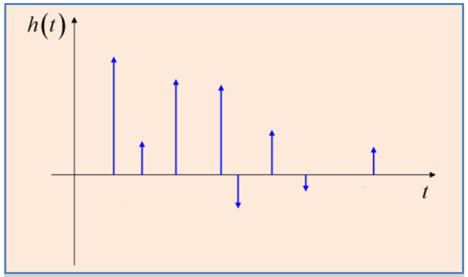
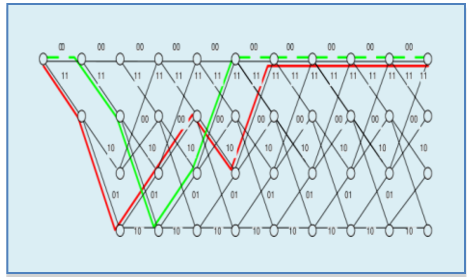
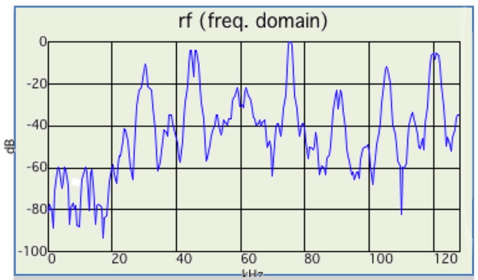
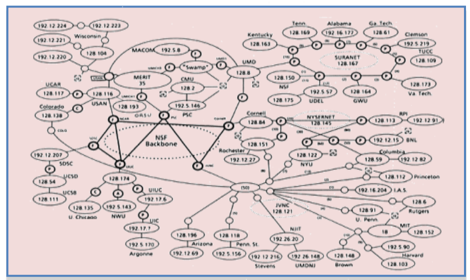




INTRODUCTION TO EECS II

DIGITAL COMMUNICATION SYSTEMS

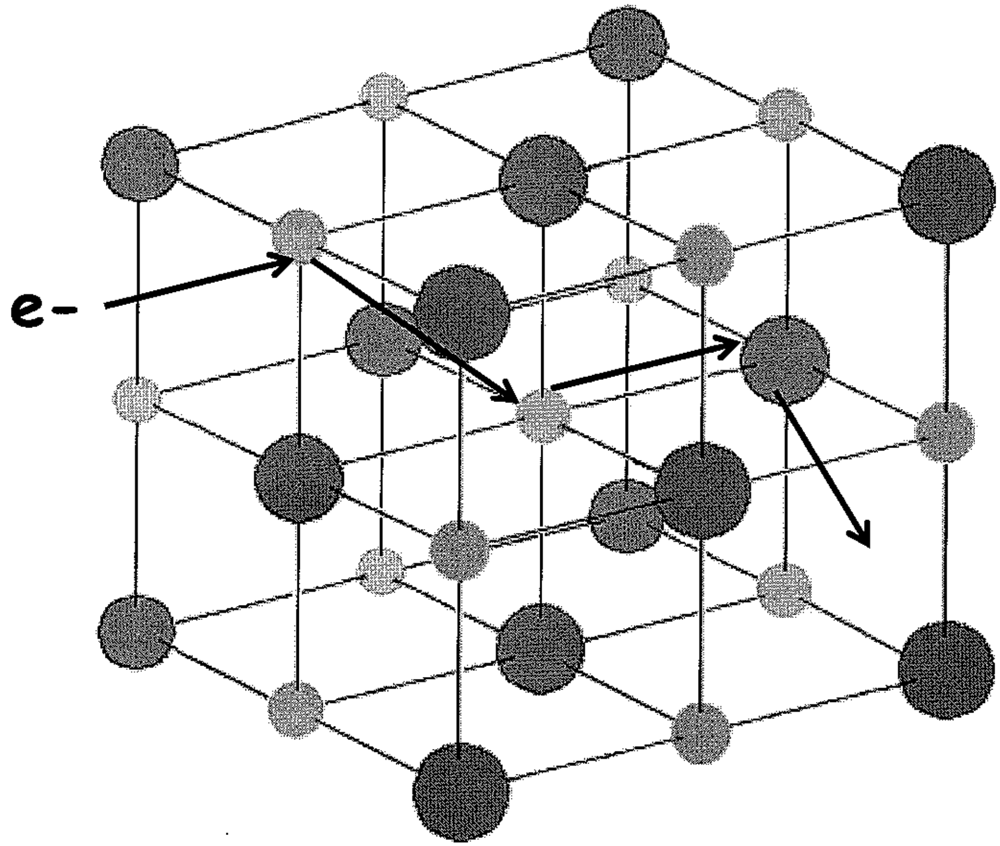
6.02 Fall 2010
Lecture #4

- Signals and Noise
    - Noise sources and bit errors
- Analyzing Noise:
    - PDF, CDF, and Noise + Noisefree decomposition
- Computing Bit Error Rates:
    - The No ISI case, the impact of the Noise PDF.

2

Noise Can Be Due to Fundamental Processes

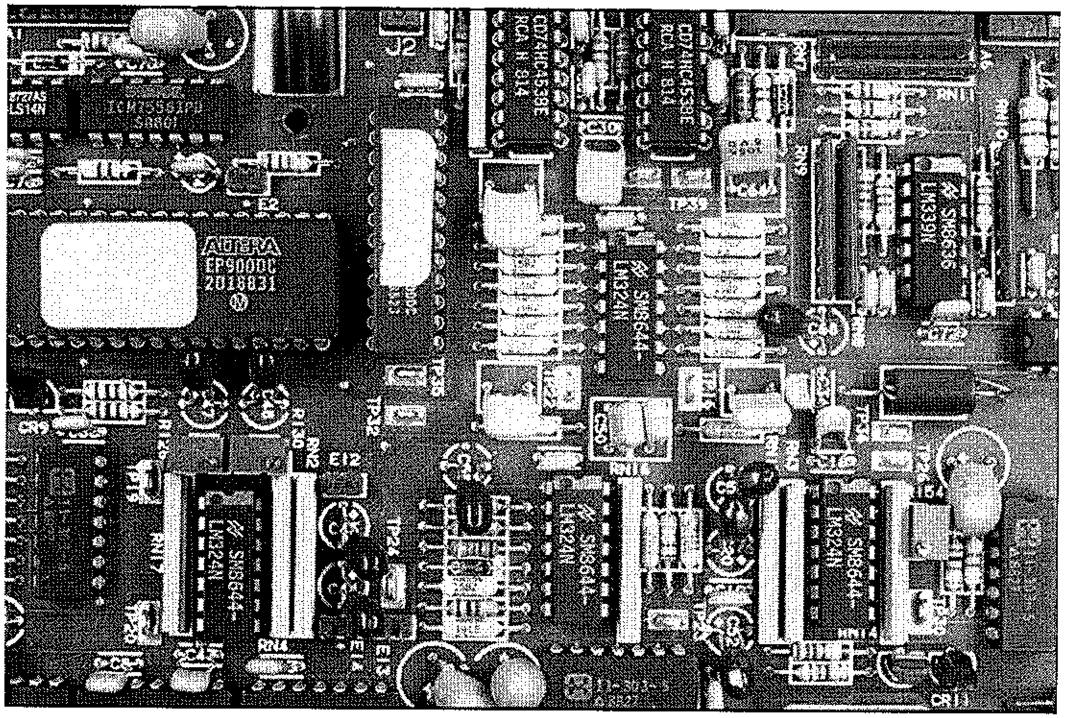
Electron Moving Through Crystal with Vibrating Atoms



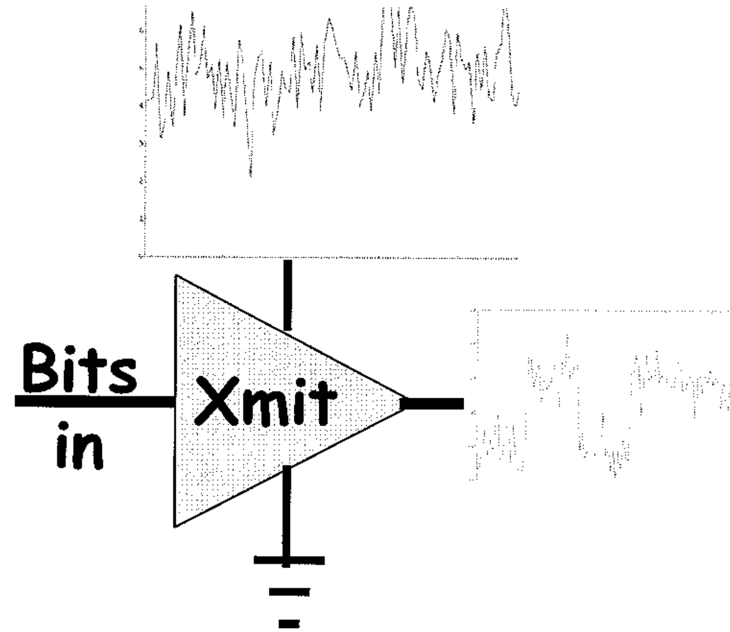
http://www4.nau.edu/meteorite/Meteorite/Images/Sodium_chloride_crystal.png

Randomized path leads to noisy current flow

Effect of Many Interactions Can Be Modeled as Noise

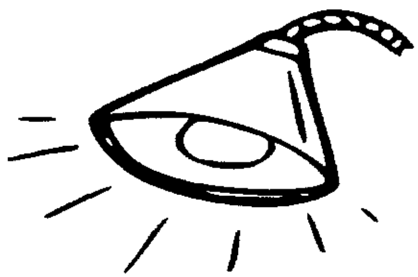


http://www.imageteck.net/PCB%20Board%203.JPG

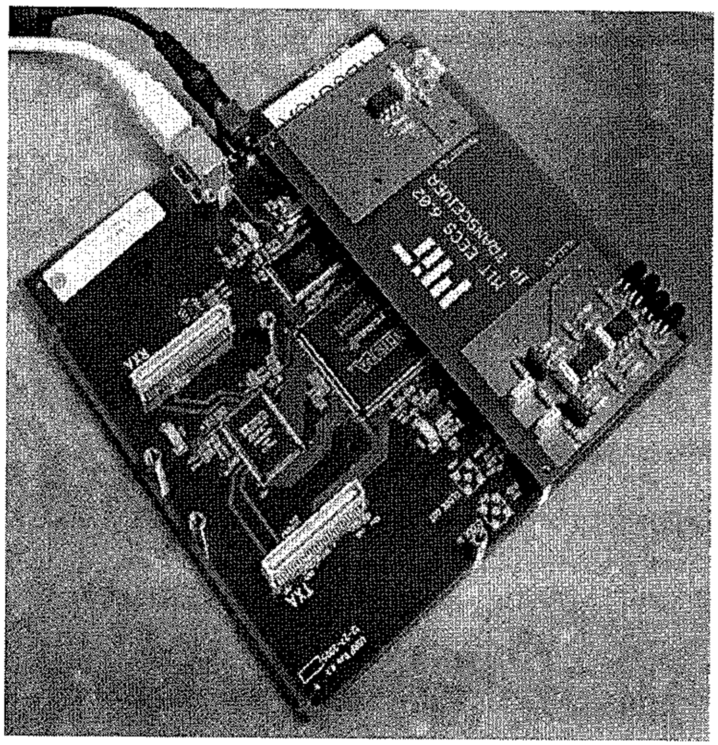


Many components connected by thin wires (that have inductance and resistance) to single power supply – 1000's of devices switching on and off creates "noisy" power supply.

6.02 IR Transceiver Noise Source - Room Light

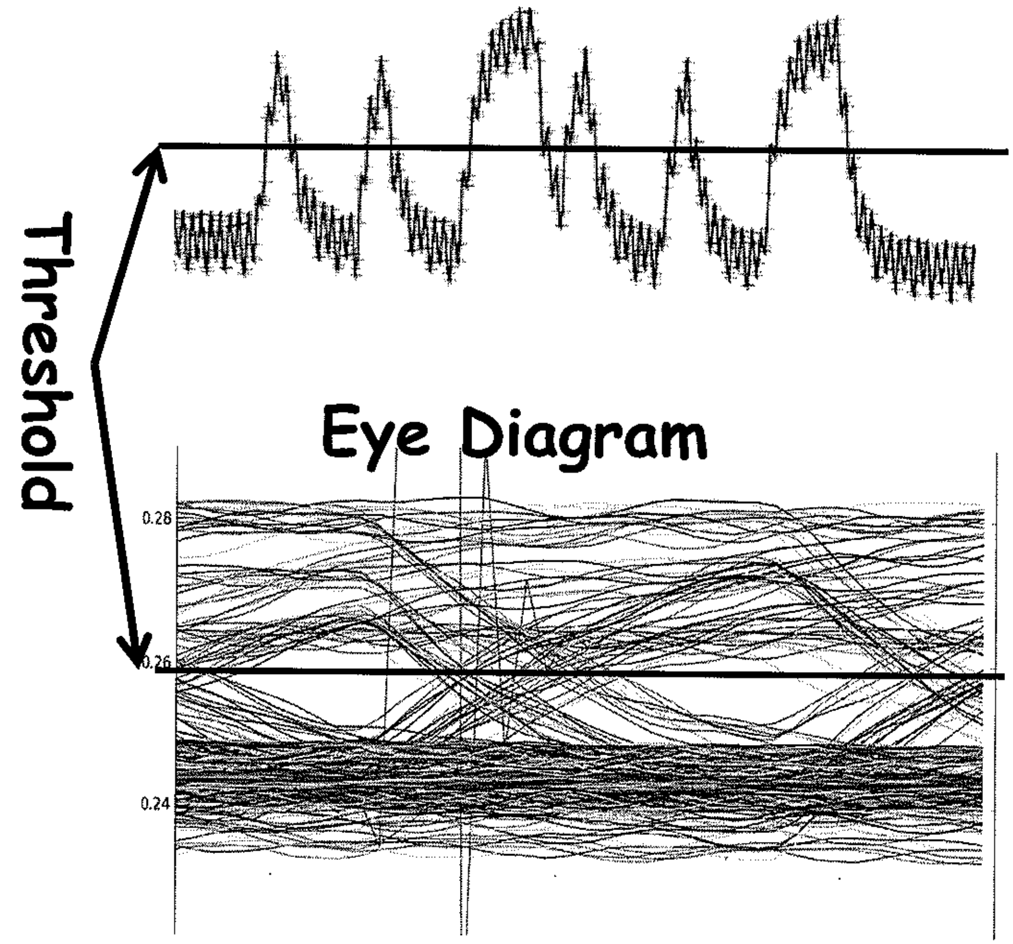


http://www.clker.com/cliparts/0/6/b/6/11954240371585298002ryanlerch_lamp_outline.svg.med.png

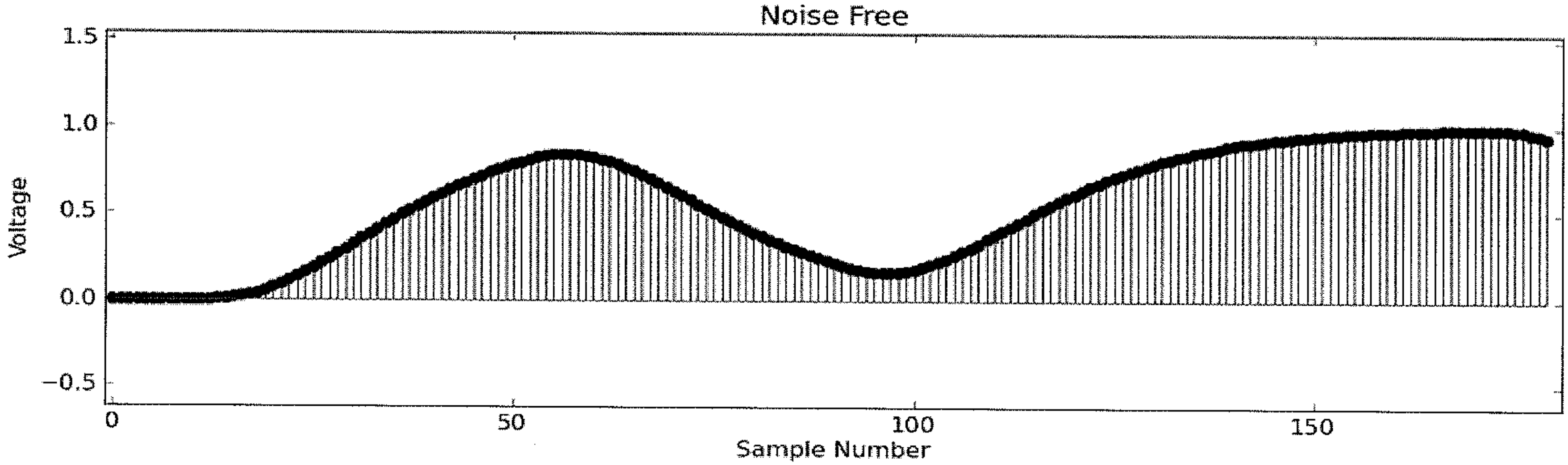


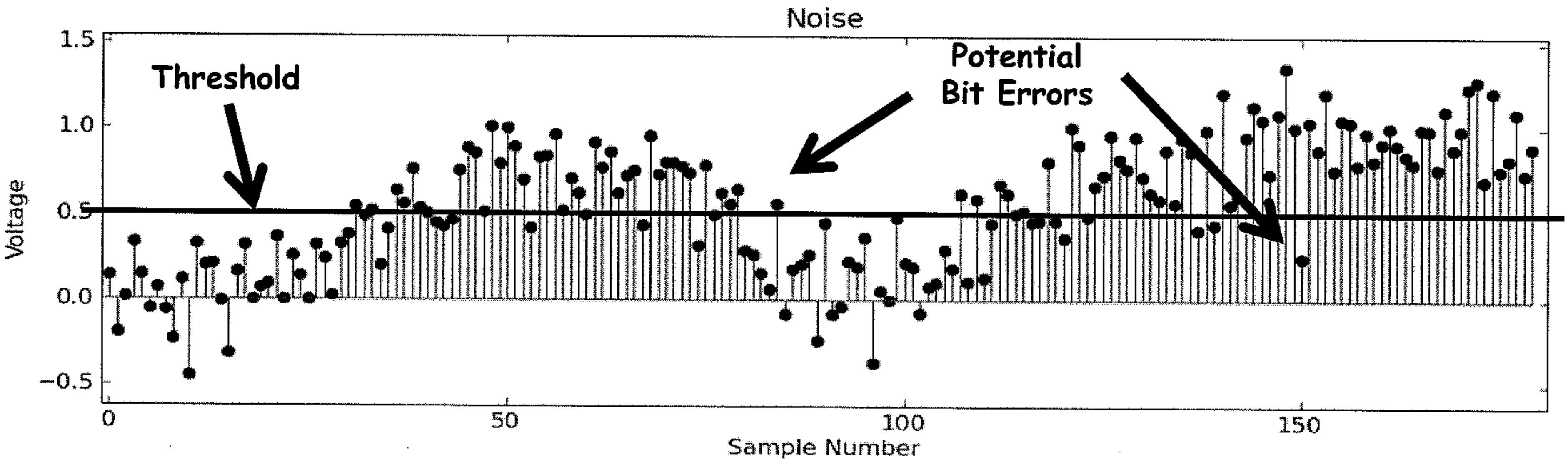
20 Samples/bit

Receiver Waveform



Noisy Signal Can Cause Bit Errors





Key Noise Questions For A Channel

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- For a given Transmission Scheme:
  - What's the Bit Error Rate (BER)?
    - BER: Fraction of erroneously received bits
- If the Signal is increased:
  - How much is BER reduced?
- If ISI is reduced:
  - Is BER reduced significantly?

To Answer these questions

- Need to Characterize the Noise
  - What "shape" (probability density function)?
  - How "big" (variance)?

Bit Error Rates

Bad Name - Really the probability that a given bit will be in error

Examples:

Really Good Channel: BER = 1 error in 10^{12} bits
O'kay Channel: BER = 1 error in 10^{4} bits
Lousy Channel: BER = 1 error in 10^{2} bits

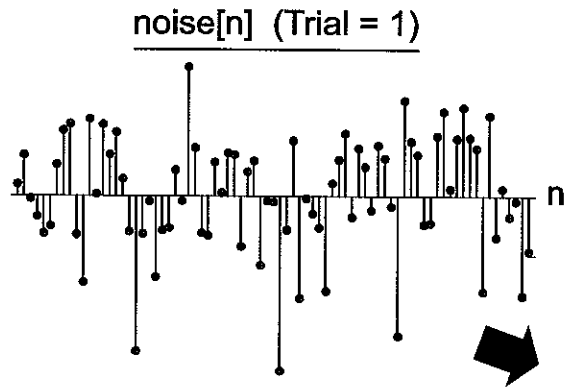
For this course: Model only additive Noise

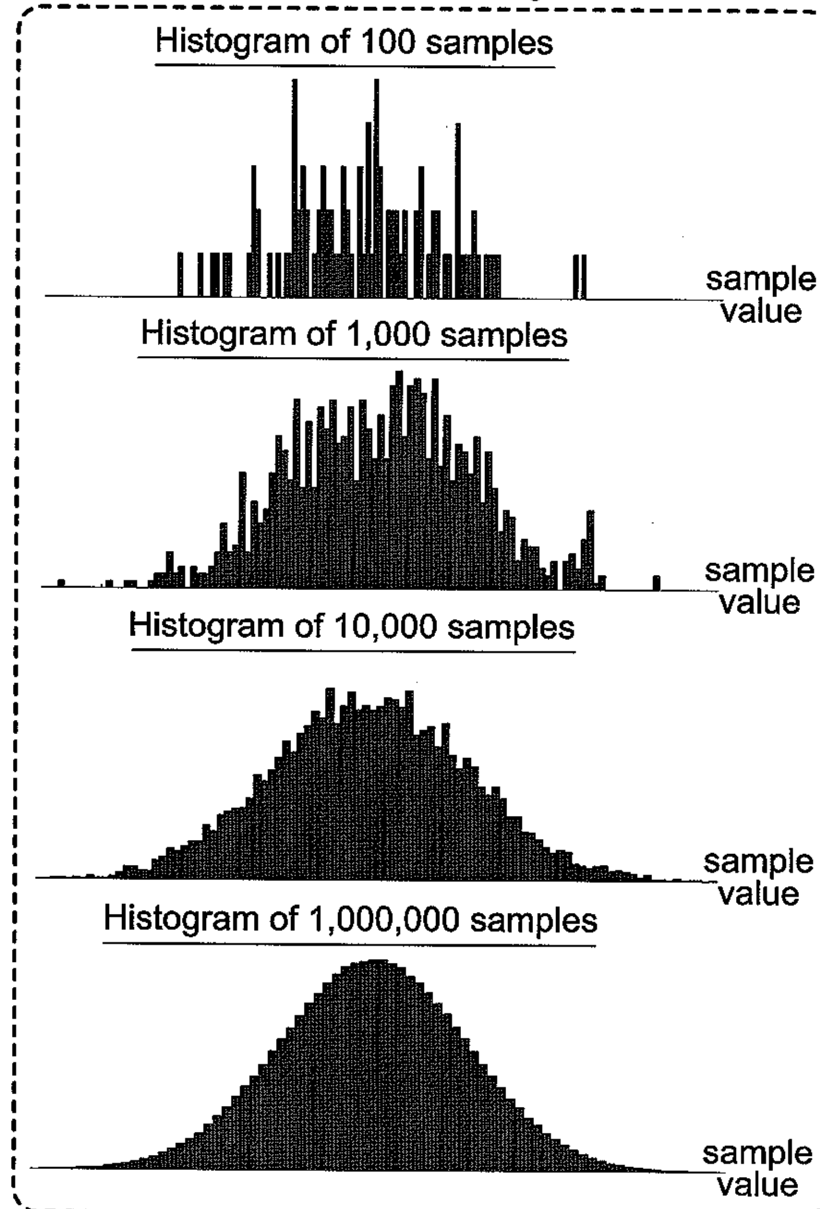
- Many other types of noise (e.g. phase noise where transmitter bit period varies)
- Additive Noise is easy to analyze

Additive noise

signal[n] = noise_free_signal[n] + noise[n]

Experiment to see Noise "Shape"





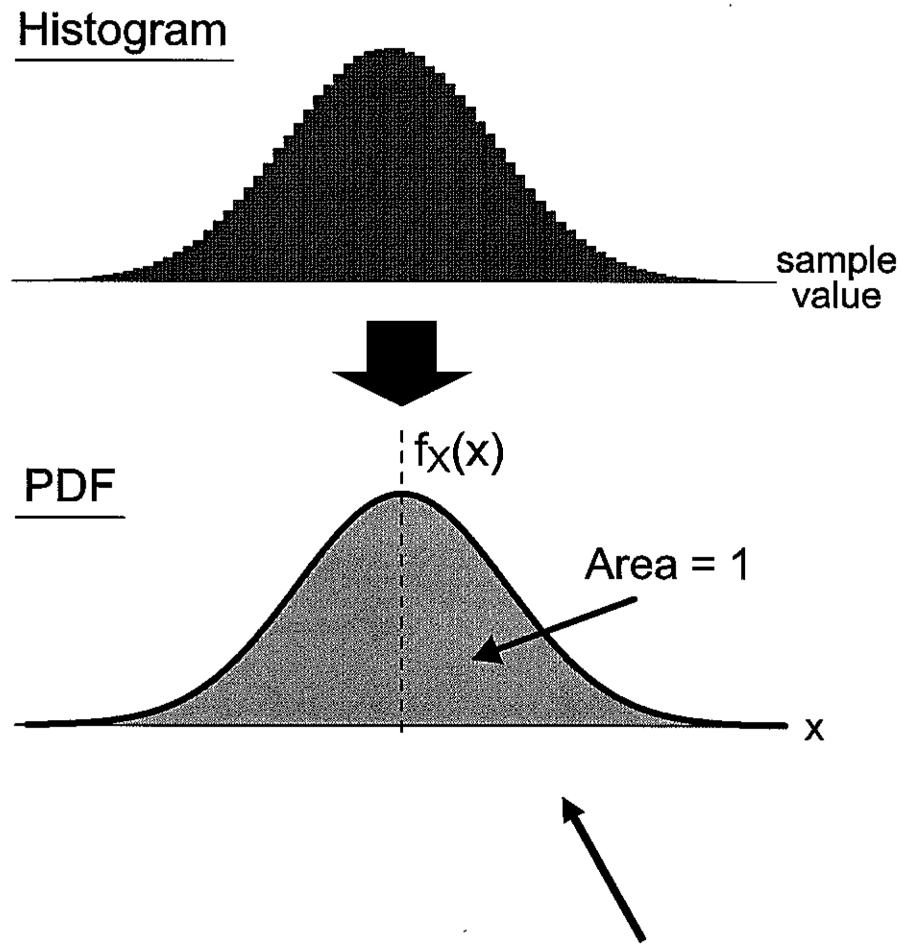
- Create histograms of sample values from trials of increasing lengths
- Assumption of independence and stationarity implies histogram should converge to a shape known as a probability density function (PDF)

Slide thanks to C. Sodini and M. Perrott

"Shape" = Probability Density Function PDF

- Define X as a random variable whose PDF has the same shape as the histogram we just obtained
- Denote PDF of X as $f_X(x)$
  - Scale $f_X(x)$ such that its overall area is 1

$$\Rightarrow \int_{-\infty}^{\infty} f_X(x) = 1$$



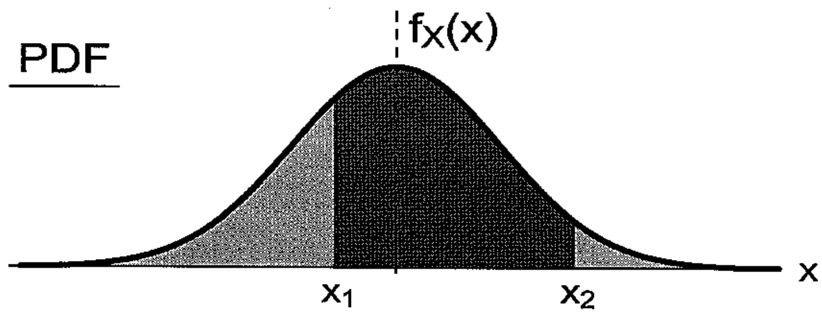
This shape is referred to as a Gaussian PDF

Slide thanks to C. Sodini and M. Perrott

PDF → Probability and PDF → CDF

- The *probability* that random variable X takes on a value in the range of x_1 to x_2 is calculated from the PDF of X as:

$$\mathrm{Prob}(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} f_X(x)dx$$



- The *probability* that random variable X takes on a value less than x_1 is the cumulative distribution function CDF(x_1)

$$\mathrm{CDF}(x_1) = \mathrm{Prob}(x \leq x_1) = \int_{-\infty}^{x_1} f_X(x)dx$$

Slide thanks to C. Sodini and M. Perrott

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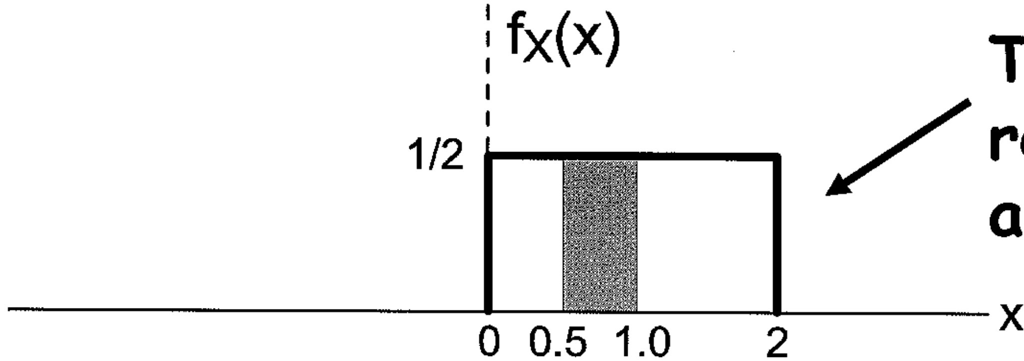
Example Probability Calculation



This shape is referred to as a uniform PDF

- **Verify that overall area is 1:**

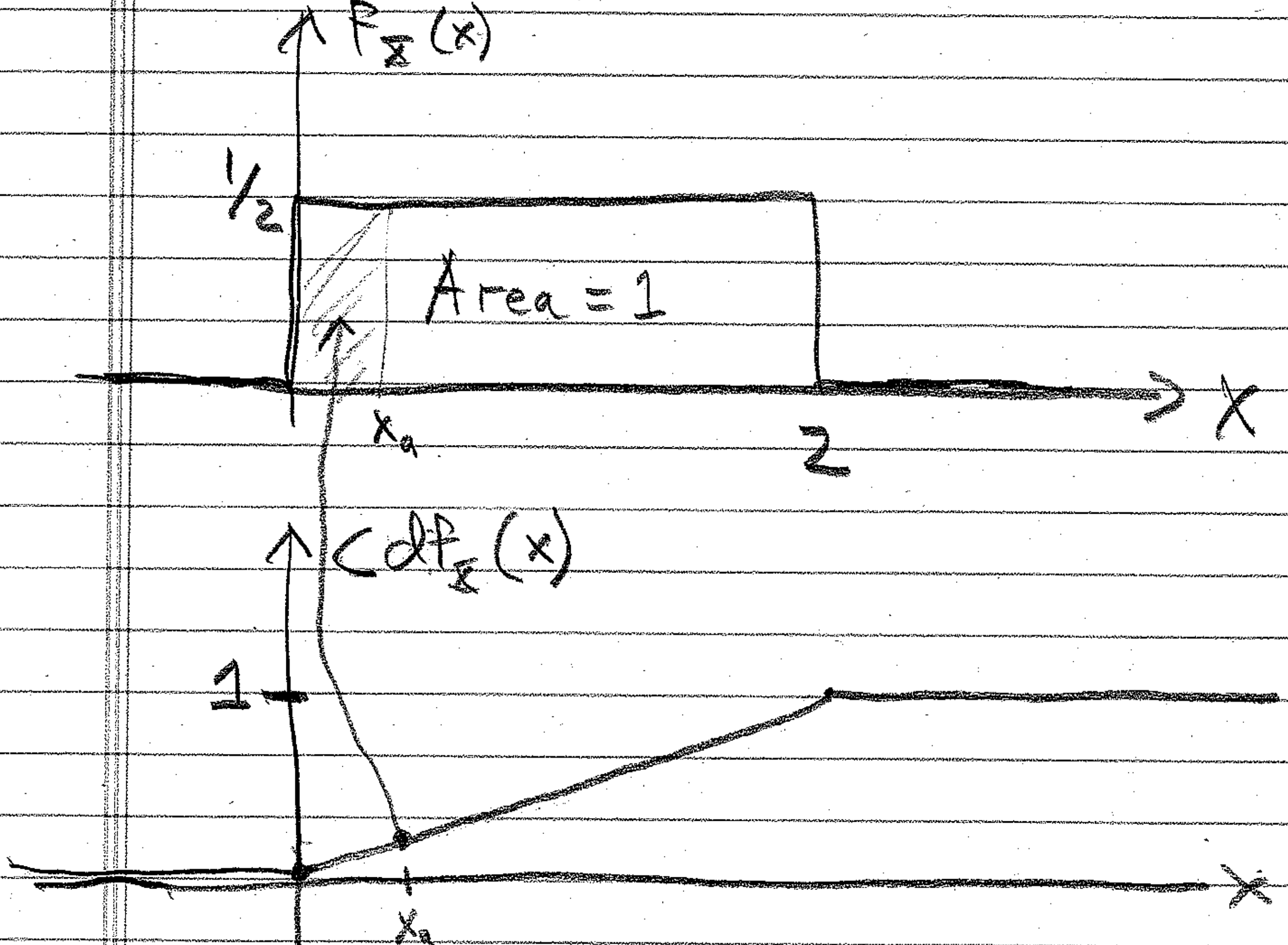
$$\int_{-\infty}^{\infty} f_X(x)dx = \int_0^2 0.5\, dx = \boxed{1}$$

- **Probability that x takes on a value between 0.5 and 1.0:**

$$\text{Prob}(0.5 \leq x \leq 1.0) = \int_{0.5}^{1.0} 0.5\, dx = \boxed{0.25}$$

Slide thanks to C. Sodini and M. Perrott

C.D.F. of a Uniform PDF



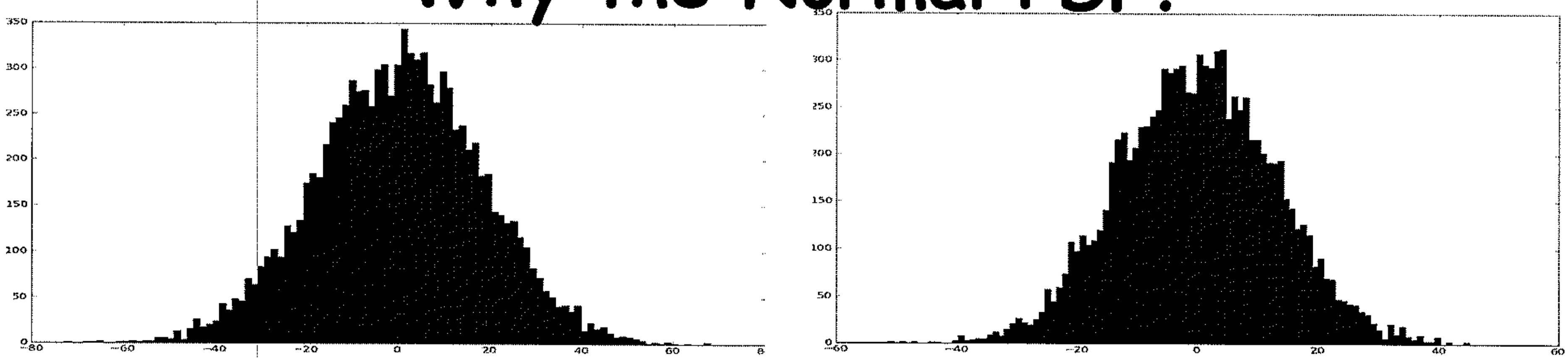
$$cdf(x_a) = \int_{-\infty}^{x_a} f_X(x)\,dx$$

Noise Modeled Using Normal (Gaussian) PDF

$$f_X(x) = \frac{1}{(\sqrt{2\pi})\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

$$\sigma = \text{standard deviation}$$

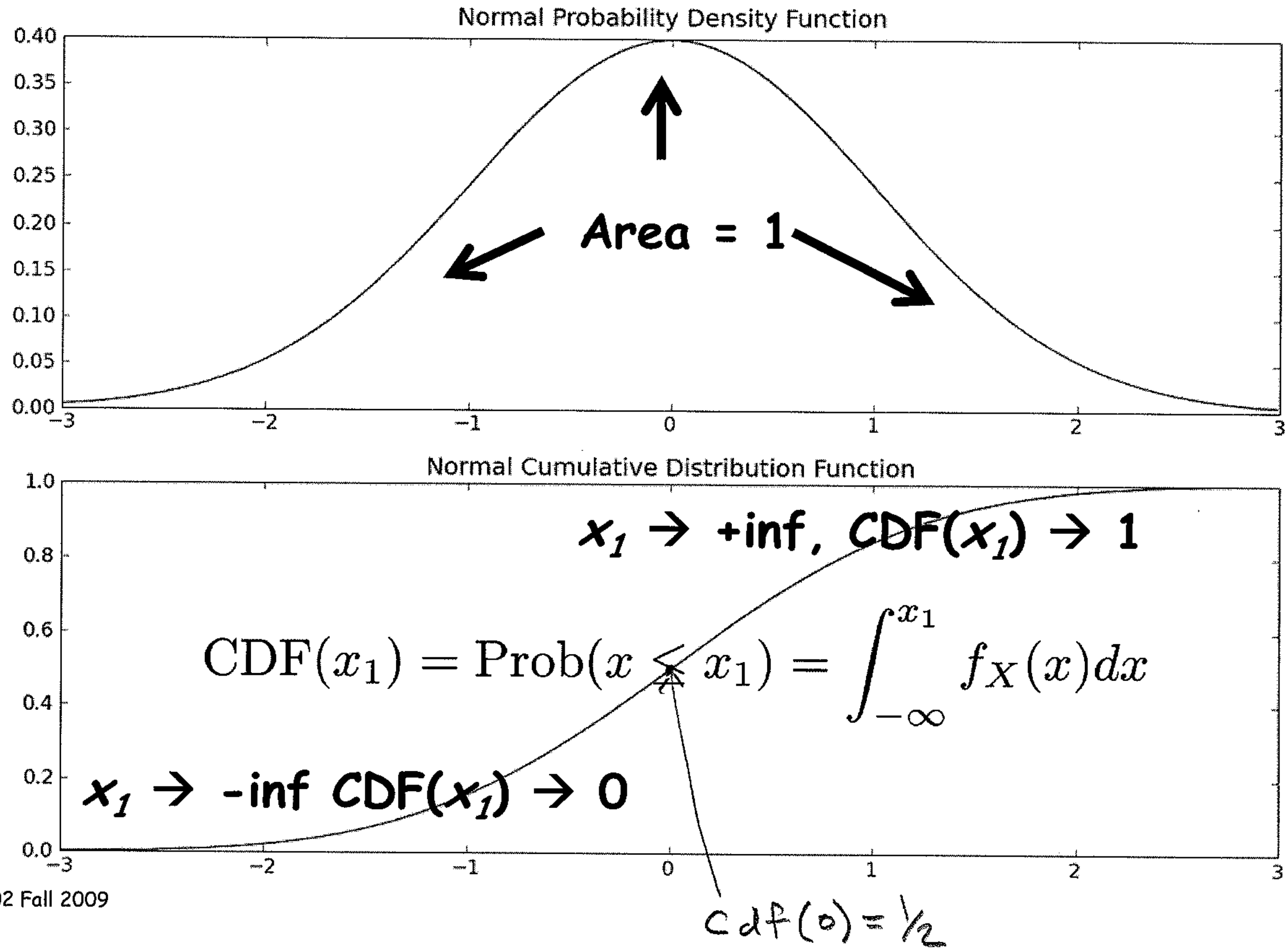
Why the Normal PDF?



Histogram for 10,000 trials of sums of 1000 uniformly (right) or triangularly (left) distributed [-1,1] random variables

Key Point: Sums of Noise from many sources well approximated by the Normal(Gaussian) PDF

Unit Normal (Gaussian) PDF and CDF



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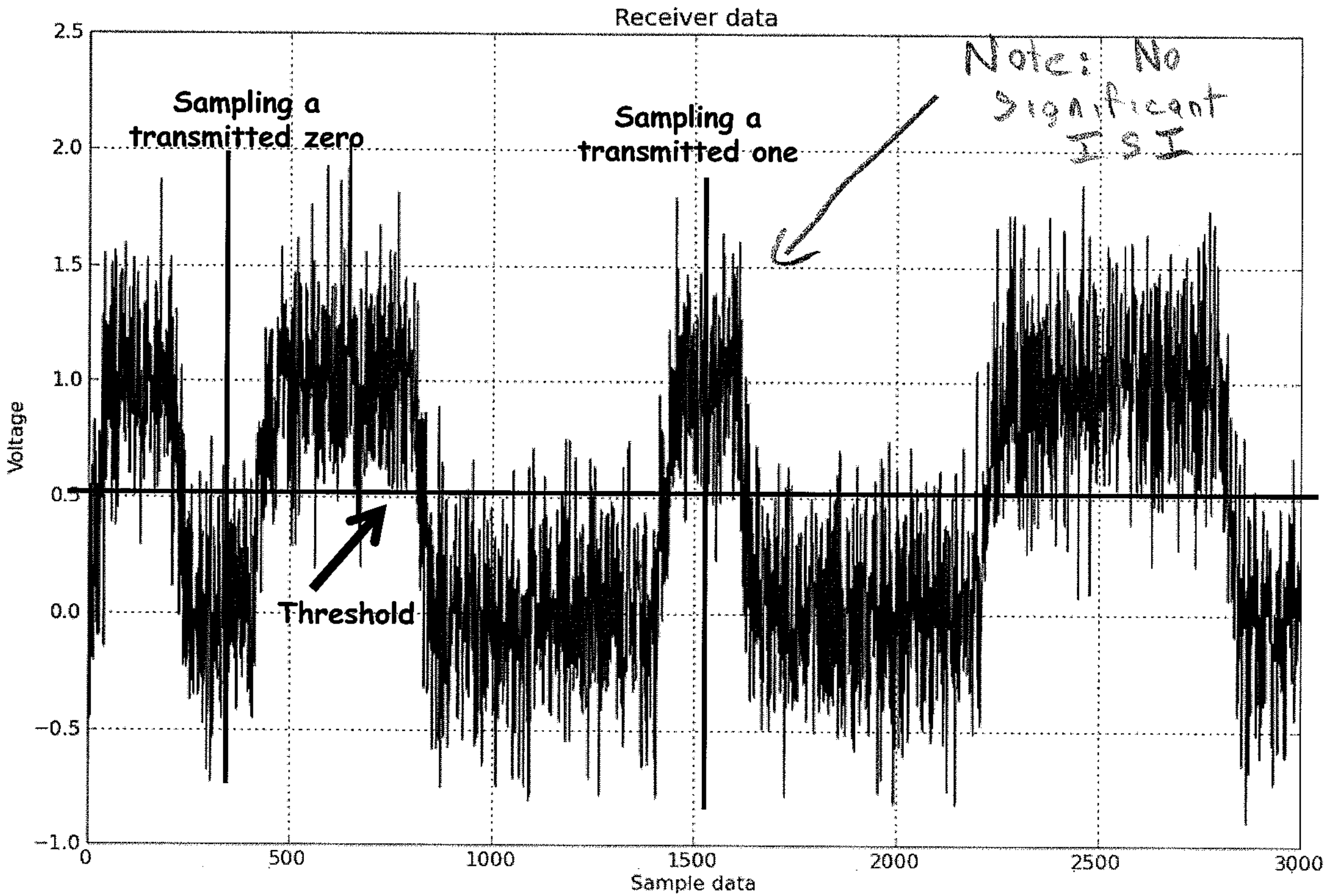
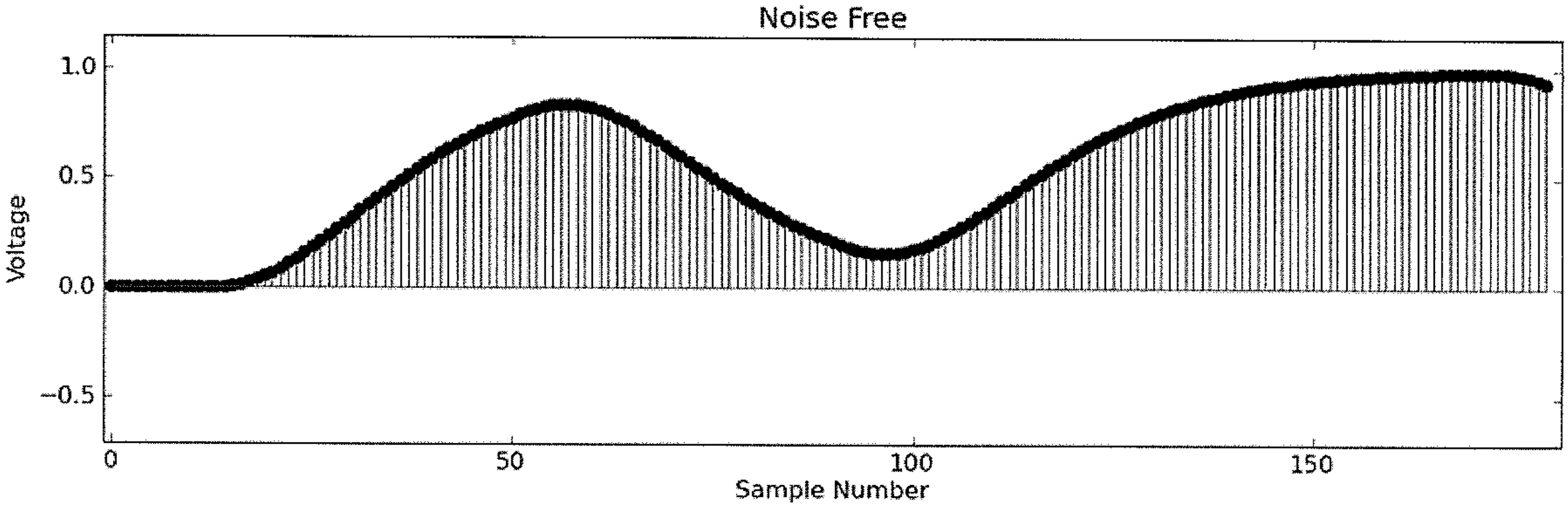
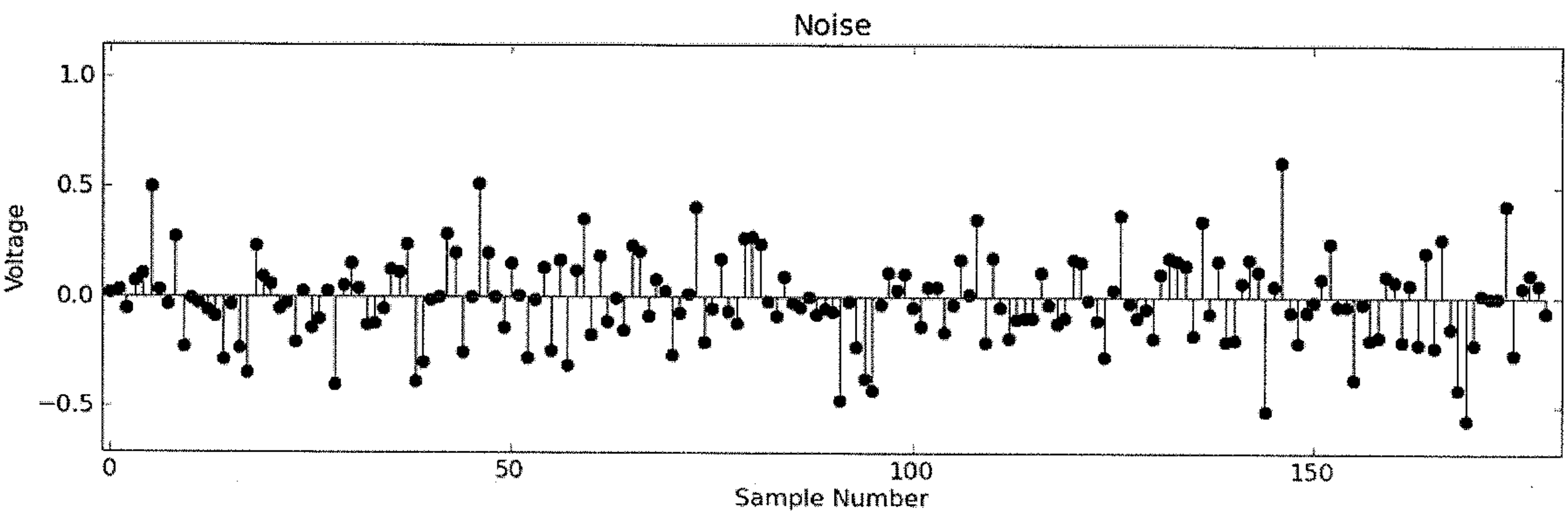
Estimating Probability of Bit Error



Decompose Into Noisefree + Noise

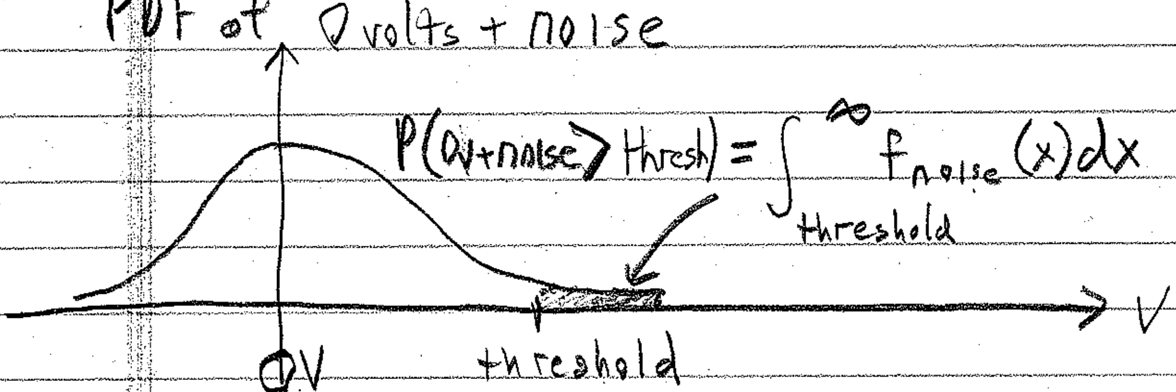




Suppose Transmitted bit = 0

Received Voltage = 0 volts + noise

has a P.D.F. $f_{noise}(x)$



$$\int_{threshold}^{\infty} f_{noise}(x)\,dx = 1 - \int_{-\infty}^{threshold} f_{noise}(x)\,dx$$

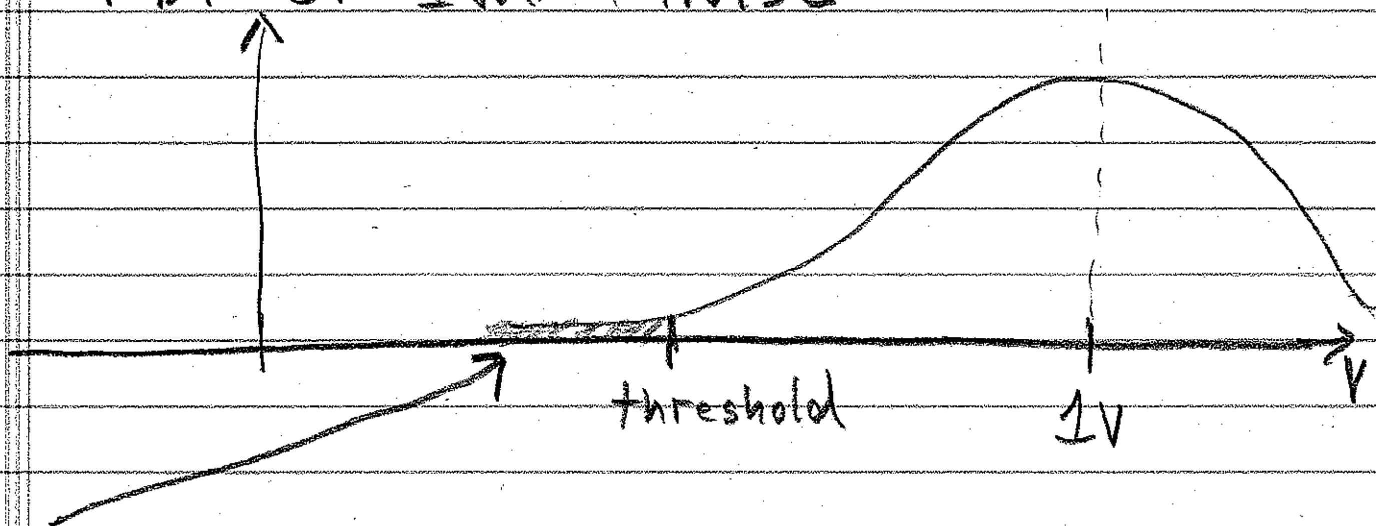
$$= 1 - CDF_{noise}(threshold)$$

$$\equiv \underbrace{P_{01}}_{\text{Probability transmitted zero, received a '1'.}}$$

Suppose Transmitted bit = 1

Received voltage = 1 volt + noise

PDF of 1 volt + noise



P_{10} = $P(1V + \text{noise} < \text{threshold})$ =

Probability transmitted a one, received as '0'.

$$\int_{-\infty}^{\text{threshold}-1} f_{\text{noise}}(x)\,dx$$

$$= \text{CDF}(\text{threshold} - 1)$$

Probability of Error: Sum Two Cases

Typically = 1/2

$\overbrace{\text{Prob(Transmitted a Zero)}}^{P_0} \cdot P_{01}$ +

$\underbrace{\text{Prob(Transmitted a one)}}_{P_1} \cdot P_{10}$

= Prob (Bit Error)

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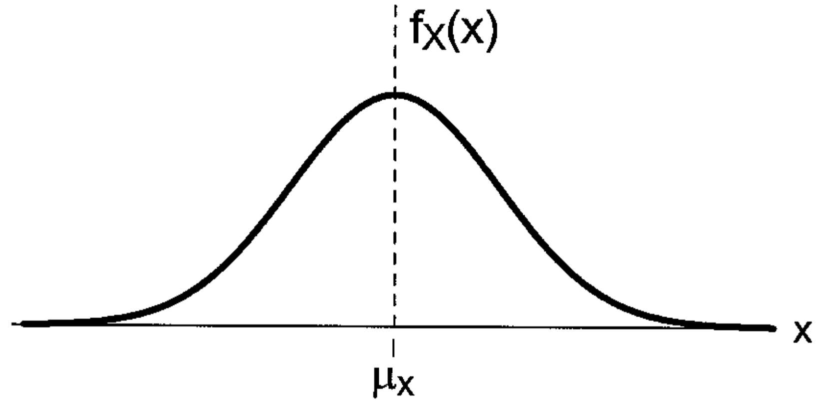
Mean and Variance



- The mean of random variable *X*, μ_x , corresponds to its average value
  - Computed as

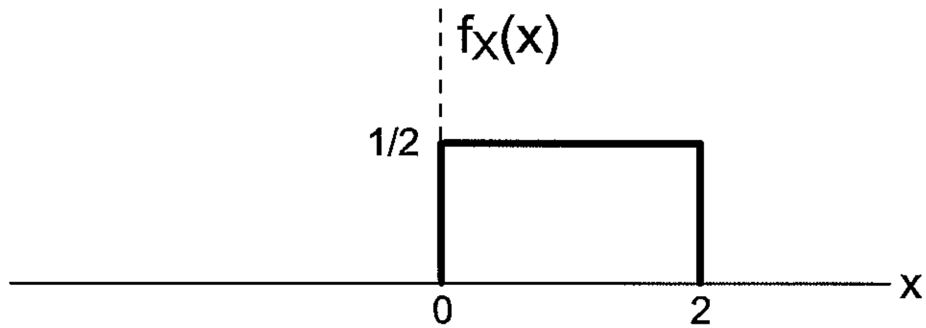
$$\mu_X = \int_{-\infty}^{\infty} x f_X(x) dx$$

- The variance of random variable *x*, σ_x^2, gives an indication of its variability
  - Computed as

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x - \mu_x)^2 f_X(x) dx$$

- The standard deviation of a random variable X, is denoted σ_X

Example Mean and Variance Calculation



- **Mean:**

$$\mu_X = \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^2 x\frac{1}{2}dx = \frac{1}{4}x^2\Big|_0^2 = \boxed{1}$$

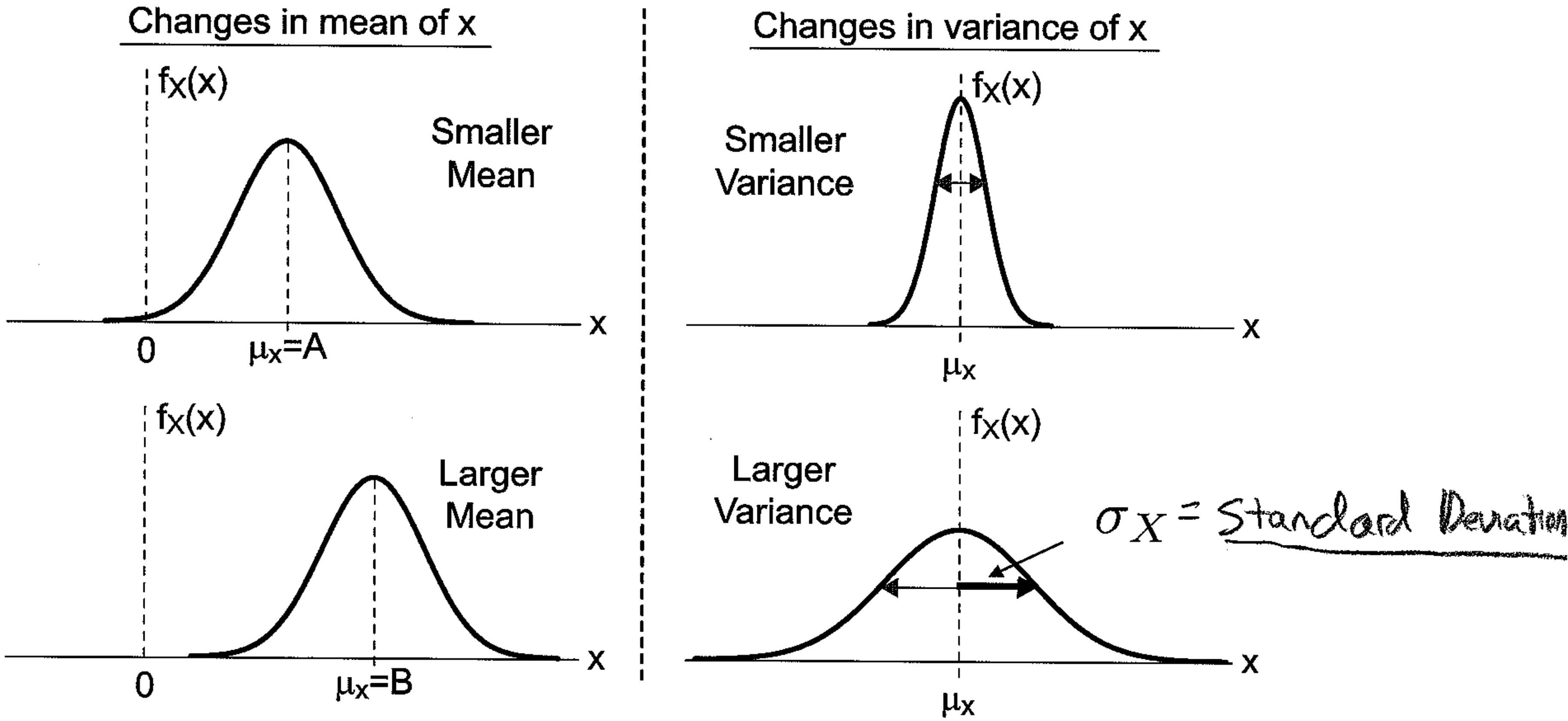
- **Variance:**

$$\sigma_X^2 = \int_{-\infty}^{\infty}(x-\mu_x)^2 f_X(x)dx = \int_0^2 (x-1)^2\frac{1}{2}dx$$

$$= \frac{1}{6}(x-1)^3\Big|_0^2 = \frac{1}{6}+\frac{1}{6} = \boxed{\frac{1}{3}}$$

Slide thanks to C. Sodini and M. Perrott

Visualizing Mean and Variance from PDF



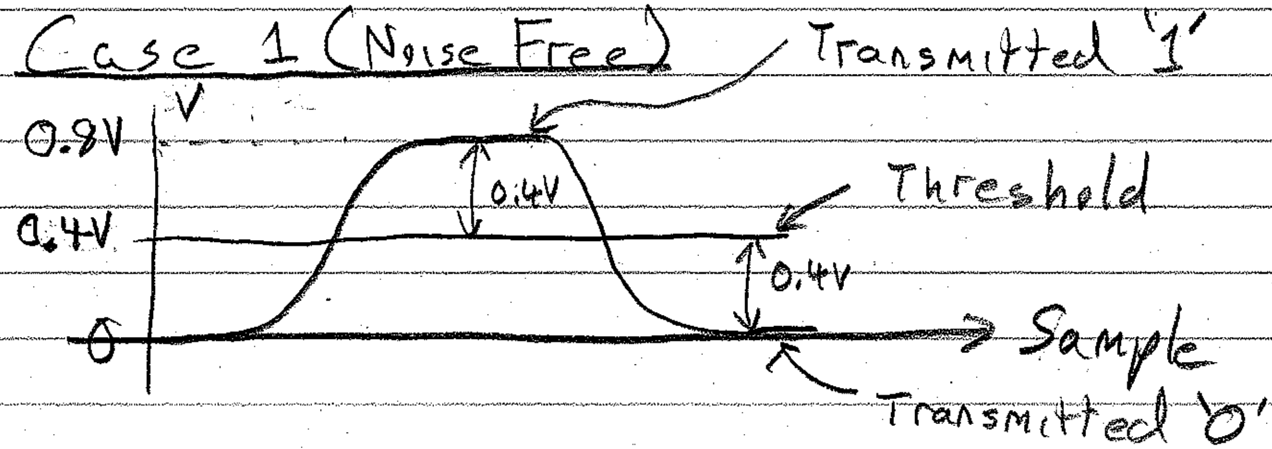
- Changes in mean shift the *center of mass* of PDF
- Changes in variance narrow or broaden the PDF
  - Variance tells us how "big" the noise is!

Slide thanks to C. Sodini and M. Perrott

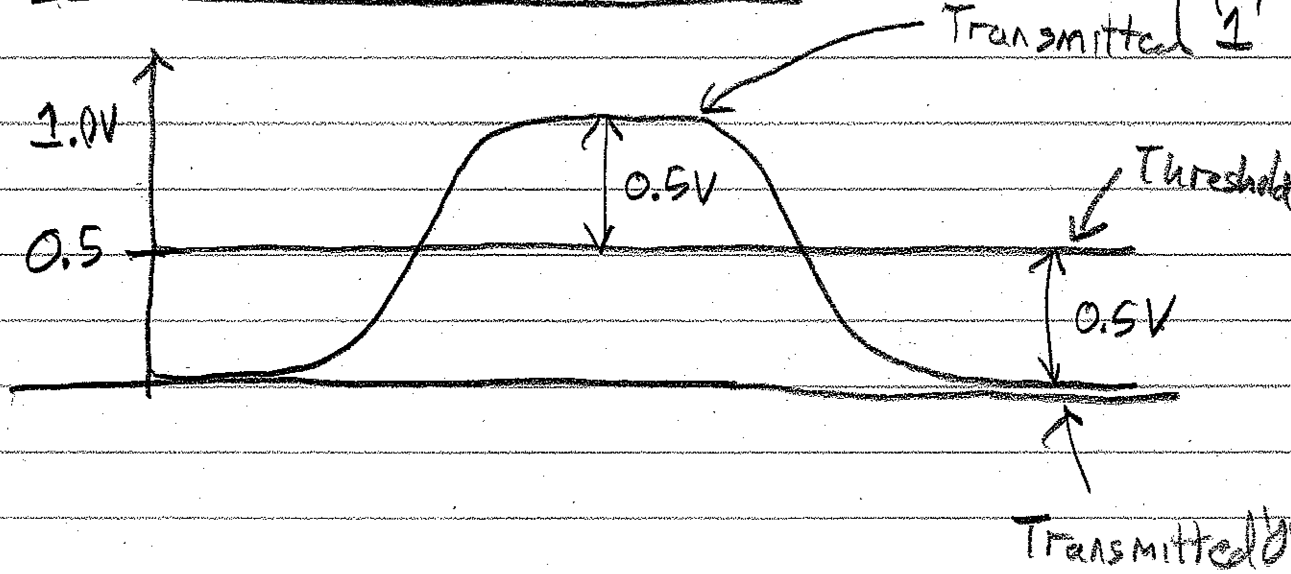
Why Does "Shape" Matter?

Question: How much better is 2 than 1?

Case 1 (Noise Free)

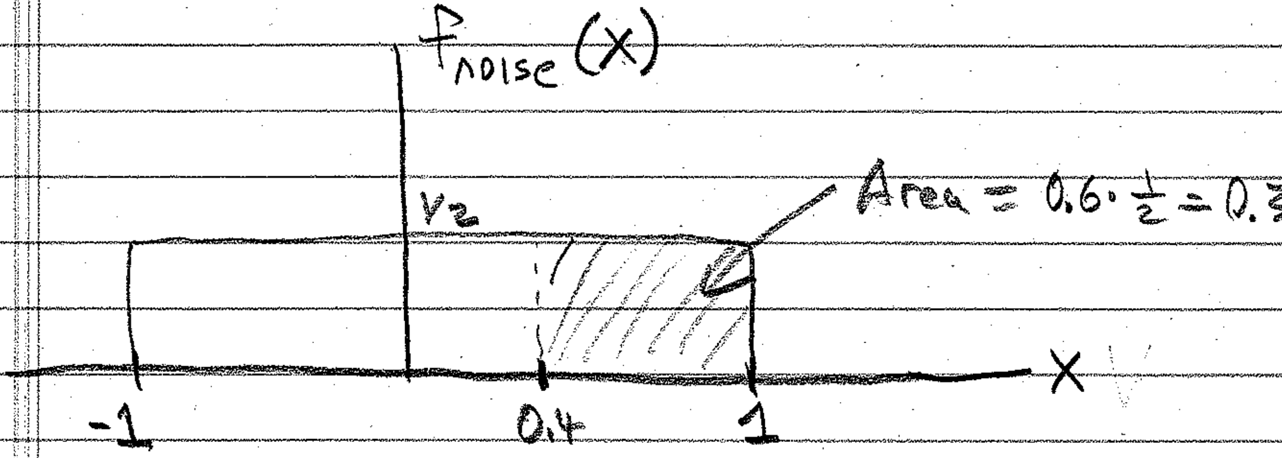


Case 2 (Noise Free)



Uniform Noise

PDF of Noise



Case 1 Bit Error

$$\underbrace{\frac{1}{2}}_{P_0} \cdot P(\text{noise} > 0.4) + \frac{1}{2} \cdot P(\text{noise} < -0.4)$$

$$= \frac{1}{2} \cdot 0.3 + \frac{1}{2} \cdot 0.3 = 0.3$$

Case 2 Bit Error

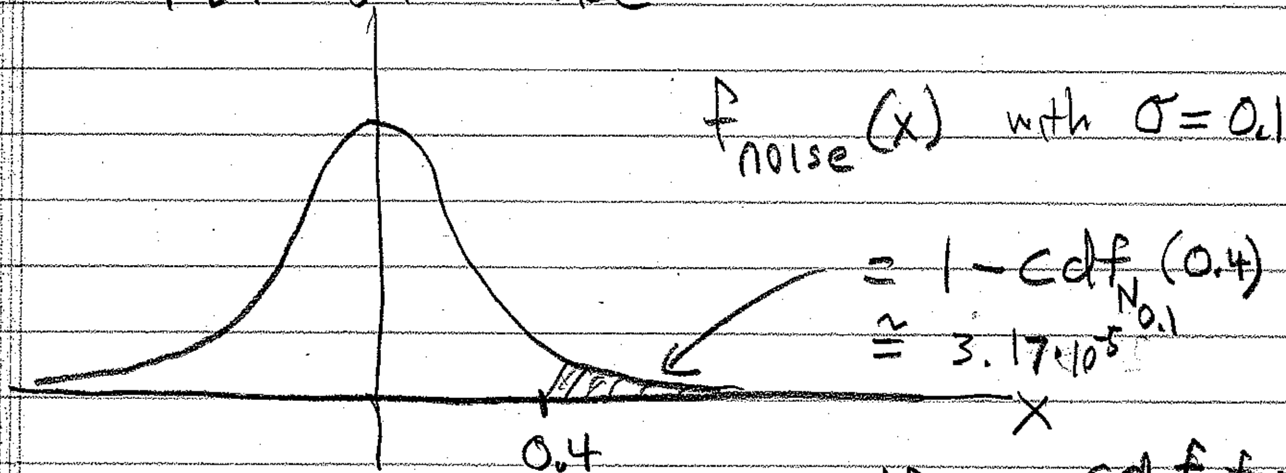
Marginally Better

$$\frac{1}{2} \cdot P(\text{noise} > 0.5) + \frac{1}{2} P(\text{noise} < -0.5)$$

$$= \frac{1}{2} \cdot 0.25 + \frac{1}{2} \cdot 0.25 = 0.25$$

Normal (Gaussian) Noise

PDF of Noise



$cdf_{N_{0.1}} \leftarrow$ cdf for Normal with $\sigma = 0.1$

Case 1 bit Error

$$P_0 \cdot P(noise > 0.4) + P_1 \cdot P(noise < -0.4)$$

$$= \frac{1}{2}\; 1 - cdf_{N_{0.1}}(0.4) + \frac{1}{2} \cdot cdf_{N_{0.1}}(-0.4)$$

$$\cong \frac{1}{2} \cdot 3.17 \cdot 10^{-5} + \frac{1}{2} \cdot 3.17 \cdot 10^{-5} = 3.17 \cdot 10^{-5}$$

100x lower bit error rate!

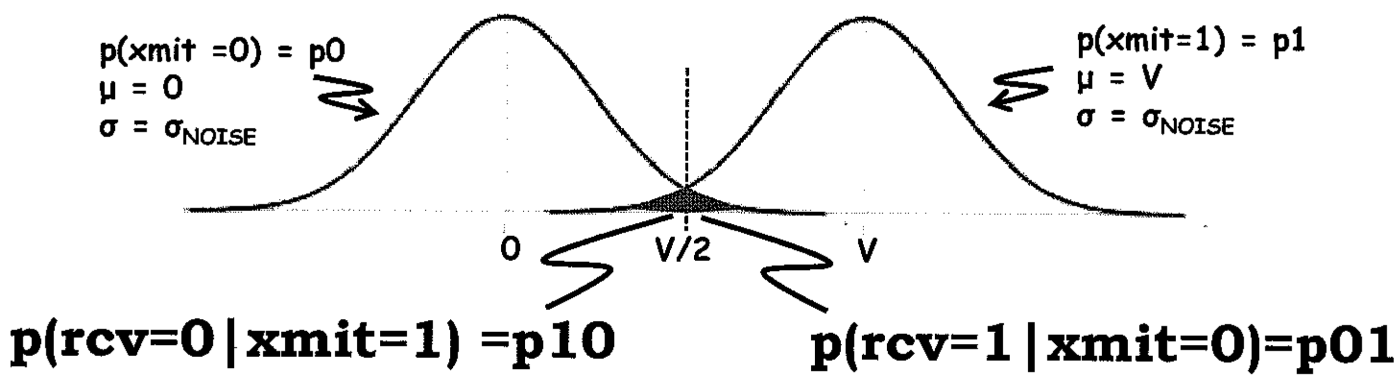
Case 2 bit Error

$$P_0 \cdot P(noise > 0.5) + P_1 \cdot P(noise < -0.5)$$

$$= \frac{1}{2} \cdot 2.87 \cdot 10^{-7} + \frac{1}{2}\; 2.87 \cdot 10^{-7} = 2.87 \cdot 10^{-7}$$

Summary

- **Assume Gaussian PDF for noise and no inter-symbol interference (ISI next time) yields the following picture:**

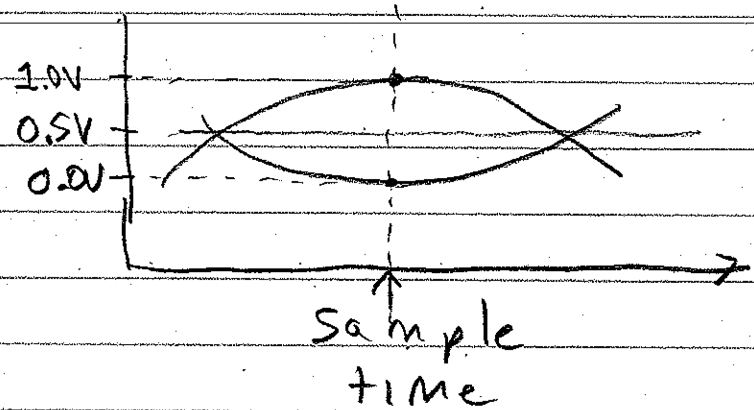


- **We can estimate the bit-error rate (BER) as**

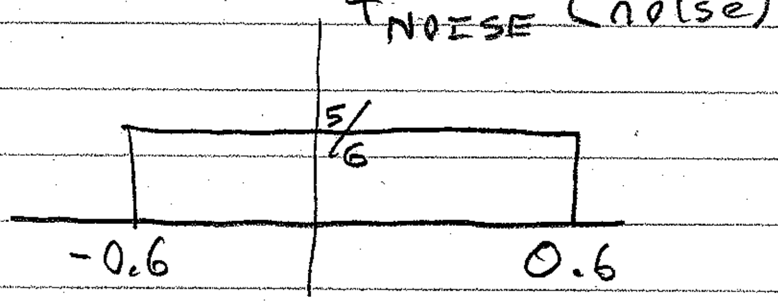
$$\boxed{p(biterror) = p0 * p01 + p1 * p10}$$

Examples

Suppose a noise-free eye diagram is:



and the noise is given by:



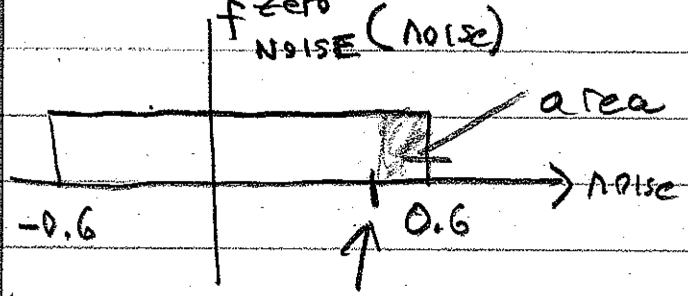
Note: $\int_{-0.6}^{0.6} f_{NOISE}(x)\,dx = 1.2 \cdot \frac{5}{6} = 1$

$\mu_{noise} = 0$

$\sigma^2_{noise} = \frac{5}{6}\left(\frac{1}{3}x^3 \Big|_{-0.6}^{0.6}\right) = \frac{3}{25}$

Probability of error

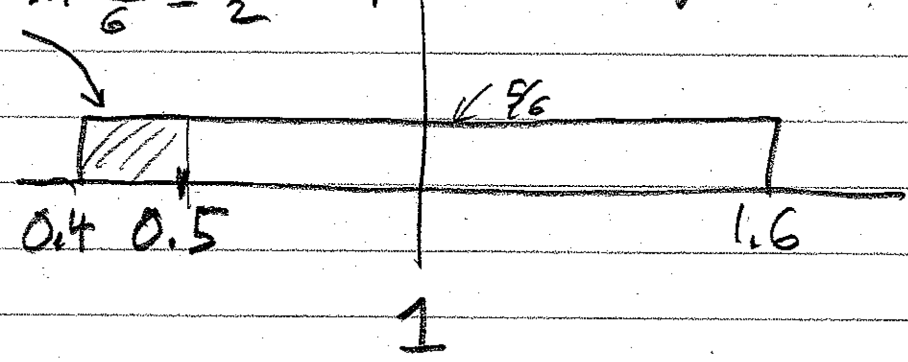
Nominal Zero



area $= 0.1 \cdot \frac{5}{6} = \frac{1}{12} =$ Prob error given bit is a zero $= p_{01}$

Nominal One

area $= 0.1 \cdot \frac{5}{6} = \frac{1}{2} =$ prob err given bit is a one $= p_{10}$



Probability of Error =

$$p_{10} \cdot \text{Prob}(\text{bit}=1) + p_{01} \cdot \text{Prob}(\text{bit}=0)$$

Equally likely case: $P(\text{bit}=\text{error}) =$

$$\frac{1}{2} \cdot \frac{1}{12} + \frac{1}{2} \cdot \frac{1}{12}$$

$$= \frac{1}{12}$$

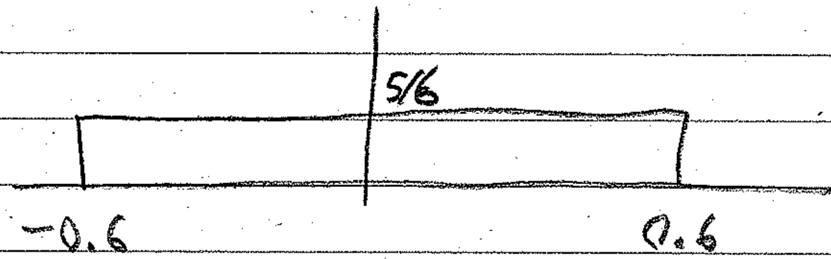
Suppose 1^s twice as likely a zeros

$$P(\text{bit}=\text{err}) = P_{10} \cdot \underset{\uparrow}{\frac{2}{3}} + P_{01} \cdot \frac{1}{3}$$

prob(bit=1)

If thresh = 0.5 V and

f_{NOISE} (noise)



then

$$P(\text{bit}=\text{err}) = \underbrace{\frac{1}{12}}_{P_{10}} \cdot \frac{2}{3} + \underbrace{\frac{1}{12}}_{P_{01}} \cdot \frac{1}{3} = \frac{1}{12}$$

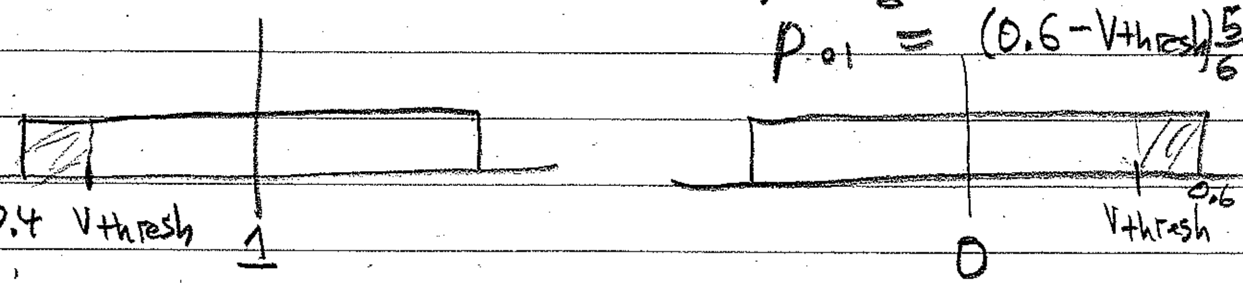
What if threshold is moved?

assuming Vthresh ≥ 0.4

$$P_{10} = (\text{Vthresh} - 0.4) \cdot \frac{5}{6}$$

assuming Vthresh ≤ 0.6

$$P_{01} = (0.6 - \text{Vthresh})\frac{5}{6}$$



Probability of Error with moved threshold

$P(\text{bit} = \text{error}) =$

$$\frac{2}{3}\,\frac{5}{6}\left(V_{thresh} - 0.4\right)\Big|_{V_{thresh} \ge 0.4}$$

$$+\ \frac{1}{3}\,\frac{5}{6}\left(0.6 - V_{thresh}\right)\Big|_{V_{thresh} \le 0.6}$$

$$=$$

$$\left(\frac{10}{18} - \frac{5}{18}\right) V_{thresh}\Big|_{0.4 \le V_{thresh} \le 0.6} + \frac{6}{10}\,\frac{5}{18} - \frac{4}{10}\,\frac{10}{18}$$

$$= \left(\frac{5}{18}\right) V_{thresh}\Big|_{0.4 \le V_{thresh} \le 0.6} - \frac{1}{18}$$

Letting $V_{thresh} = 0.4$ (No bit errors if transmitted bit = 1)

$$P(\text{bit} = \text{error}) = \frac{2}{18} - \frac{1}{18} = \frac{1}{18}$$

$$\frac{1}{18} < \frac{1}{12}$$

prob(bit = error) if $V_{thresh} = 0.5$

Suppose One Tried to balance the errors when transmitted 1's and 0's

That is $\quad p01 \cdot P_0 = p10 \cdot P_1$

Then

$$\frac{2}{3}\left(V_{thresh} - 0.4\right)\frac{5}{6} = \frac{1}{3}\left(0.6 - V_{thresh}\right)\frac{5}{6}$$

$$\Rightarrow \left(\frac{10}{18} + \frac{5}{18}\right) V_{thresh} = \frac{10}{18}\frac{4}{10} + \frac{5}{18}\frac{6}{10}$$

$$\frac{5}{6} V_{thresh} = \frac{4}{18} + \frac{3}{18}$$

$$V_{thresh} = \frac{7}{15}$$

Probability of error with Vthresh = 7/15

$$P_{10} = \left(\underbrace{\tfrac{14}{30}}_{\text{Vthresh}} - \tfrac{12}{30}\right)\cdot\tfrac{5}{6} = \tfrac{1}{18}$$

$$P_{01} = \left(\tfrac{18}{30} - \overset{\text{Vthresh}}{\tfrac{14}{30}}\right)\cdot\tfrac{5}{6} = \tfrac{2}{18}$$

$$P(\text{bit} = \text{error}) = \underbrace{P_{10}\cdot\tfrac{2}{3}} + \underbrace{P_{01}\cdot\tfrac{1}{3}}$$

Note Equality

$$= \tfrac{1}{27} + \tfrac{1}{27} = \tfrac{2}{27}$$

Note

$$\underbrace{\tfrac{1}{18}} < \underbrace{\tfrac{2}{27}} < \underbrace{\tfrac{1}{12}}$$

P(bit=error) for error minimizing Vthresh $= \frac{12}{30}$

P(bit=error) when p01·p0 = p10·p1 Vthresh $= \frac{14}{30}$

P(bit=error) Vthresh = 0.5 $= \frac{15}{30}$