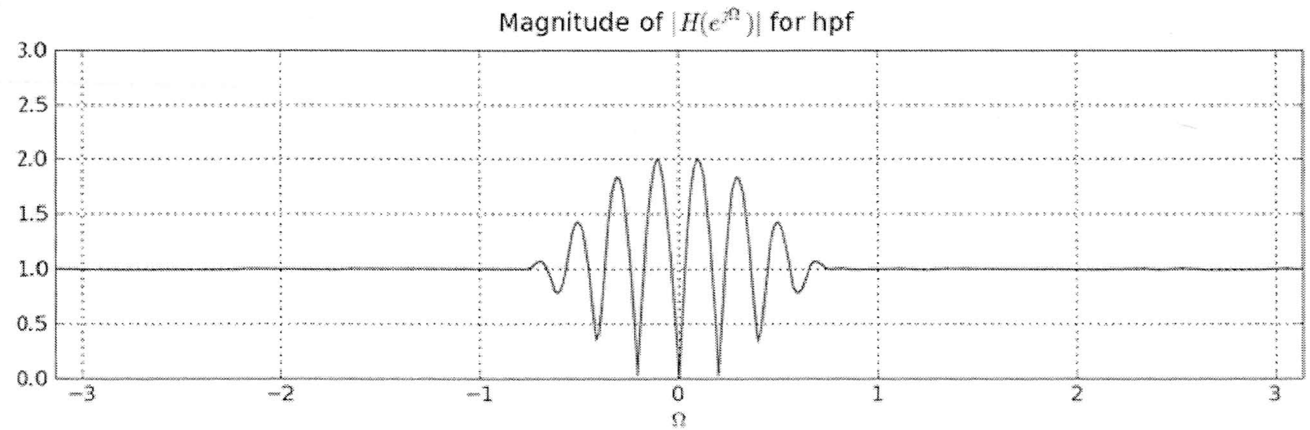
“Wrong” High Pass Filter

$H_1 : \delta[n]$

$H_2 : lpf[n]$



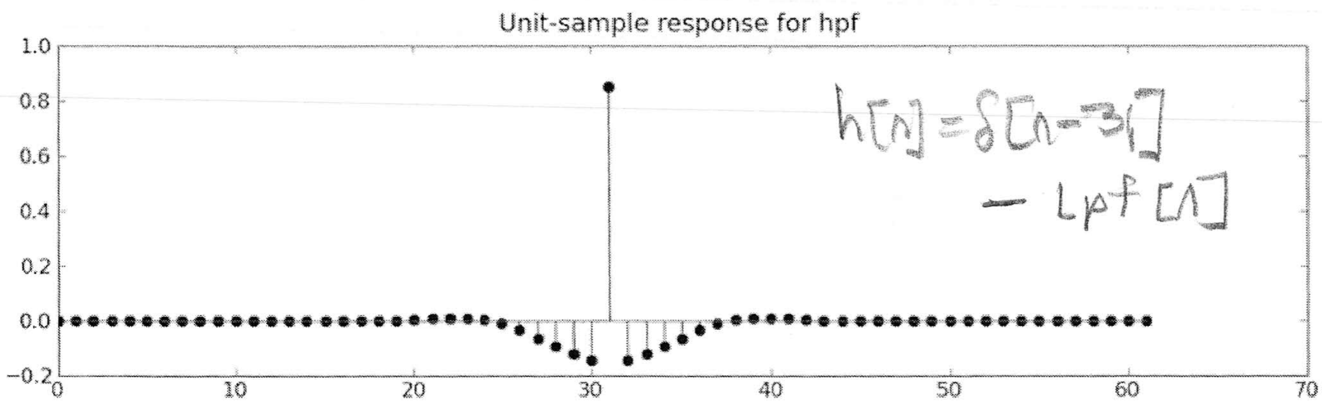
$h[n] = h_1[n] - h_2[n]$

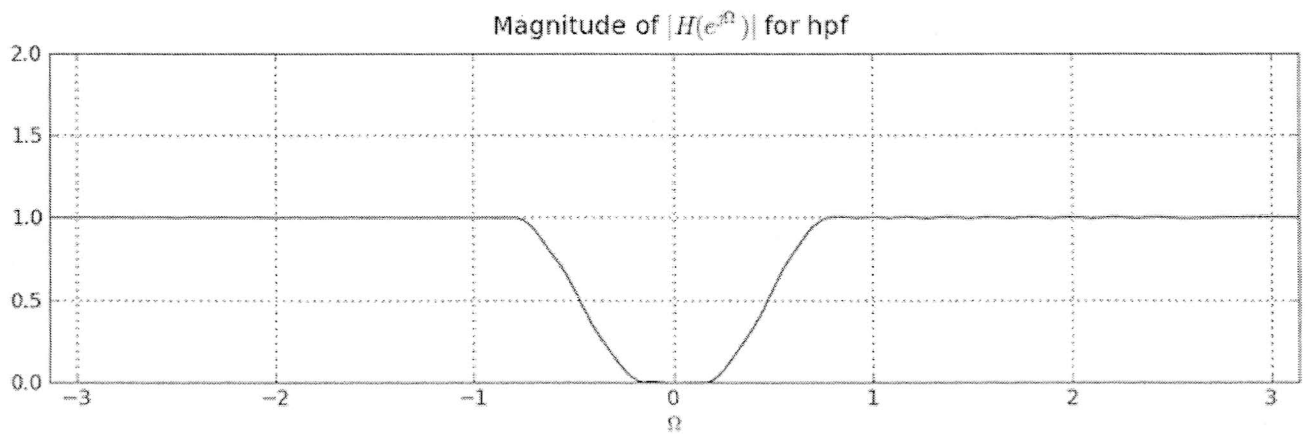


(15)

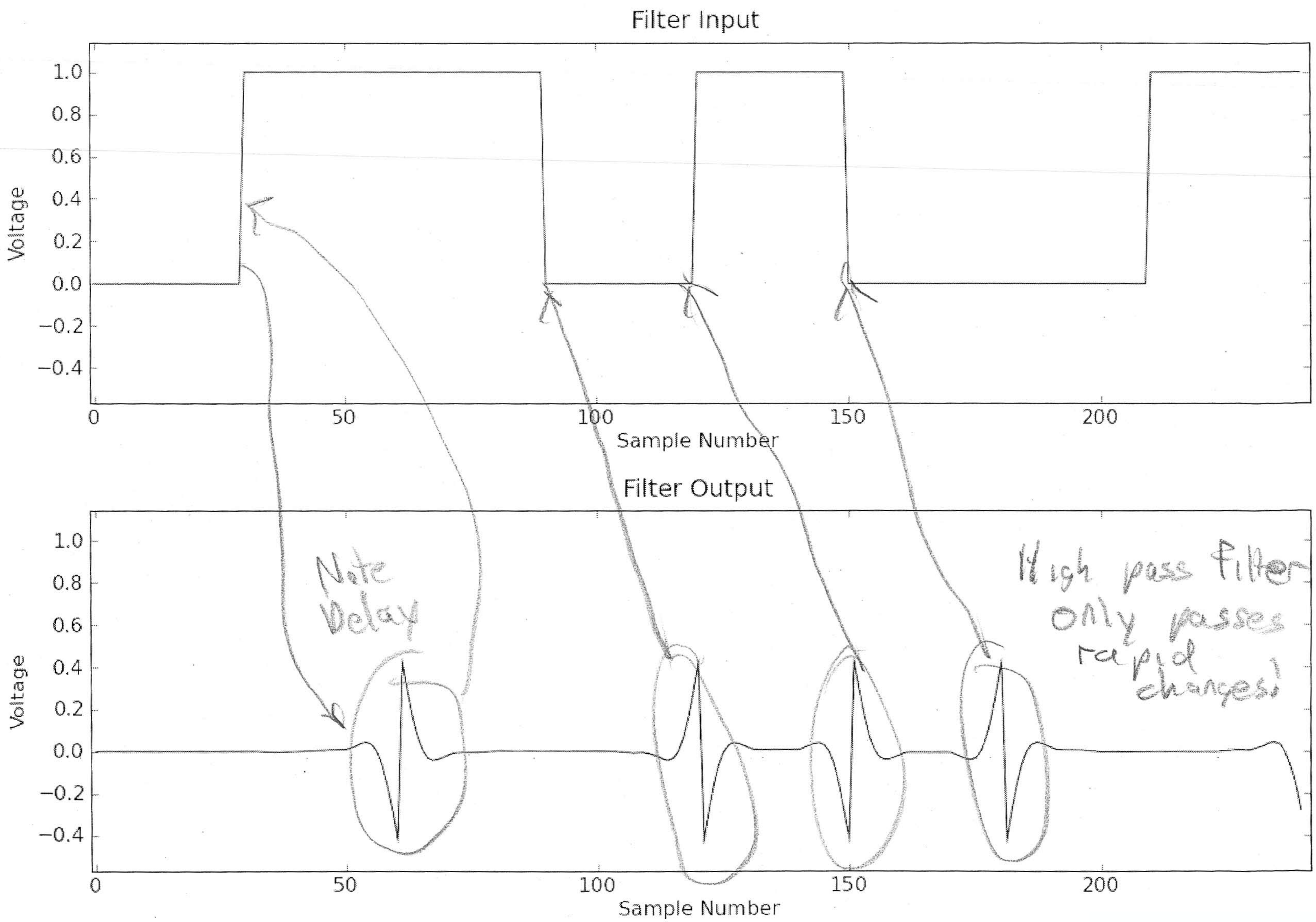
High Pass Filter

H_1: $\delta[n-31]$
H_2: $Lpf[n]$

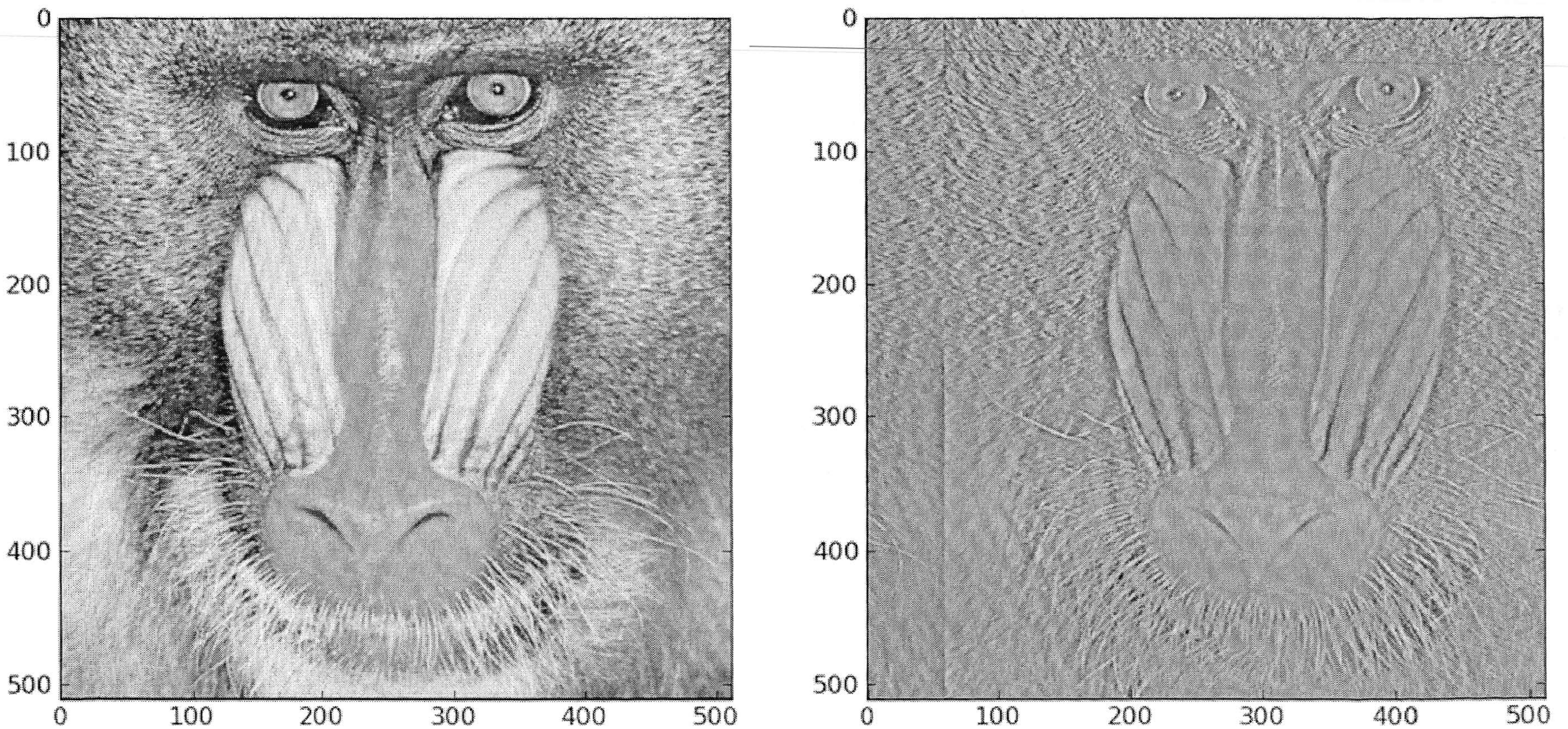




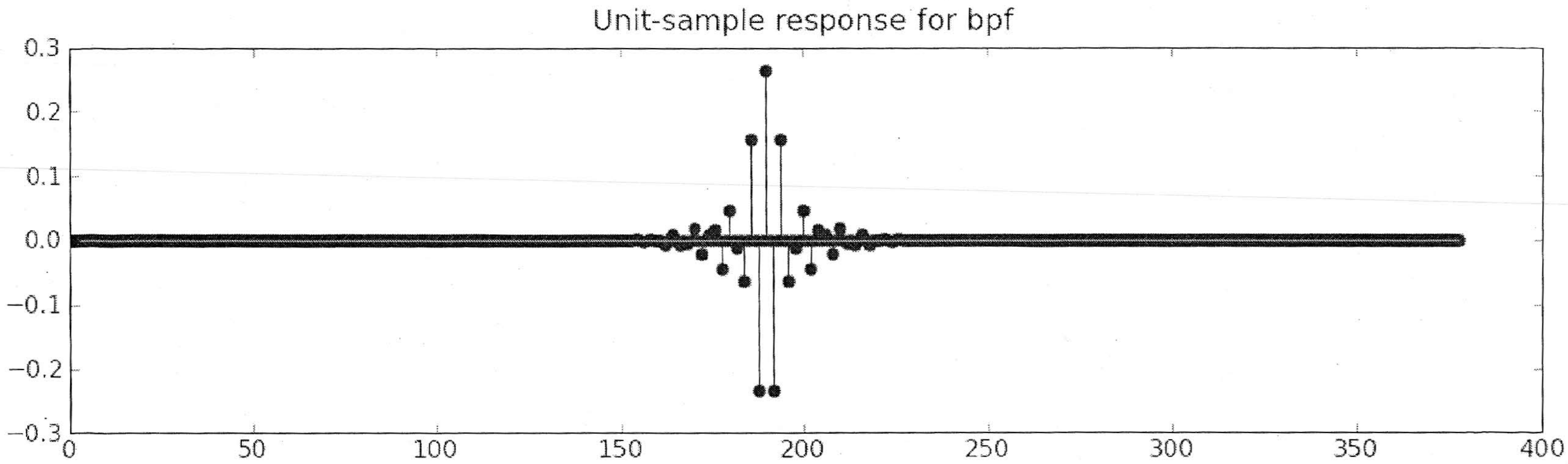
High Pass Filter Applied to Bits

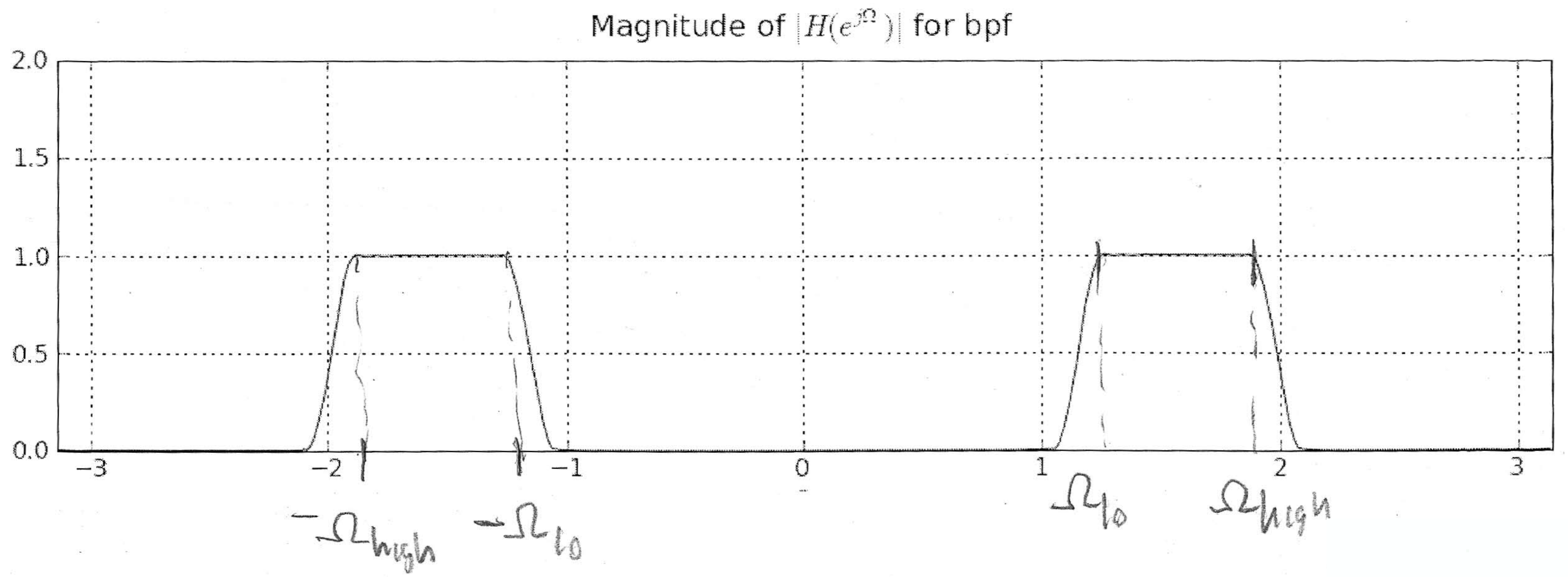


High Pass Filter Applied to Image

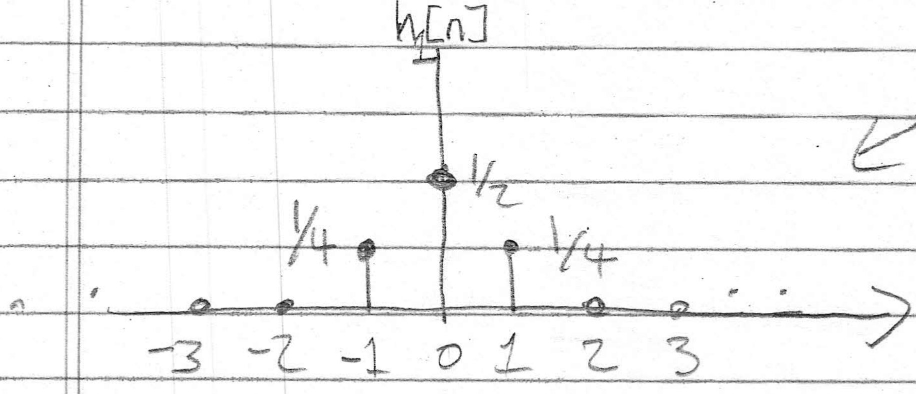


Band Pass Filter





Low Pass Filter Example



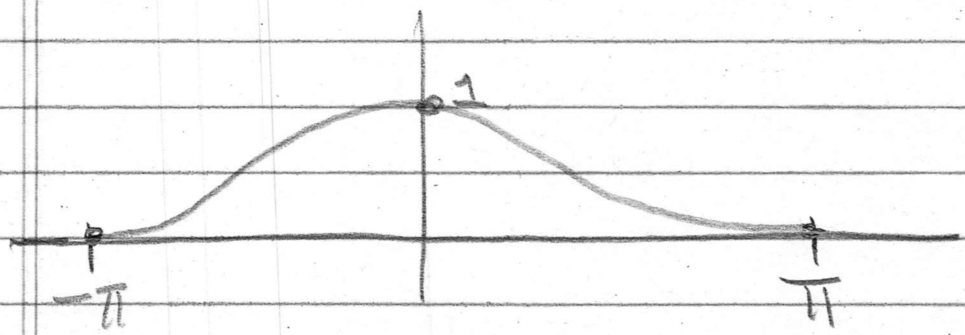
Note
Non causal

$$H_1(e^{j\Omega}) = h_1[-1]e^{j\Omega} + h_1[0] + h_1[1]e^{-j\Omega}$$

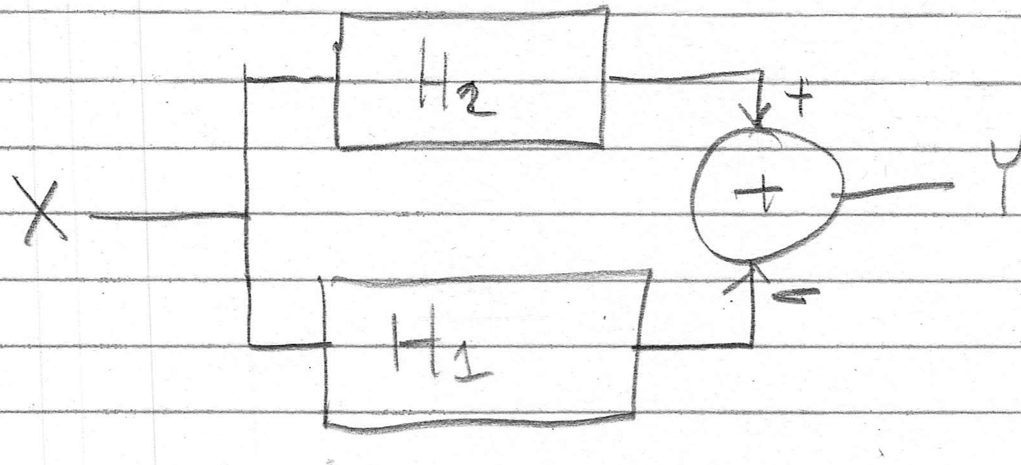
$$= \frac{1}{4}\left(e^{j\Omega} + e^{-j\Omega}\right) + \frac{1}{2}$$

$$= \frac{1}{2}\cos\Omega + 1/2$$

$|H_1(e^{j\Omega})|$



Making a High Pass Filters

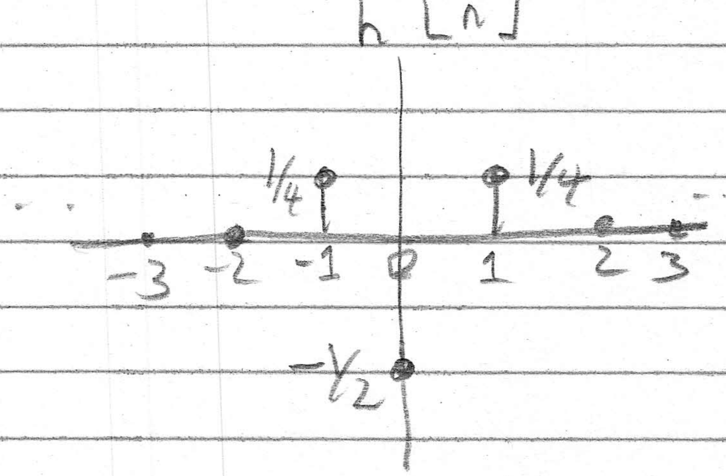


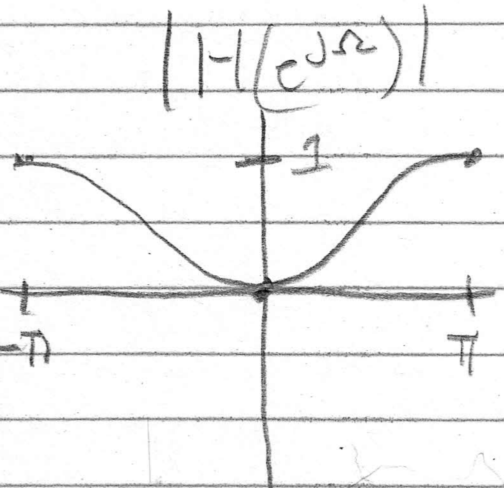
$$Y = (H_2 - H_1) * X = H * X$$

Try $h_2[n] = \delta[n]$

$$H_2(e^{j\Omega}) = 1$$

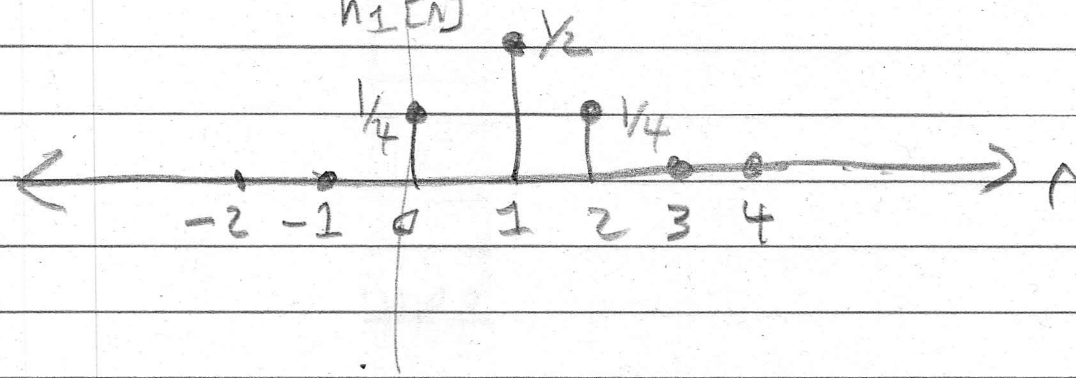
$$h[n] = h_2[n] - h_1[n]$$





$$H(e^{j\Omega}) = \underset{\uparrow \atop H_2(e^{j\Omega})}{1} - \underbrace{\left(\tfrac{1}{2}\cos\Omega + \tfrac{1}{2}\right)}_{H_1(e^{j\Omega})} = \tfrac{1}{2} - \tfrac{1}{2}\cos\Omega$$

Now Suppose H_1 is causal

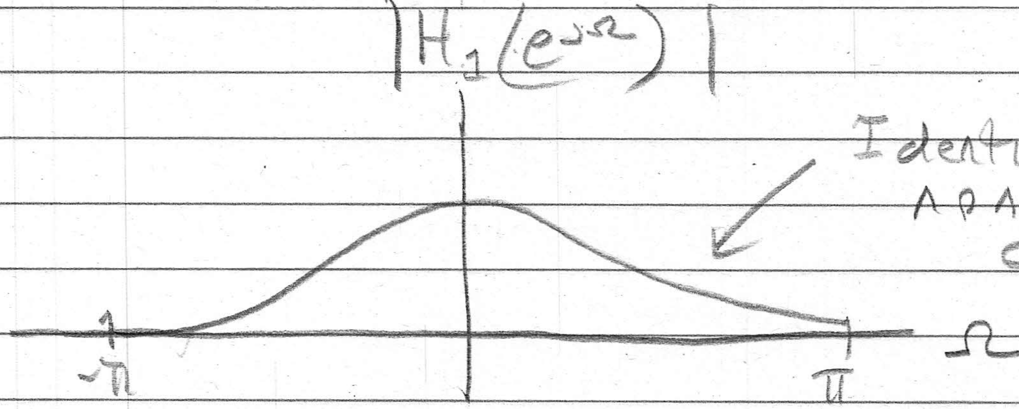


$$H_1(e^{j\Omega}) = \frac{1}{4} + \frac{1}{2}e^{-j\Omega} + \frac{1}{4}e^{-2j\Omega}$$

$$= e^{-j\Omega}\left(\frac{1}{4}\left(e^{+j\Omega} + e^{-j\Omega}\right) + \frac{1}{2}\right)$$

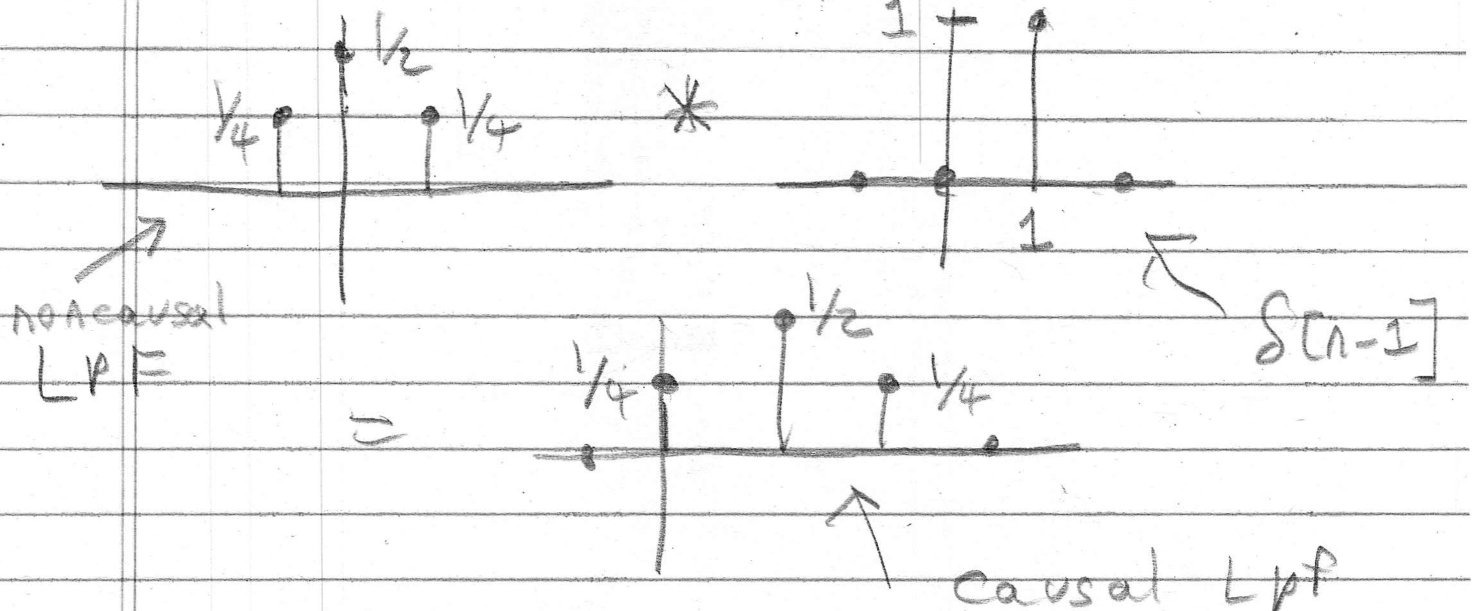
$$= e^{-j\Omega}\left(\frac{1}{2}\cos\Omega + \frac{1}{2}\right)$$

$$|H_1(e^{j\Omega})| = \underbrace{|e^{-j\Omega}|}_{=1}\left|\frac{1}{2}\cos\Omega + \frac{1}{2}\right|$$

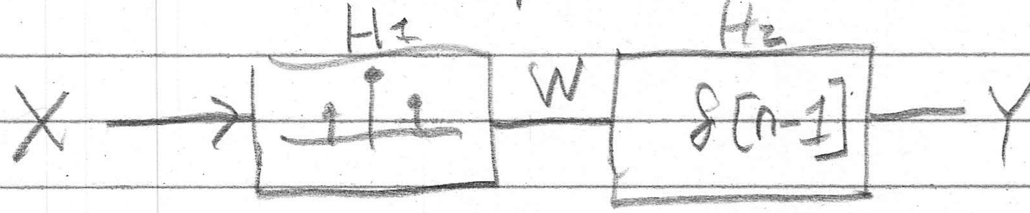


Identical to non causal case

Another Approach



Series composition

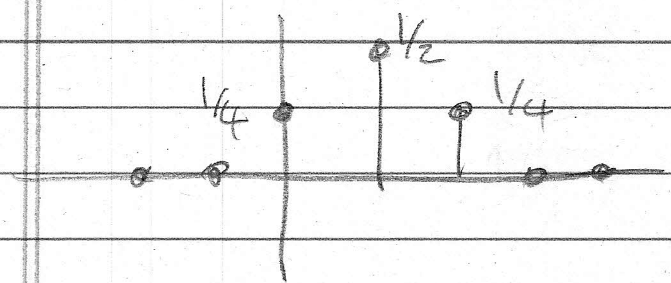


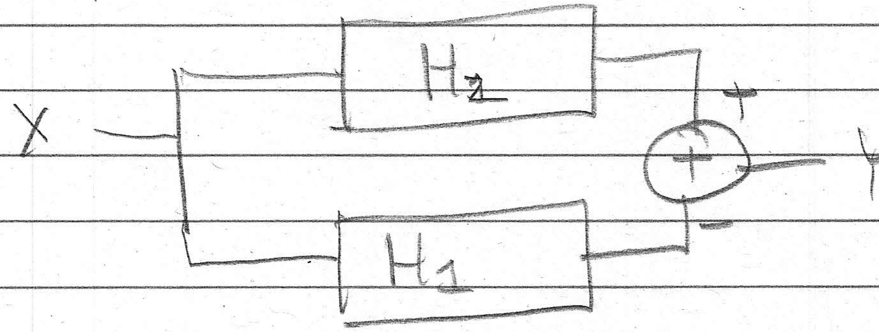
$$|H(e^{j\Omega})| = |H_1(e^{j\Omega})|\,|H_2(e^{j\Omega})|$$

$$= \underbrace{\left|\tfrac{1}{2}\cos\Omega + \tfrac{1}{2}\right|\,\left|e^{-j\Omega}\right|}_{\text{same result}}$$

Making a High Pass Filter

$h_1[n]$ $\qquad\qquad\qquad\qquad$ $h_2[n] = ?$

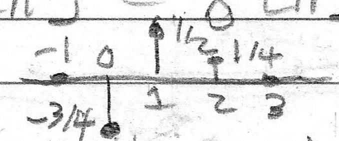




$$Y = (H_2 - H_1) * X = H * X$$

Suppose $h_2[n] = \delta[n]$?

$h[n] =$

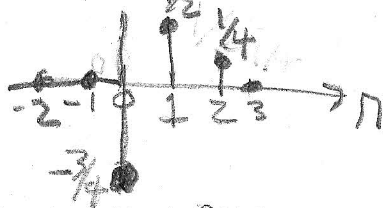


$$H(e^{j\Omega}) = H_2(e^{j\Omega}) - H_1(e^{j\Omega})$$

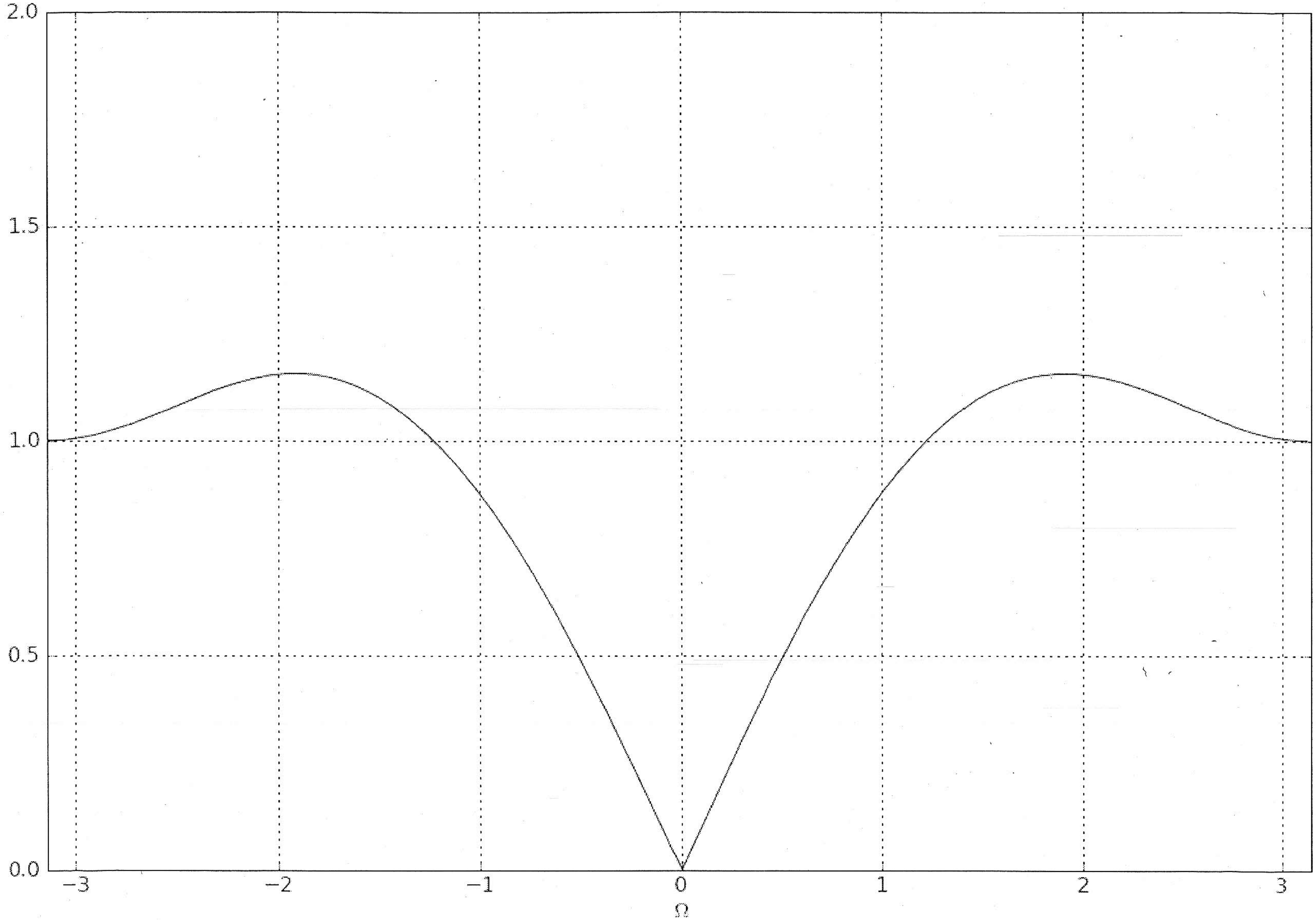
$$= 1 - e^{-j\Omega}\left(\tfrac{1}{2}\cos\Omega + \tfrac{1}{2}\right)$$

$$|H(e^{j\Omega})| = \left|1 - \left(\cos\Omega\left(\tfrac{1}{2}\cos\Omega + \tfrac{1}{2}\right)\right) - j\sin\Omega\left(\tfrac{1}{2}\cos\Omega + \tfrac{1}{2}\right)\right|$$

23A

h[n] = 

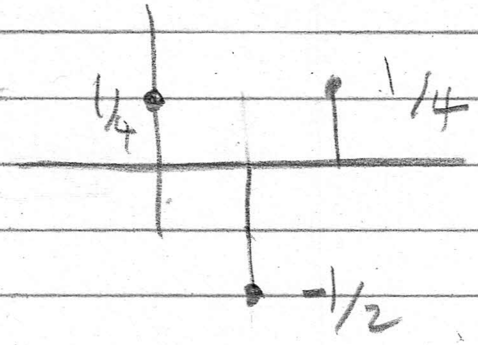
Magnitude of $|H(e^{j\Omega})|$ for Wrong HPF



Try Again

Suppose $h_2[n] = \delta[n-1]$

$h[n] =$

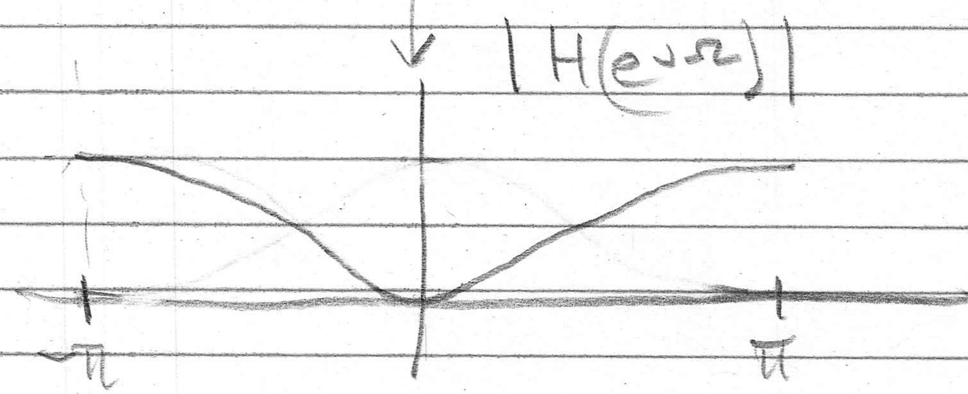


$$|H(e^{j\Omega})| = |H_2(e^{j\Omega}) - H_1(e^{j\Omega})|$$

$$= |e^{-j\Omega} - e^{-j\Omega}(\tfrac{1}{2}\cos\Omega + \tfrac{1}{2})|$$

$$= |e^{-j\Omega}(1 - (\tfrac{1}{2}\cos\Omega + \tfrac{1}{2}))|$$

$$= \underbrace{|e^{-j\Omega}|}_{=1}\ \underbrace{|\tfrac{1}{2} - \tfrac{1}{2}\cos\Omega|}$$



Filter Example 1

$$y[n] = A_1 \cos\frac{\pi}{6}n + A_2 \underbrace{\cos 0n}_{=1}$$

$$w[n] = \sum h[m]\, y[n-m]$$

Design a filter to eliminate $\cos\frac{\pi}{6}n$

Pick $h[i]$'s so that

1) Eliminate $e^{j\frac{\pi}{6}n}$ & $e^{-j\frac{\pi}{6}n}$

$$H(e^{j\pi/6 n}) = H(e^{-j\pi/6 n}) = 0$$

2) Pass Through $e^{j0n} = 1$ & $e^{-j0n} = 1$

$$H(e^{j0n}) = H(e^{-j0n}) = H(1) = 1$$

Three conditions: $H(e^{j\pi/6 n}) = 0 \quad H(e^{-j\pi/6 n}) = 0$

$$H(1) = 1$$

Use Formula

$$H(e^{j\Omega n}) = \sum_{m=0}^{L} h[m]\, e^{-j\Omega m}$$

$L = 2$ ($h[0]$, $h[1]$, $h[2]$ can match 3 conditions)

Eqn 1 $\quad \underbrace{e^{-j\pi/6\cdot 0}}_{=1} h[0] + e^{-j\pi/6\cdot 1} h[1] + e^{-j\pi/6\cdot 2} h[2] = 0$

Eqn 2 $\quad \underbrace{e^{+j\pi/6\cdot 0}}_{=1} h[0] + e^{+j\pi/6\cdot 1} h[1] + e^{+j\pi/6\cdot 2} h[2] = 0$

Eqn 3 $\quad \underbrace{e^{-j0\cdot 0}}_{=1} h[0] + \underbrace{e^{-j0\cdot 1}}_{=1} h[1] + \underbrace{e^{-j0\cdot 2}}_{=1} h[2] = 1$

II

Rewriting

$$H(e^{j\pi/6}) = h[0] + e^{-j\pi/6}h[1] + e^{-j2\pi/6}h[2] = 0$$

$$H(e^{-j\pi/6}) = h[0] + e^{+j\pi/6}h[1] + e^{+j2\pi/6}h[2] = 0$$

$$H(1) \quad h[0] + h[1] + h[2] = 1$$

Soln:

$$h[0] = 2+\sqrt{3} = h[2] \qquad h[1] = -(3+2\sqrt{3})$$

Example 2

$$y[n] = A_1 \cos \frac{\pi}{2} + A_2$$

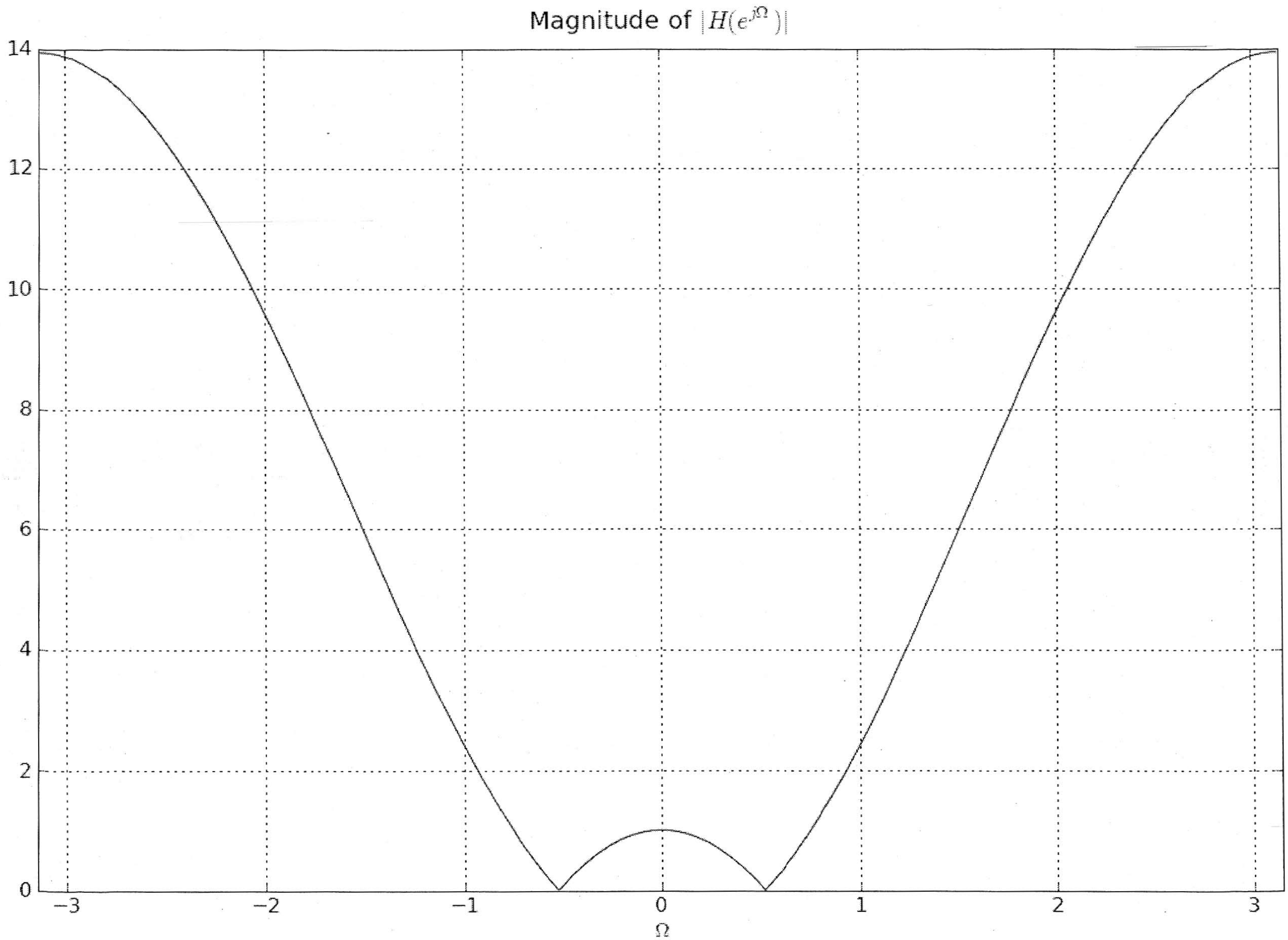
Eliminate $e^{j\pi/2\, n}$ & $e^{-j\pi/2\, n}$

$$H(e^{j\pi/2}): \quad h[0] + e^{-j\pi/2\cdot 1}h[1] + \overbrace{e^{-j\pi/2\cdot 2}}^{=-1}h[2] = 0$$

$$H(e^{-j\pi/2}): \quad h[0] + e^{+j\pi/2\cdot 1}h[1] + \overbrace{e^{+j\pi/2\cdot 2}}^{=-1}h[2] = 0$$

$$H(1) \quad h[0] + h[1] + h[2] = 1$$

Example1 II A



Rewriting

$$h[0] + e^{-j\pi/2} h[1] - h[2] = 0$$

$$h[0] + e^{+j\pi/2} h[1] - h[2] = 0$$

$$h[0] + h[1] + h[2] = 1$$

$$h[0] = 1/2 \quad h[1] = 0 \quad h[2] = 1/2$$

Example 3

$$y[n] = A_1 \cos\frac{\pi}{6}n + A_2 \cos\frac{\pi}{2}n + A_3$$

Eliminate: $e^{j\pi/6 n}$ $e^{-j\pi/6 n}$, $e^{j\pi/2 n}$, $e^{-j\pi/2 n}$

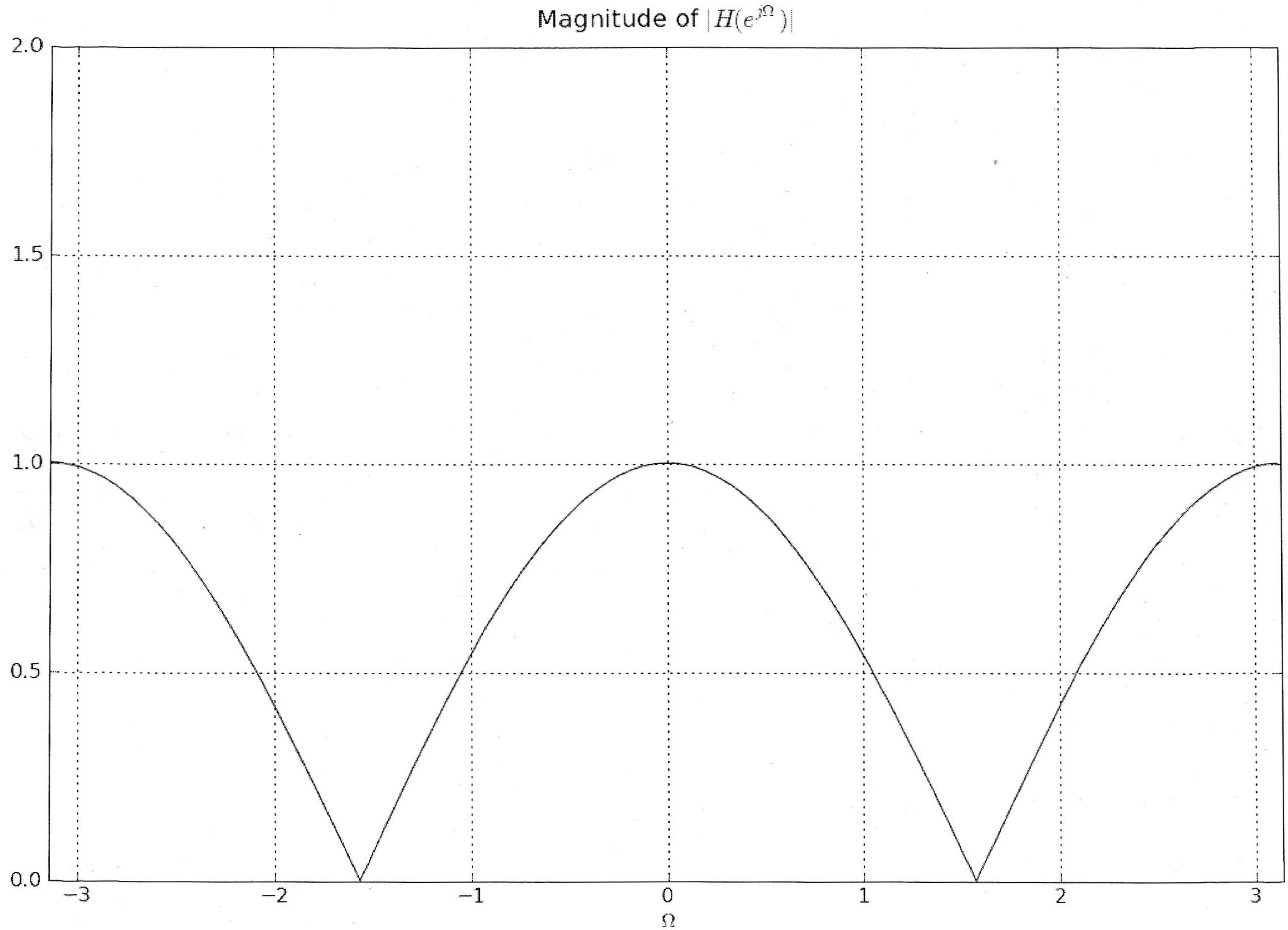
Pass: e^{j0n}

5 Constraints $H(e^{j\pi/6}) = H(e^{-j\pi/6}) = H(e^{j\pi/2}) = H(e^{-j\pi/2}) = 0$

$$H(1) = 1$$

L = 4 $\quad \underbrace{h[0], h[1], h[2], h[3], h[4]}_{\text{unknowns}}$

Example 2 IIA



IV

<u>Using Earlier simplifications</u>

$$0 = h[0] + e^{j\pi/6} h[1] + e^{j\pi/3} h[2] + e^{j\pi/2} h[3] + e^{j\frac{2\pi}{3}} h[4]$$

$$0 = h[0] + e^{-j\pi/6} h[1] + e^{-j\pi/3} h[2] + e^{-j\pi/2} h[3] + e^{-j\frac{2\pi}{3}} h[4]$$

$$0 = h[0] + e^{j\pi/2} h[2] + h[2] + e^{j3\pi/2} h[3] - h[4]$$

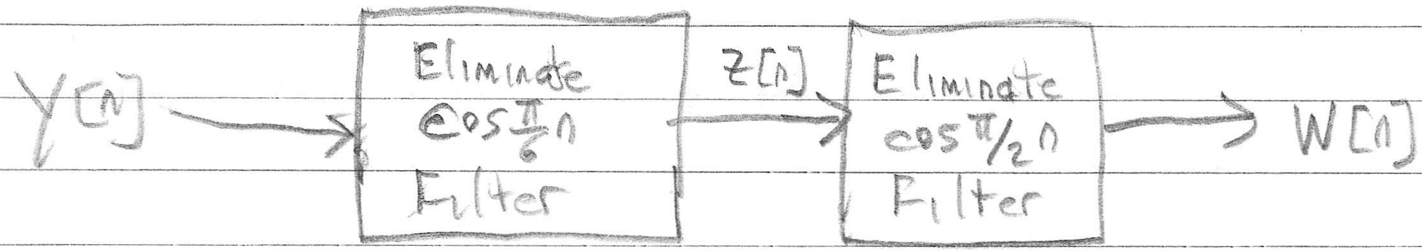
$$0 = h[0] + e^{-j\pi/2} h[2] + h[2] + e^{-3\pi/2} h[3] - h[4]$$

$$1 = h[0] + h[1] + h[2] + h[3] + h[4]$$

$h[0] = 1 + \frac{\sqrt{3}}{2}$ $\quad h[1] = -\left(\frac{3}{2} + \sqrt{3}\right)$ $\quad h[2] = 2 + \sqrt{3}$

$h[3] = -\left(\frac{3}{2} + \sqrt{3}\right)$ $\quad h[4] = 1 + \frac{\sqrt{3}}{2}$

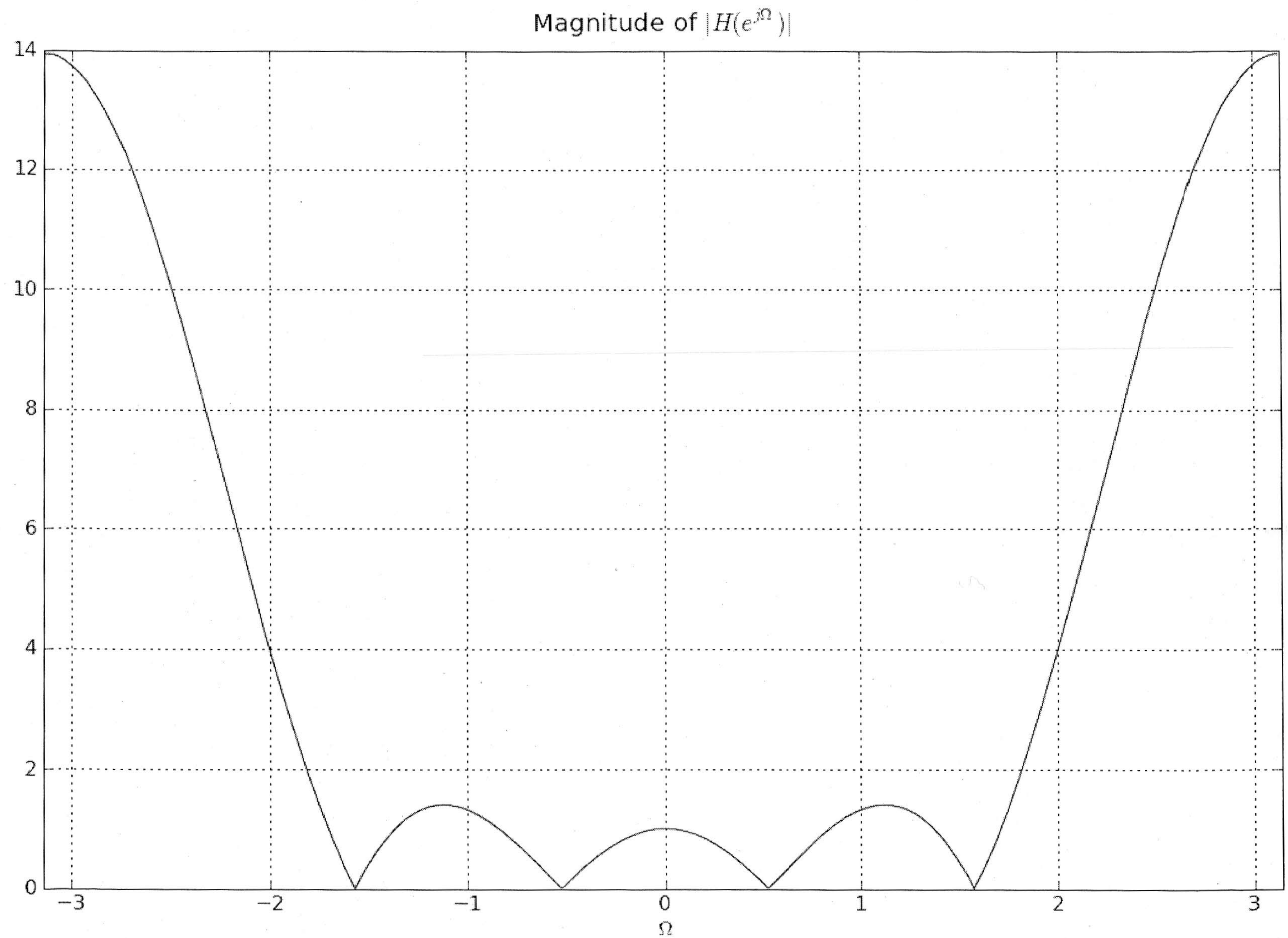
<u>OR</u>



$z = h_{\pi/6} * y \qquad w = h_{\pi/2} * z \Rightarrow w = h_{\pi/6} * h_{\pi/2} * y$

$h_{\pi/6} * h_{\pi/2} = [2+\sqrt{3}, -(3+2\sqrt{3}), 2+\sqrt{3}] * [\frac{1}{2}, 0, \frac{1}{2}]$

Example 3 IVA



$$h_{\pi/6} * h_{\pi/2} = \left[1+\frac{\sqrt{3}}{2}, -\left(3/2+\sqrt{3}\right), 2+\sqrt{3}, -\left(3/2+\sqrt{3}\right), 1+\frac{\sqrt{3}}{2}\right]$$

Can Compose two Filters!