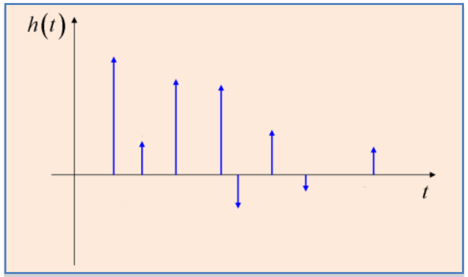
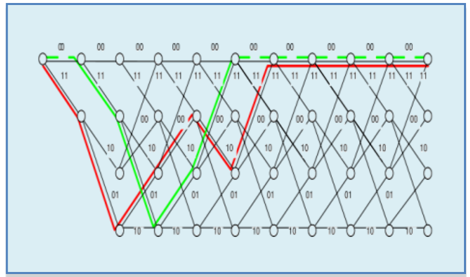
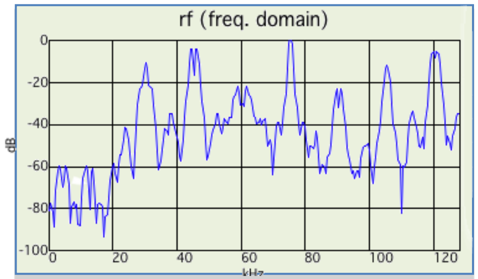
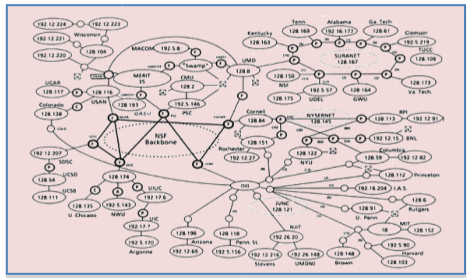








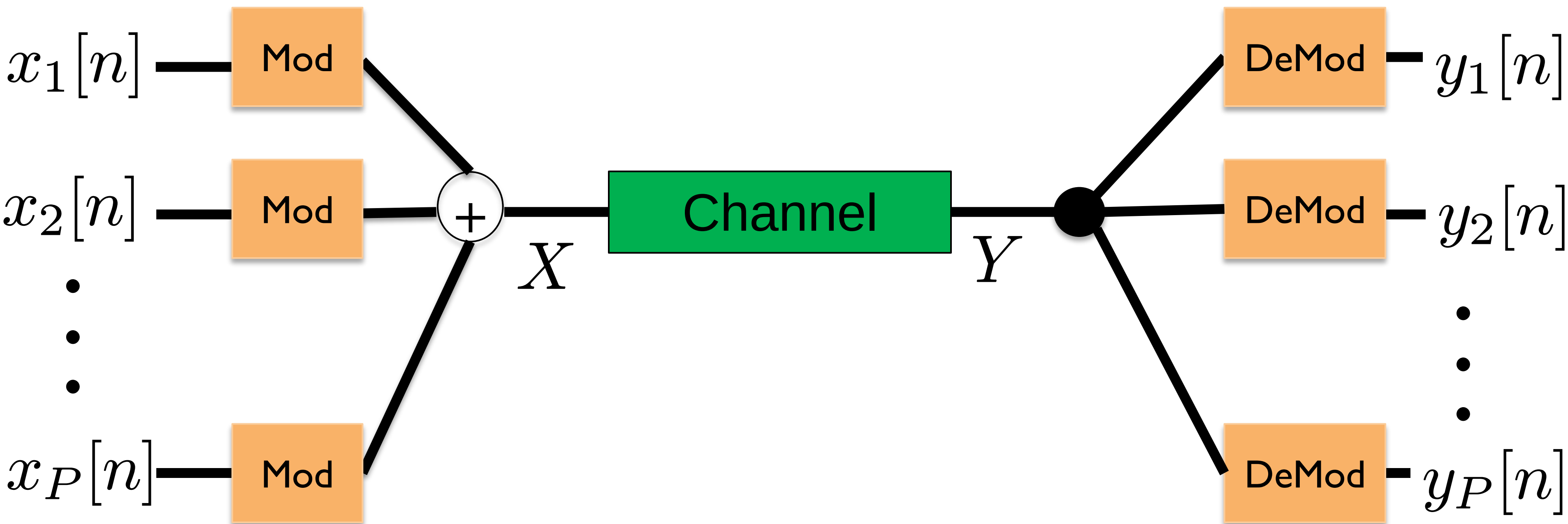
INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

6.02 Fall 2010
Lecture #14

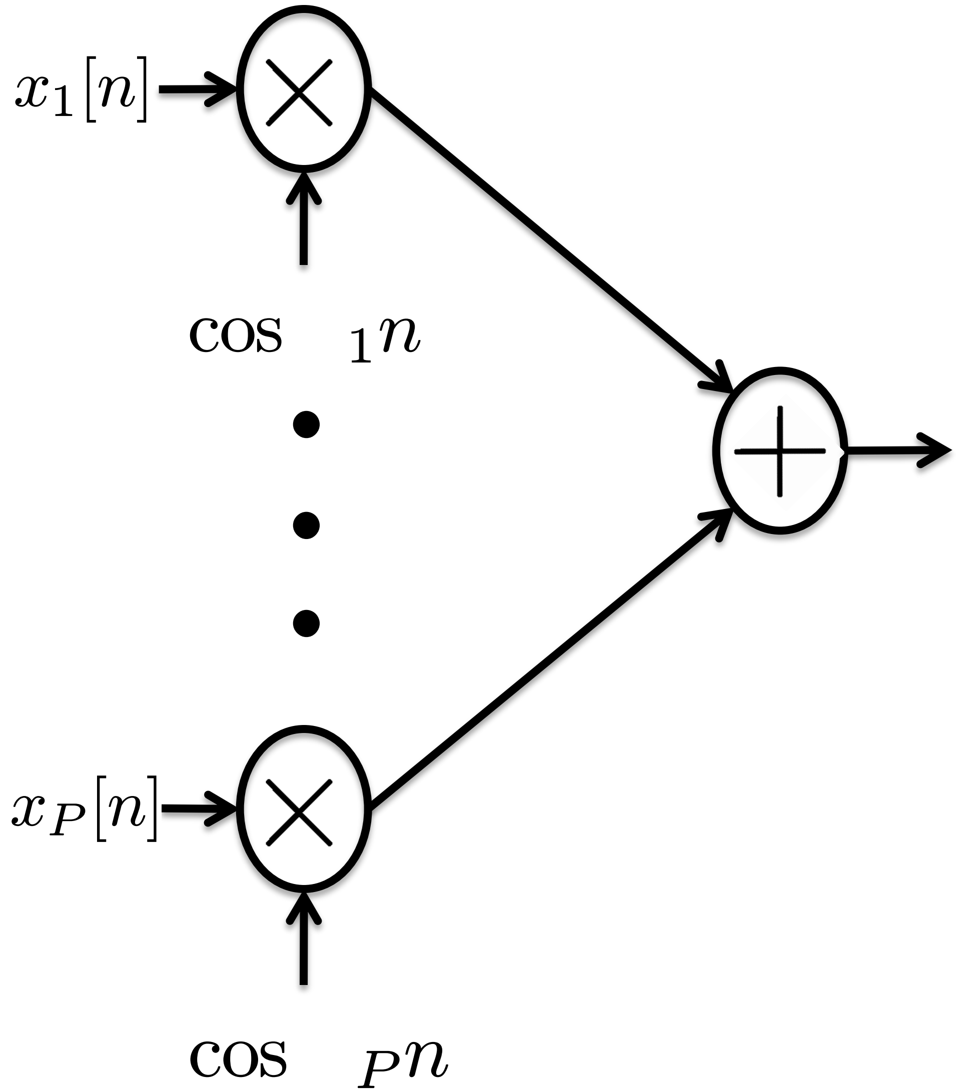
- Frequency Domain Sharing
- Spectrum and the Fourier Series
- Fourier Series Examples
- Rise Time and Spectrum

Frequency Domain Sharing - Big Picture



- Questions
  - What is Modulation (Mod in figure)?
  - What do typical X's look like?
  - What should the Demodulator be?
  - What is the relation between $x_i[n]$ and $y_i[n]$

Cosine Modulation

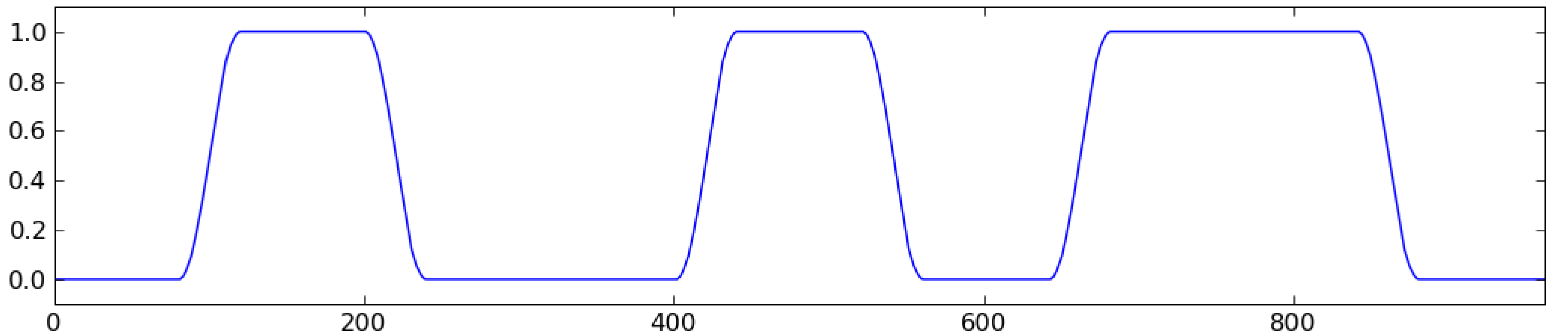


$$x[n] = \sum_{i=1}^{P} x_i[n] \cos \; {}_i n$$

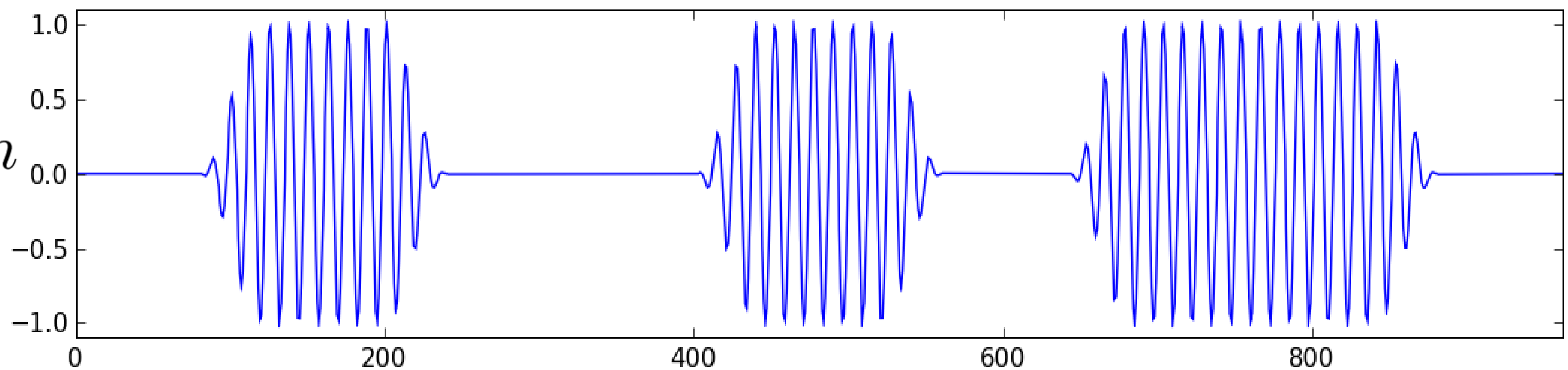
Ideal Channel Case $Y = X$

$$y[n] = \sum_{i=1}^{P} x_i[n] \cos \Omega_i n$$

$x_i[n]$

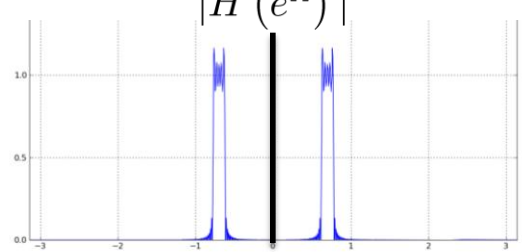


$x_i[n] \cos 0.5n$



$y[n] = \sum_{i=1}^{P} x_i[n] \cos \Omega_i n$

$|H\left(e^{\Omega}\right)|$



$-\Omega_k \quad \Omega_k$

?

What we have so far

If $x[n] = e^{j\Omega n}$ is the input to an LTI channel,

Then

$$y[n] = \underbrace{H(e^{j\Omega})}_{\substack{\text{Channel} \\ \text{Frequency} \\ \text{Response}}} e^{j\Omega n}$$

Problem: using F.D.M. to send messages

$$x[n] = \sum_{i=1}^{p} \underbrace{X_i[n]}_{i^{th}\text{ message}} \; \underbrace{e^{j\Omega_i [n]}}_{\substack{\text{frequency} \\ \text{for the} \\ i^{th} \\ \text{message}}}$$

How do we analyze $X_i[n] e^{j\Omega_i n}$

$$X_i[n] e^{j\Omega_i n} \to \boxed{H} \to \cancel{H(e^{j\Omega_i}) X_i[n] e^{j\Omega_i n}}$$

Why?

Need a new Tool: Fourier Series

Periodic Approximation

For some large value of N,

Assume $x_i[n+N]\, e^{j\Omega_i(n+N)} = x_i[n] e^{j\Omega_i n}$

Why? leads to tractable analysis

1st Consequence of periodic assumption

Since $e^{j\Omega_i(n+N)} = e^{j\Omega_i n}$

Then $\cancel{e^{j\Omega_i n}}\, e^{j\Omega_i N} = \cancel{e^{j\Omega_i n}}$

$e^{j\Omega_i N}$ must equal 1

$$e^{j\Omega_i N} = \underbrace{\cos \Omega_i N}_{\text{Must be } 1} + j \underbrace{\sin \Omega_i N}_{\text{Must be Zero}}$$

$\Omega_i N$ must equal $2\pi, 4\pi, 8\pi, \ldots$

Since $-\pi \leq \Omega_i \leq \pi$

$$\Omega_i = 0, \pm \frac{2\pi}{N}\cdot 1, \pm \frac{2\pi}{N}\cdot 2, \ldots, \pm \frac{2\pi}{N}\cdot\frac{N}{2}$$

Assuming N is even!

2^{nd} Consequence of Periodic Assumption

If $x[n]$ is real and periodic with period N, and N is <u>even</u>:

No $k=\frac{N}{2}$ or $k=0$ term for sin, as $\sin(0)=\sin\pi=0$

$$x[n] = \underbrace{\sum_{\cancel{k=0}\; k=1}^{\cancel{k=\frac{N}{2}}\; k=\frac{N}{2}-1} a_k \sin\frac{2\pi}{N}kn + \sum_{k=0}^{k=\frac{N}{2}} b_k \cos\frac{2\pi}{N}kn}_{\text{Discrete-Time Fourier Series. (DTFS)}}$$

$\left(\frac{N}{2}-2\right) + \left(\frac{N}{2}+1\right)$ coeffs

$= N-1$ coeffs

DTFS using Complex Exponentials

Removes a redundancy

$$x[n] = \sum_{k=-\frac{N}{2}}^{N/2-1} \underbrace{X[k]}_{\text{complex coefficients}} e^{j\frac{2\pi}{N}kn}$$

<u>Note</u>: $2\cdot\frac{N}{2} = N$ <u>complex</u> coeffs in DTFS, and only N <u>real</u> unique values for x, $x[0], x[1], \ldots x[N-1]$.

DTFS Degrees of Freedom

N is even case

If $x[n]$ is real for all n

then $X[k]e^{j\frac{2\pi}{N}kn} + X[-k]e^{-j\frac{2\pi}{N}kn}$

must be real!

$\Rightarrow$ $Re(X[k]) = Re(X[-k])$

$Im(X[k]) = -Im(X[-k])$

So $\left(2 \cdot \frac{N}{2}\right) = N$ complex coefficients have only N degrees of freedom (

N-2 constraints $\begin{cases} Re(X[1]) = Re(X[-1]), \ldots, Re(X[\frac{N}{2}-1]) = Re(X[-\frac{N}{2}+1]) \\ Im(X[1]) = -Im(X[-1]), \ldots, Im(X[\frac{N}{2}-1]) = -Im(X[-\frac{N}{2}+1]) \end{cases}$

And

1 constraint $\begin{cases} Re(X[0]) = Re(X[0]) \quad Im(X[0]) = -Im(X[0]) \\ \Rightarrow X[0] \text{ must be real} \end{cases}$

3 last constraints $\begin{cases} \text{Finally} \quad e^{j\frac{2\pi}{N}\left(\frac{-N}{2}\right)n} = e^{-j\frac{2\pi}{N}\left(\frac{N}{2}\right)n} = -1^n \\ X[\frac{N}{2}] \text{ is real} \end{cases}$

Computing Fourier Coefficients

Multiply by test exponential

$$e^{-j\frac{2\pi}{N}\ell n}\, x[n] = e^{-j\frac{2\pi}{N}\ell n} \underbrace{\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} X[k]\, e^{j\frac{2\pi}{N}kn}}_{\text{DTFS}}$$

Never zero!

Sum both sides over $0 \le n \le N-1$

$$\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}\ell n}\, x[n] = \sum_{n=0}^{N-1} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} X[k]\, e^{j\frac{2\pi}{N}(k-\ell)n}$$

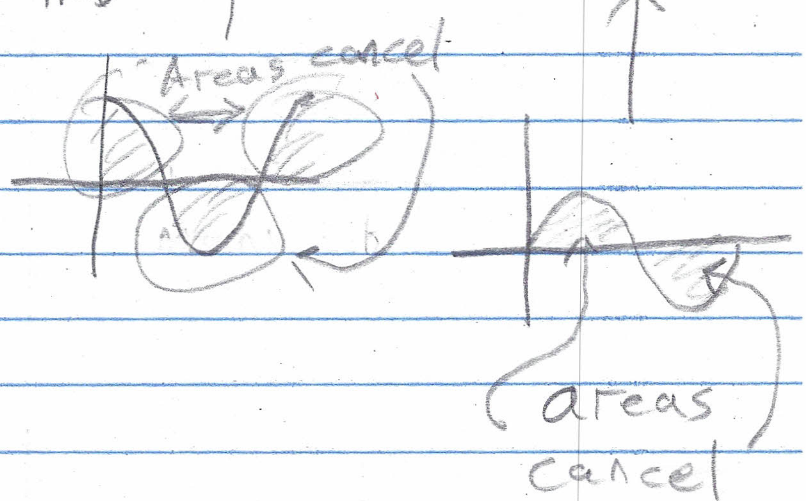
$$= \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} X[k] \underbrace{\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-\ell)n}}_{\substack{=0 \quad k \neq \ell \\ =N \quad k=\ell}}$$

$$\Rightarrow \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}\ell n}\, x[n] = X[\ell]$$

Why is $\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-l)n} = 0 \quad k \neq l$

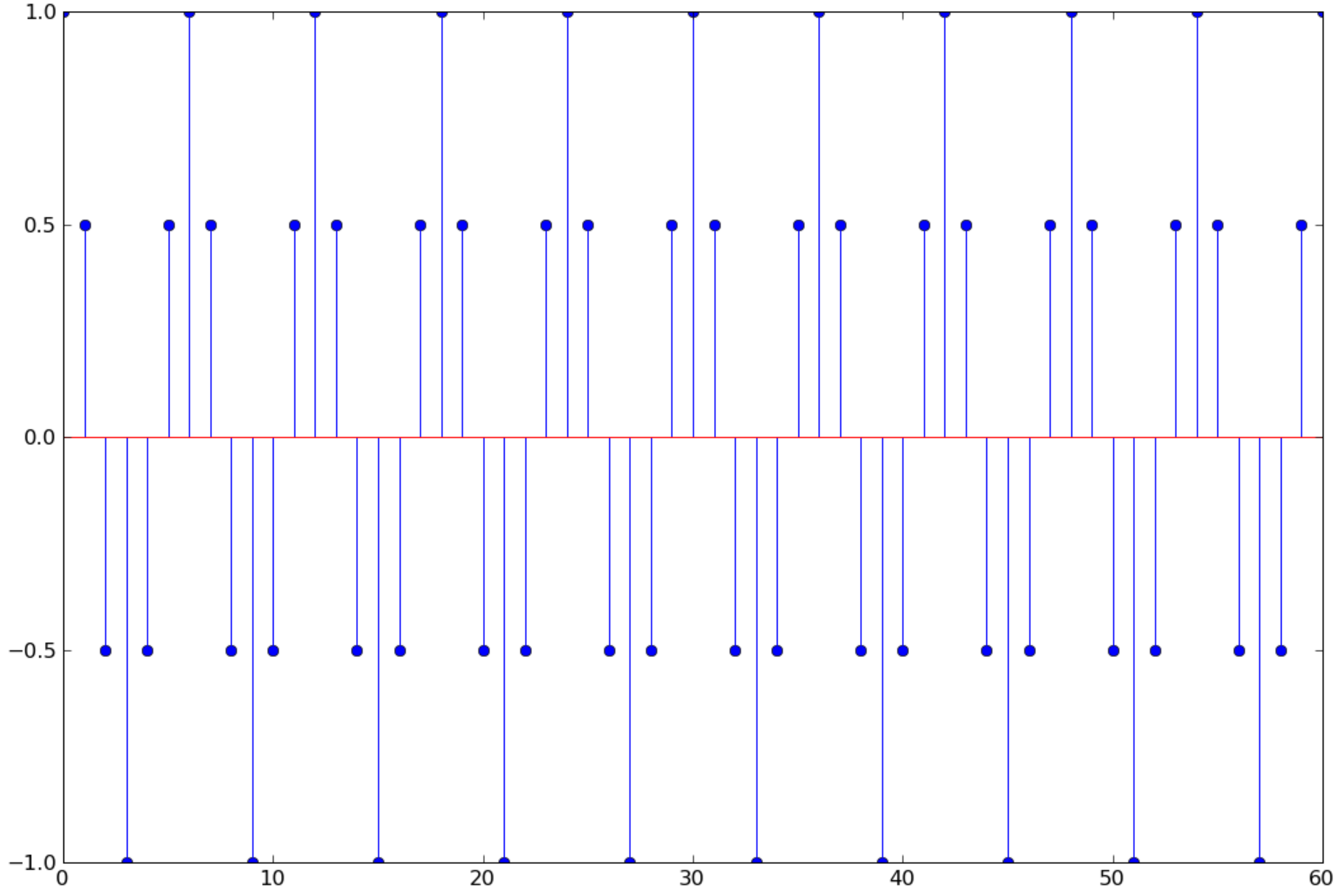
Suppose $k-l = 1$

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}n} = \sum_{n=0}^{N-1}\left(\cos\frac{2\pi}{N}n + j\sin\frac{2\pi}{N}n\right)$$



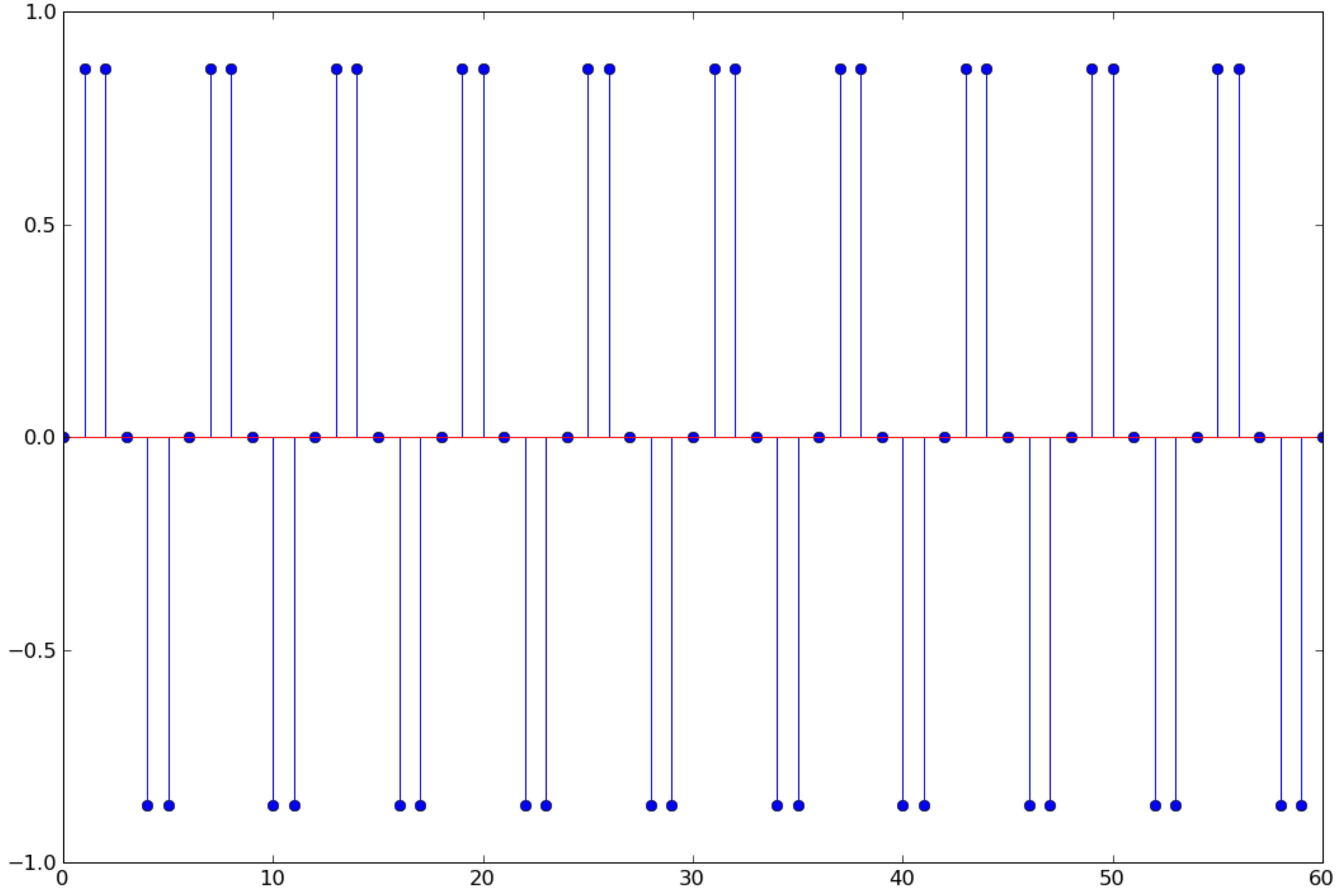
Cosine in Time

$$x[n] = \cos\frac{\pi}{3}n = \frac{1}{2}e^{j\frac{\pi}{3}n} + \frac{1}{2}e^{-j\frac{\pi}{3}n}$$



Sine in Time

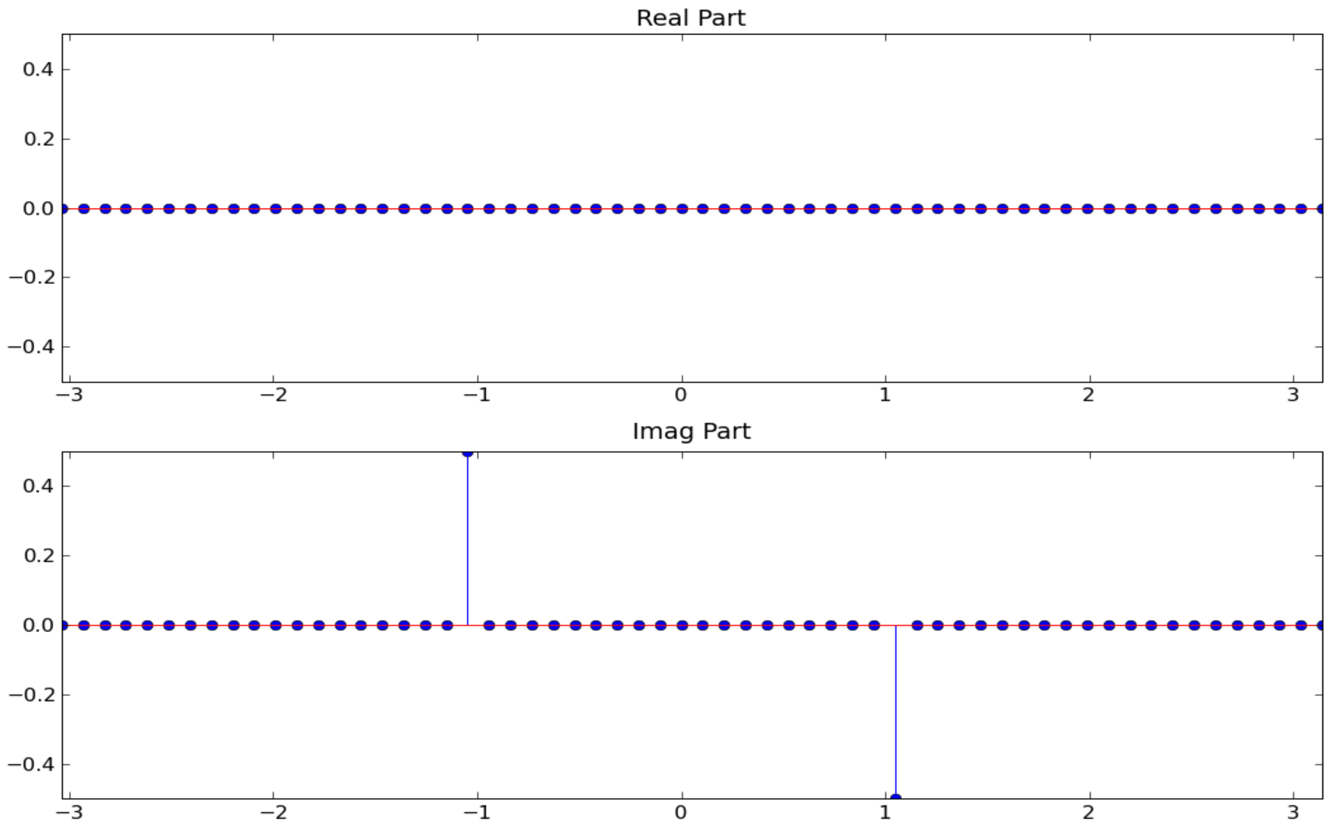
$$x[n] = \sin \tfrac{\pi}{3} n = \frac{-j}{2} e^{j \frac{\pi}{3} n} + \frac{j}{2} e^{-j \frac{\pi}{3} n}$$



Sine Fourier Series, 2 Imaginary values

$$x[n] = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} X[k] e^{j\frac{2\pi}{N}kn} = \frac{-j}{2} e^{j\frac{\pi}{3}n} + \frac{j}{2} e^{-j\frac{\pi}{3}n}$$

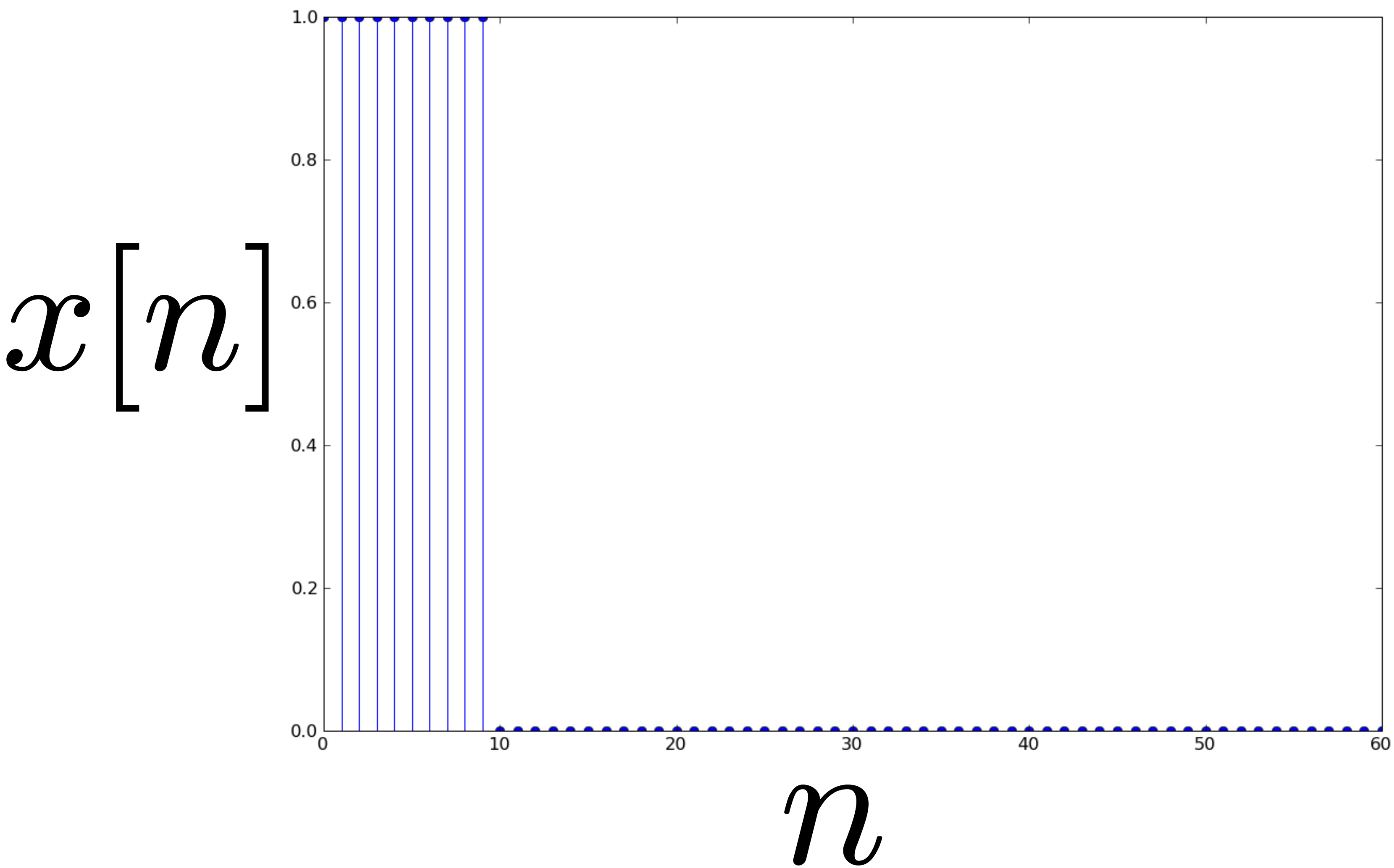
$$X[k]$$



$$\frac{2\pi}{N}k$$

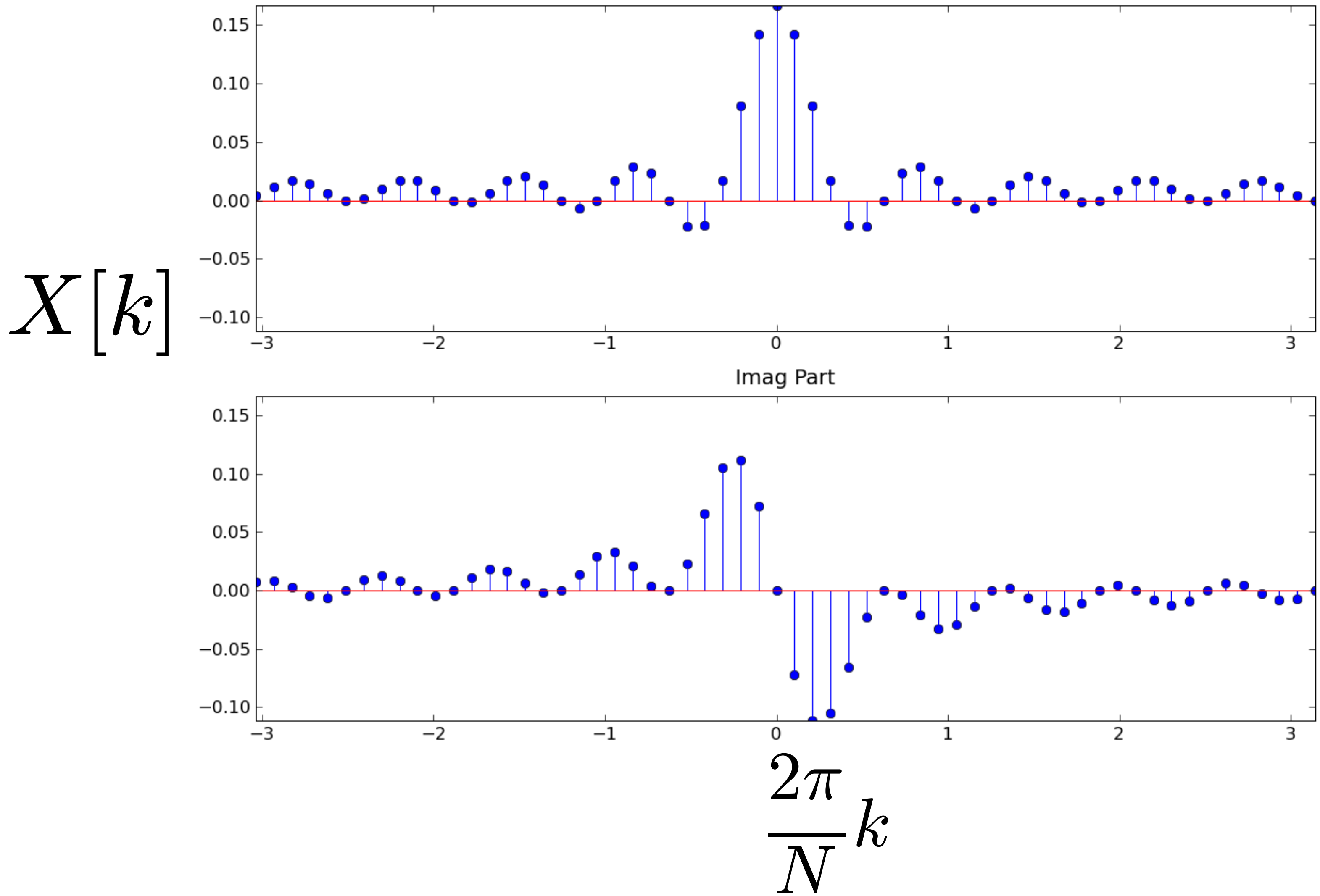
Note horizontal axis ranges over –pi to pi!

Fast Rise Pulse

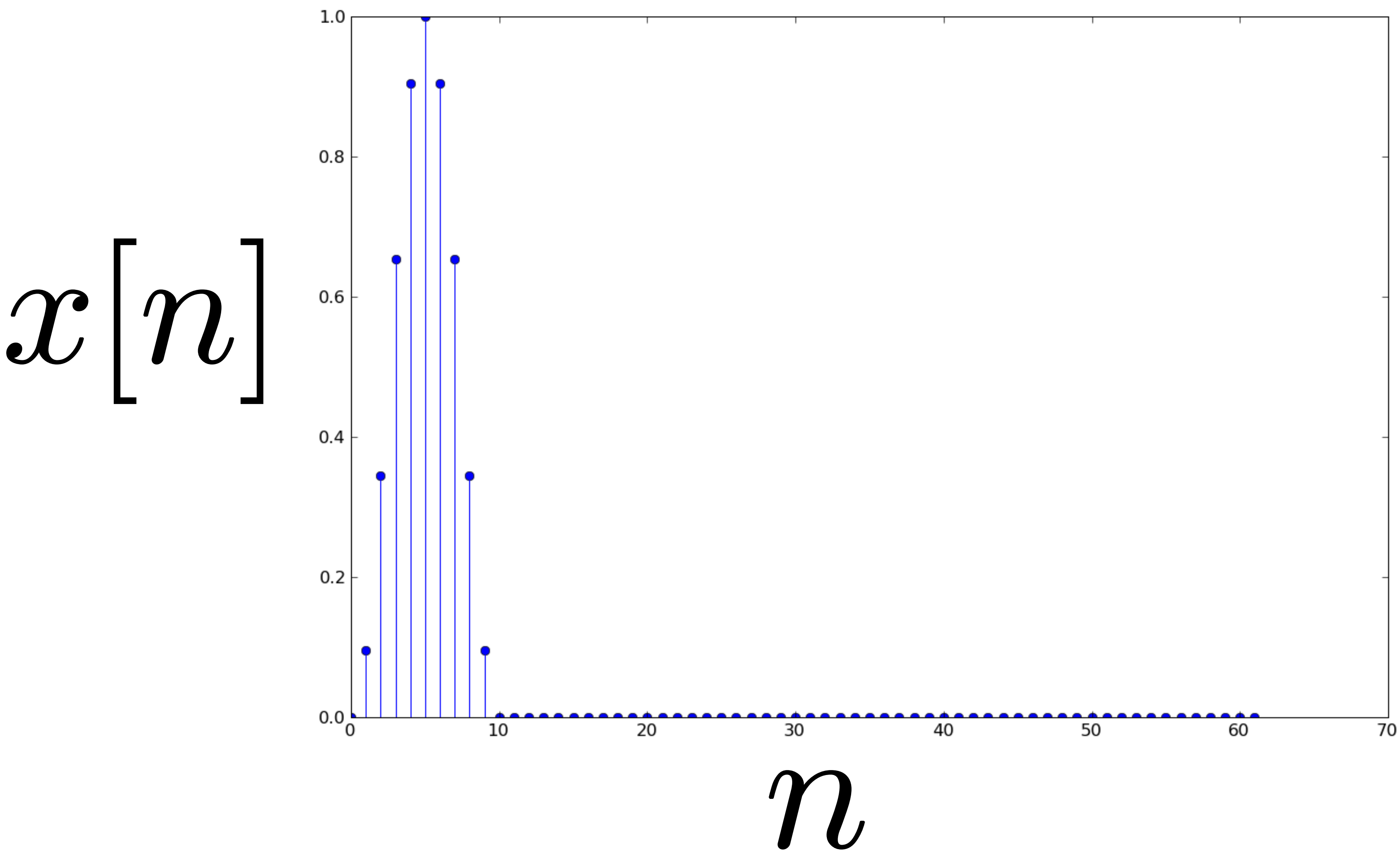


Spectrum of Fast Rise Pulse

$$x[n] = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} X[k] e^{j\frac{2\pi}{N}kn}$$



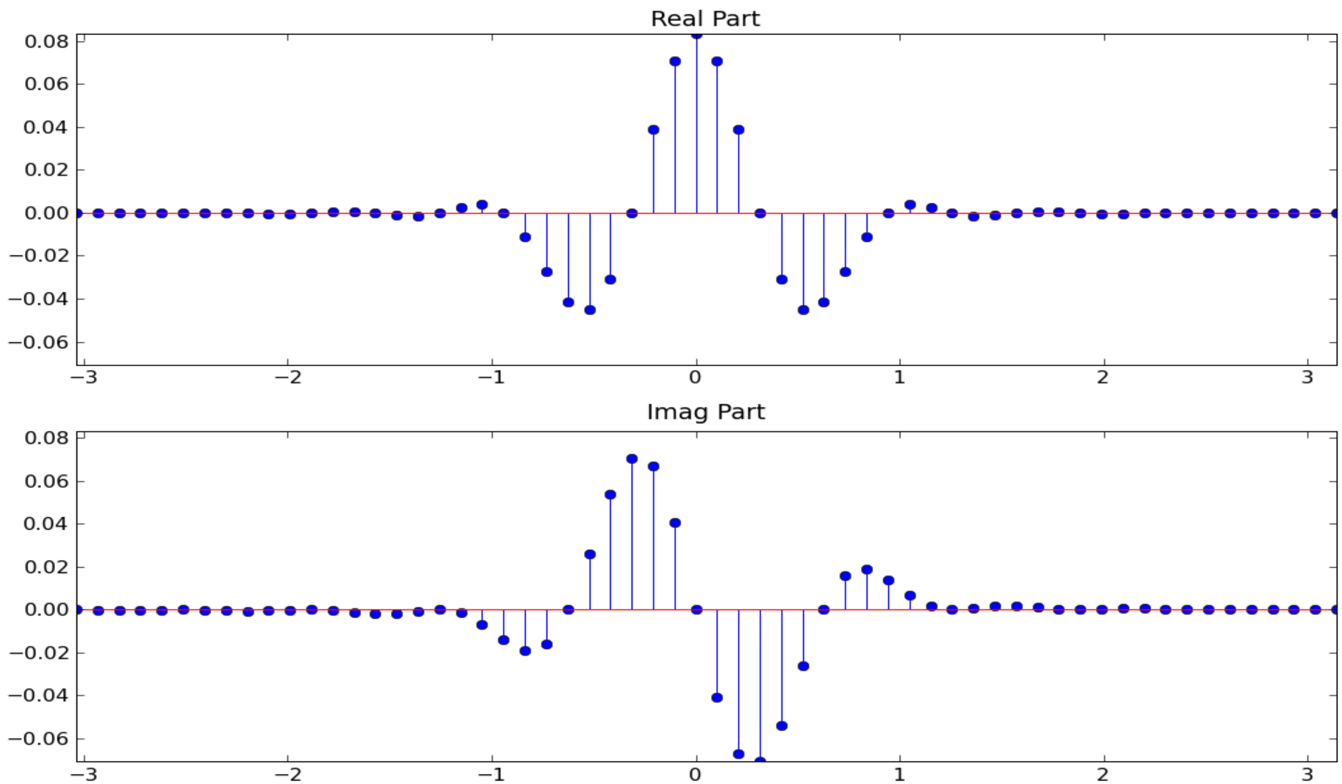
Slow Rise Pulse



Spectrum Slow Rise Pulse

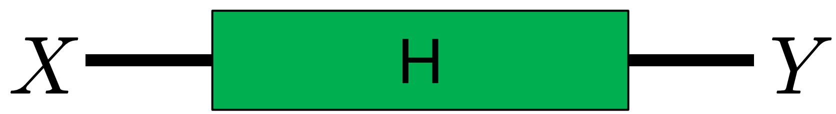
$$x[n] = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} X[k] e^{j\frac{2\pi}{N}kn}$$

$X[k]$



$$\frac{2\pi}{N}k$$

Key Fourier Series Advantage



$$y[n] = \sum_{m=0}^{n} h[m]x[n-m]$$

OR

$$x[n] = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} X[k]e^{j\frac{2\pi}{N}kn} \Rightarrow y[n] = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} H(e^{j\frac{2\pi}{N}k})X[k]e^{j\frac{2\pi}{N}kn}$$

$$H(e^{j\quad}) = \sum_{m=0}^{L} h[m]e^{-j\quad m}$$

Examples

Suppose $N = 20$

1) What are the allowed Ω_i's

$$\Omega_i = 0,\ \pm\frac{\pi}{10},\ \pm\frac{\pi}{5},\ \pm\frac{3\pi}{10},\ \ldots,\ \pm\frac{4\pi}{5},\ \pm\frac{9\pi}{10},\ \pm\pi$$

We will only use $-\pi$ Ω

$$\left(e^{j\frac{2\pi}{N}\cdot\frac{N}{2}n} = e^{j\pi n} = e^{-j\frac{2\pi}{N}\frac{N}{2}n} = e^{-j\pi n}\right)$$

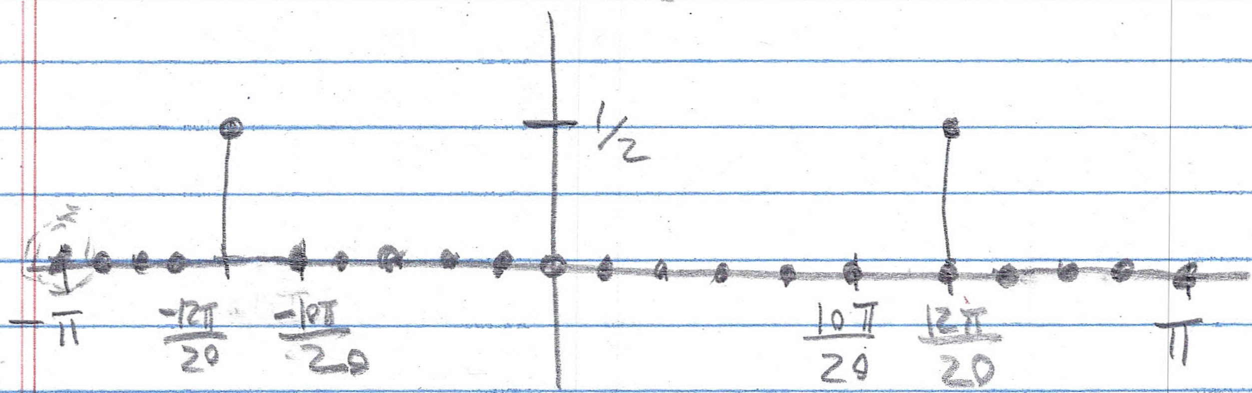
2)
$$x[n] = \cos\left(\frac{3\pi}{5}n + \phi\right)$$
$$= \frac{1}{2}\left(e^{j\left(\frac{2\pi}{20}\cdot 6n + \phi\right)}\right) + \frac{1}{2}\left(e^{-j\left(\frac{2\pi}{20}\cdot 6n + \phi\right)}\right)$$
$$= \frac{e^{j\phi}}{2}e^{j\frac{2\pi}{20}6n} + \frac{e^{-j\phi}}{2}e^{-j\frac{2\pi}{20}6n}$$

$$X[6] = \frac{e^{j\phi}}{2} \qquad X[-6] = e^{-j\phi}/2$$

$$X[k] = 0 \qquad k \neq 6$$

If $\phi = 0$ $\qquad Im(X[K]) = 0$

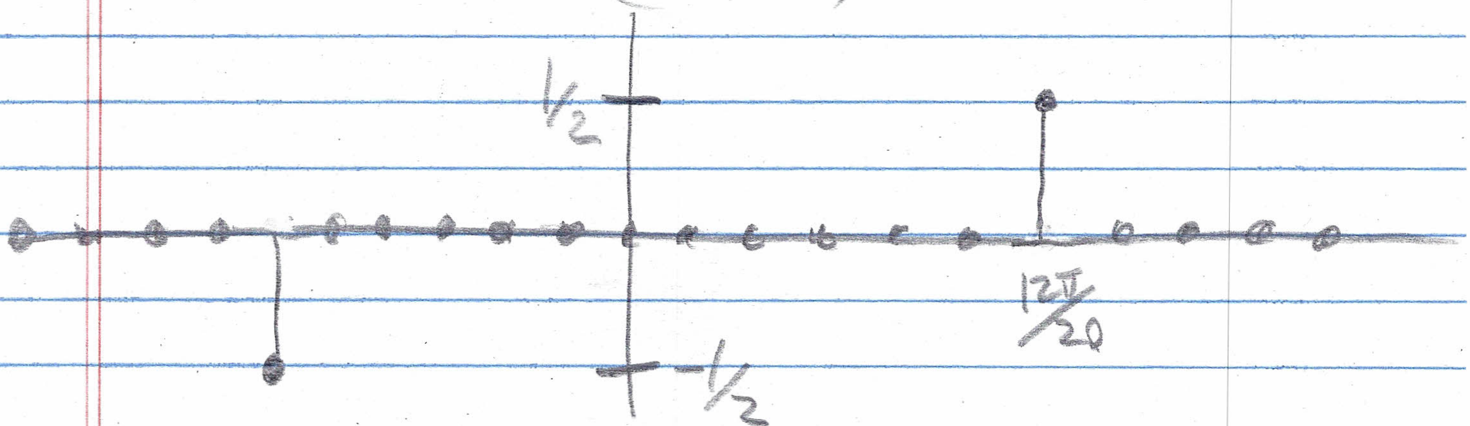
$Re\, X[K]$



If $\phi = \pi/2$ $\qquad \frac{e^{j\phi}}{2} = \frac{1}{2}j \quad \frac{e^{-j\phi}}{2} = -\frac{1}{2}j$

$Re(X[K]) = 0$

$Im(X[K])$

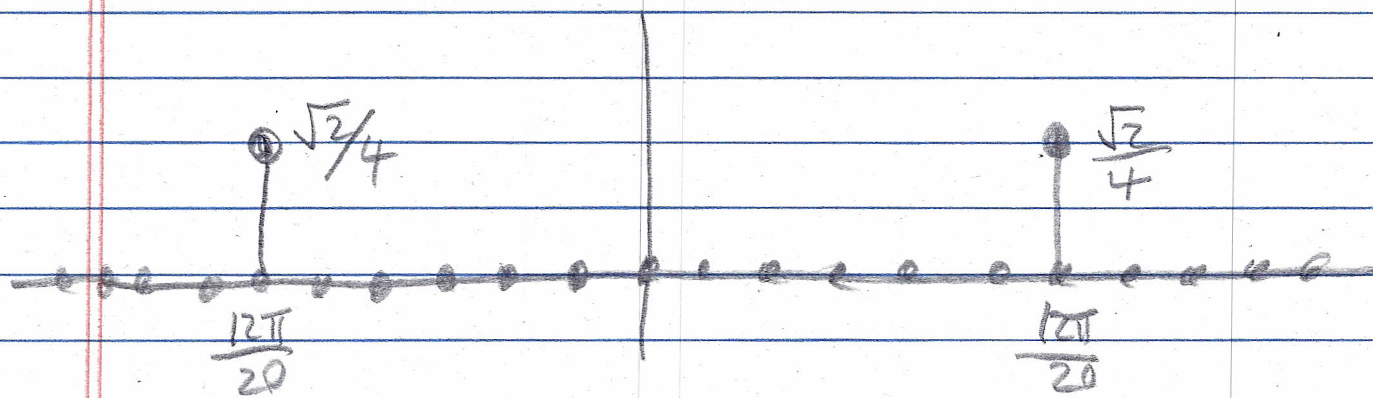


Note $\cos(\Omega n + \pi/2) = -\sin(\Omega n)$

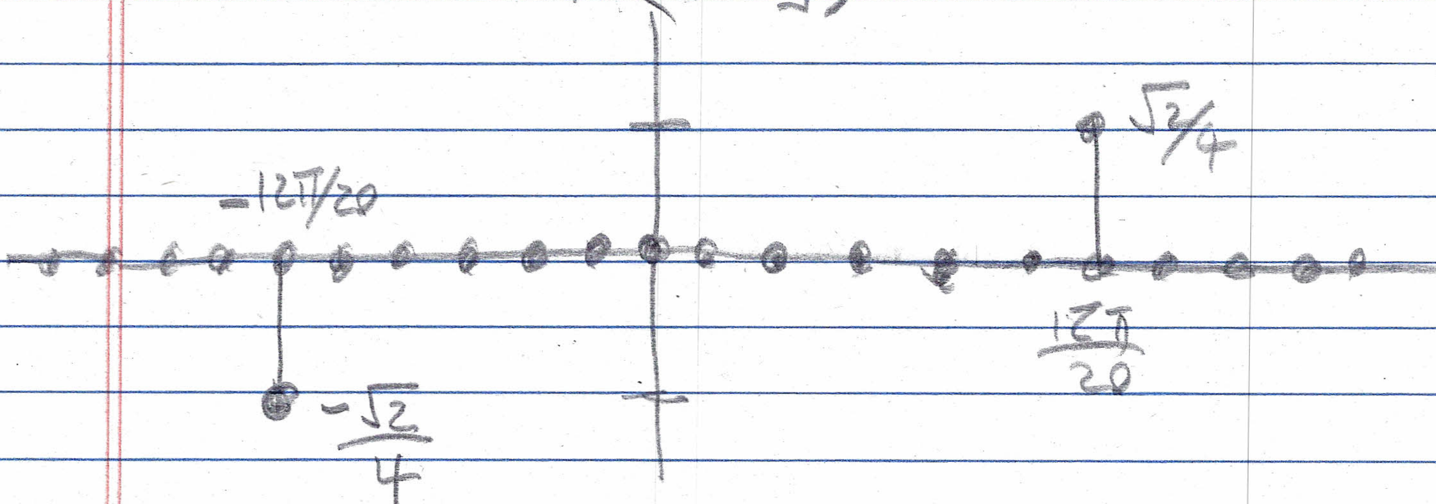
$\underline{\phi = \pi/4}$ $\quad \frac{1}{2}e^{j\phi} = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4}j$

$$\frac{1}{2}e^{-j\phi} = \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4}j$$

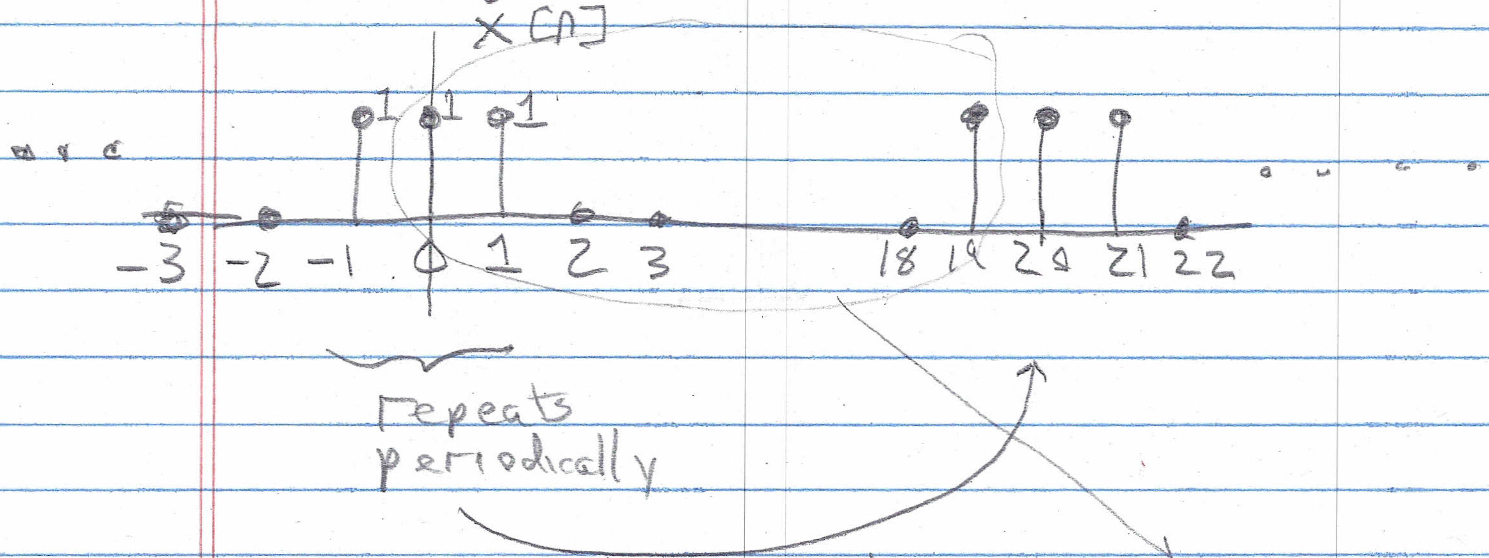
Re(X[K])



Im(X[K])



Square Pulse Case (N=20)



$$\underline{X}[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{20}kn}$$

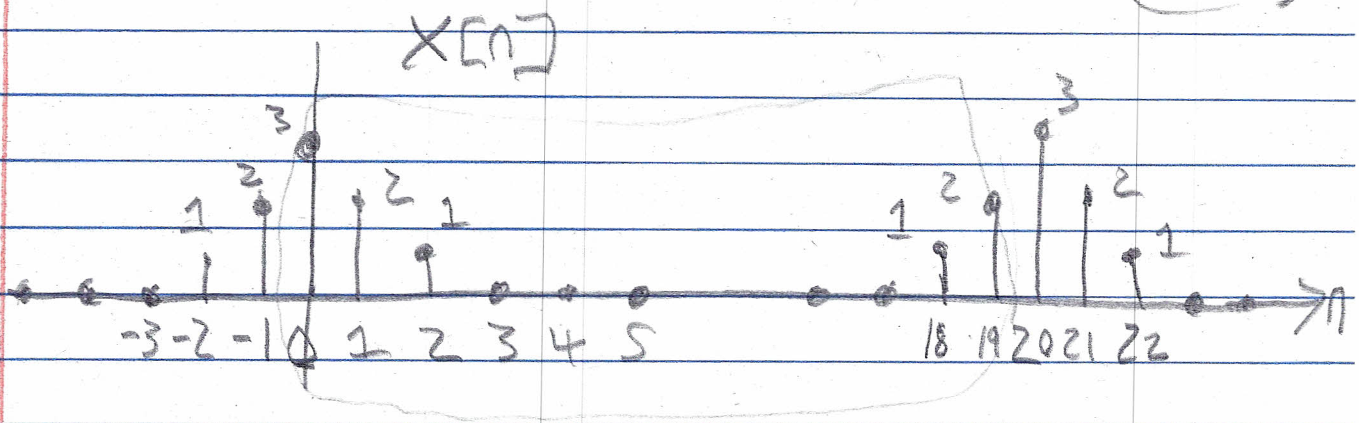
$$= \frac{1}{20}\left(1\cdot e^{-j\frac{2\pi}{20}k\cdot 0} + 1\cdot e^{-j\frac{2\pi}{20}k\cdot 1} + 1\cdot e^{-j\frac{2\pi}{20}k\cdot 19}\right)$$

$$= \frac{1}{20}\left(1 + e^{-j\frac{2\pi}{20}k} + \underbrace{e^{-j\frac{2\pi\cdot 19}{20}k}}_{= e^{+j\frac{2\pi}{20}k}}\right)$$

$$= \frac{1}{20}\left(1 + 2\cos\left(\frac{2\pi}{20}k\right)\right)$$

For arbitrary N $\quad \underline{X}[k] = \dfrac{1 + 2\cos\left(\frac{2\pi}{N}k\right)}{N}$

Triangular Pulse Case (N=20)



$$X[k] = \frac{1}{20}\left(3 \cdot 1 + 2 \cdot e^{-j\frac{2\pi}{20}k} + 1 \cdot e^{-j\frac{4\pi}{20}k} + 1\, e^{-j\frac{2\pi \cdot 18}{20}k} + 2\, e^{-j\frac{2\pi \cdot 19}{20}k}\right)$$

$$= \frac{1}{20}\left(3 + 4\cos\left(\frac{2\pi}{20}k\right) + 2\cos\left(\frac{4\pi}{20}k\right)\right)$$

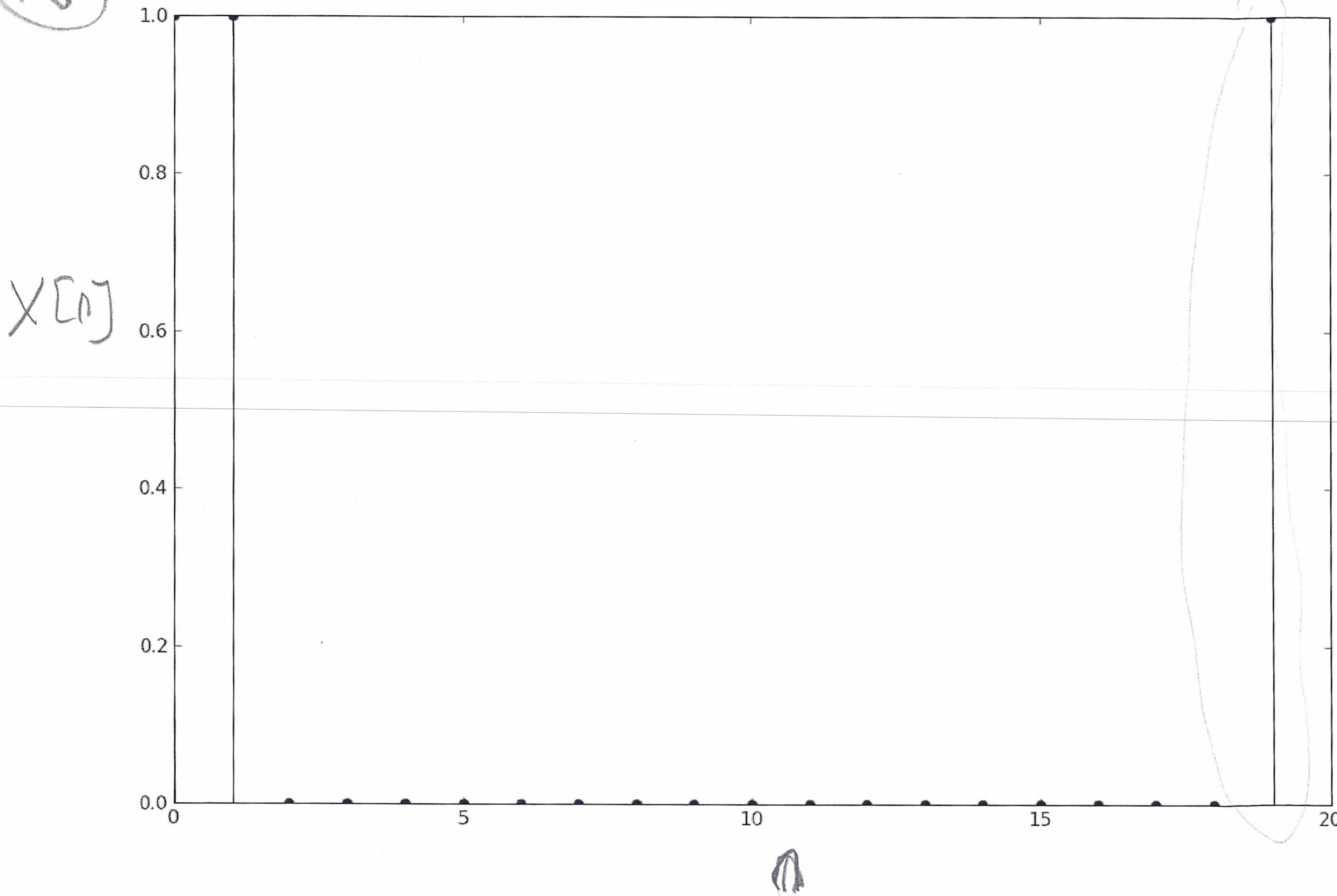
Note: Compare to product of square pulses

$$\left(\frac{1}{20}\left(1 + e^{-j\frac{2\pi}{20}k} + e^{+j\frac{2\pi}{20}k}\right)\right) \cdot \left(\frac{1}{20}\left(1 + e^{-j\frac{2\pi}{20}k} + e^{+j\frac{2\pi}{20}k}\right)\right)$$

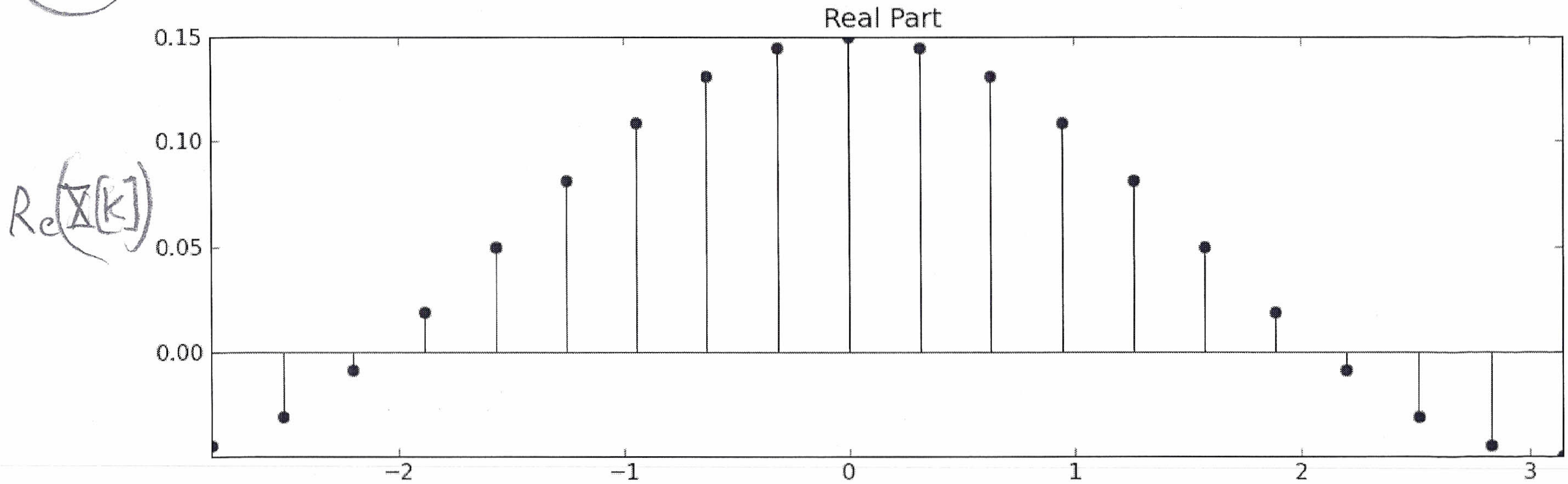
$$= \frac{1}{20^2}\left(3 + 2e^{-j\frac{2\pi}{20}k} + e^{-j\frac{4\pi}{20}k} + e^{j\frac{4\pi}{20}k} + 2e^{j\frac{2\pi}{20}k}\right)$$

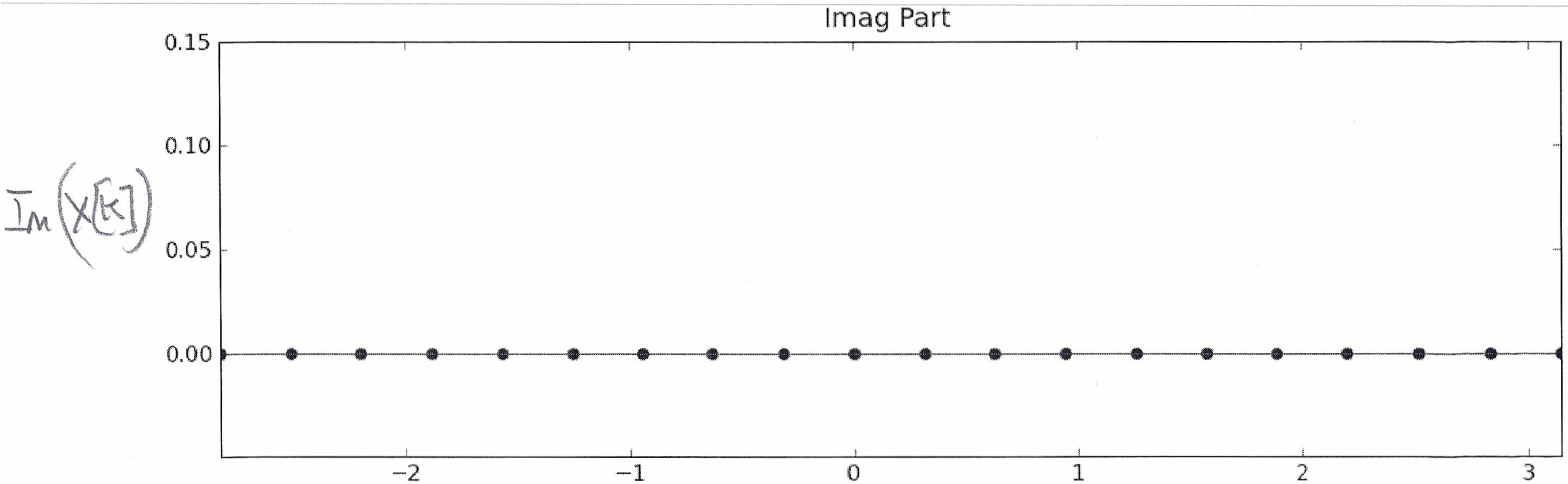
$$= \frac{1}{(20)^2}\left(3 + 4\cos\left(\frac{2\pi}{20}k\right) + 2\cos\left(\frac{4\pi}{20}k\right)\right)$$

Square pulse case



Square Pulse DTFS Coefficients





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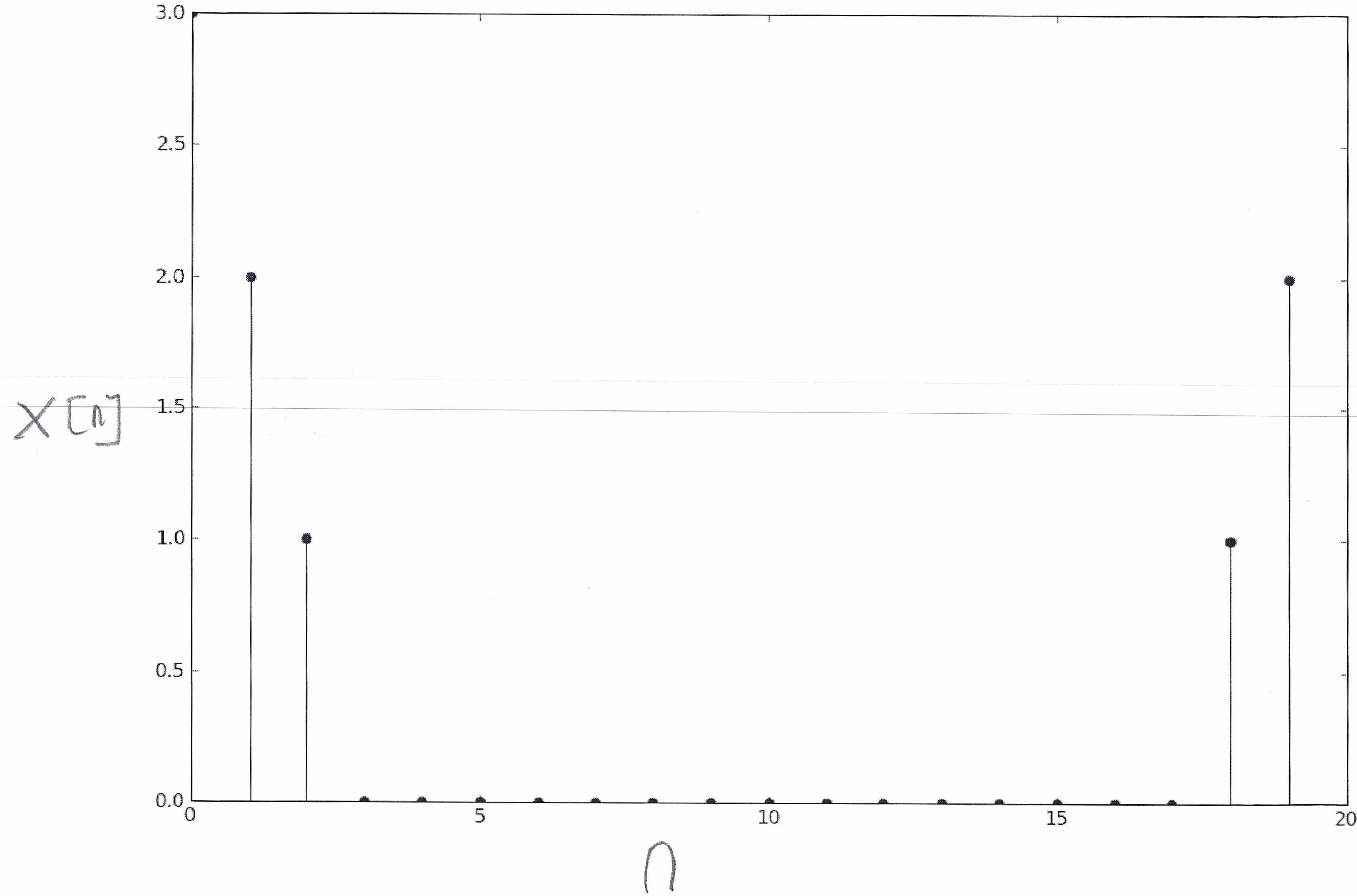
Triangular pulse Case



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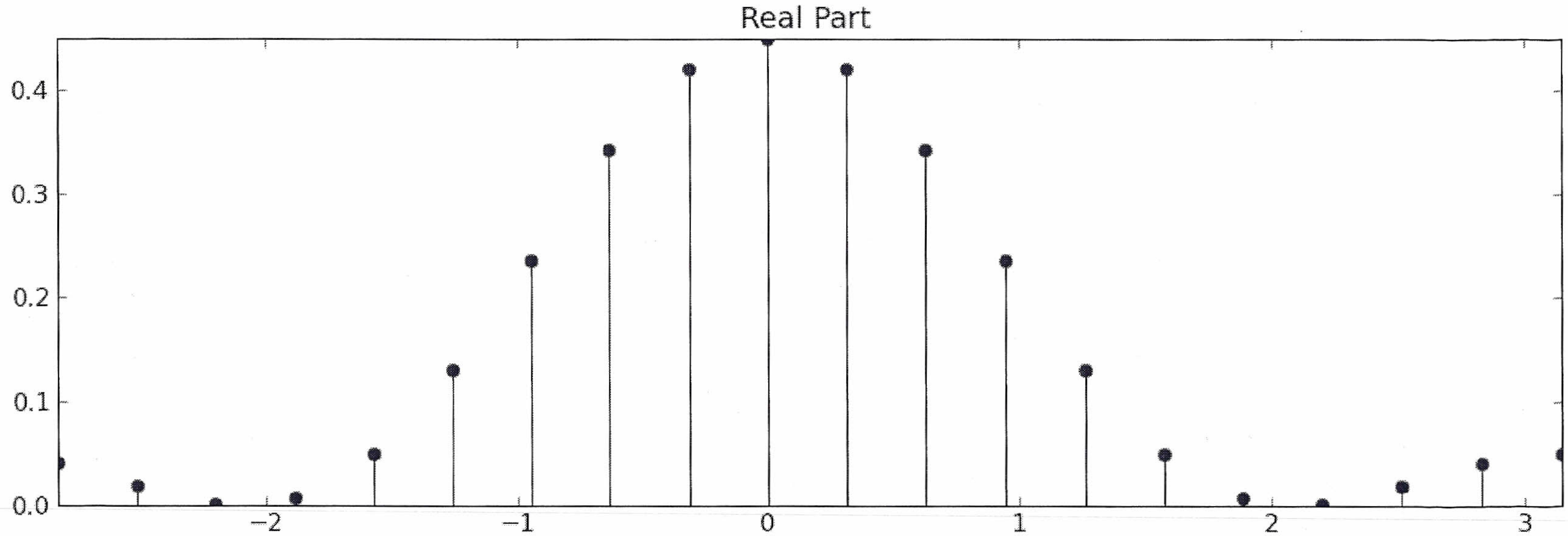
Triangular Pulse DTFS coefficients

Re(X[k])



Im(X[k])