Department of Electrical Engineering and Computer Science

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.02 Fall 2010

Practice Problems on Channel Coding, MAC Protocols, Filters

I Block Codes

In addition to the problems listed here, please study and solve the problems in the notes for Lecture 6 and the online problem set for Lab 4.

1. The Registrar has asked for an encoding of class year (”Freshman”, ”Sophomore”, ”Junior”, ”Senior”) that will allow single error correction. In the table below, please fill in an appropriate 5-bit binary encoding for each of the four years.

Year	Multi-bit binary encoding for year
Freshman	
Sophomore	
Junior	
Senior	

Solution: We want a (5,2,3) block code; we may as well try a linear one, as that's what we have learned about. For such a code, 00000 is a codeword by definition. Every other codeword must have weight at least 3, and 00111 is an obvious choice (or any permutation thereof).

We now need only two more codewords and each must have at least three ones, and must also have a Hamming distance of 3 from the second codeword above. A little bit of trial and error shows that 11011 works (as do some others). And the last word we want "writes itself" by just adding the two previous codewords: 11100.

So: 00000, 00111, 11011, 11100 should satisfy the Registrar.

2. Dos Equis Encodings, Inc. specializes in codes that use 20-bit transmit blocks. They are trying to design a $(20, 16)$ linear block code for single error correction. Explain whether they are likely to succeed or not.

Solution: A single error correcting code must be able to uniquely identify $n+1$ patterns (n error patterns and one without any errors). So 2^{n-k} must be $\geq n+1$. That is not true when $n = 20$ and $k = 16$.

3. For any linear block code over $\mathbb{F}_2$ with minimum Hamming distance at least $2t+1$ between code words, show that:

$$2^{n-k} \geq 1 + \binom{n}{1} + \binom{n}{2} + \ldots \binom{n}{t}.$$

Hint: How many errors can such a code always correct?

Solution: An (n, k) block code can represent in its parity bits at most 2^{n-k} patterns, and these must cover all the error cases we wish to correct, as well as the one case with no errors. When the minimum Hamming distance is $2t+1$, the code can correct up to t errors. The number of ways in which the transmission can experience $0, 1, 2, \ldots, t$ errors is equal to $1 + \binom{n}{1} + \binom{n}{2} + \ldots \binom{n}{t}$, and clearly this number must not exceed 2^{n-k}. QED.

4. The following matrix shows a rectangular single error correcting code consisting of 9 data bits, 3 row parity bits and 3 column parity bits. For each of the examples that follow, please indicate the correction the receiver must perform: give the position of the bit that needs correcting (e.g., D7, R1), or "no" if there are no errors, or "M" if there is a multi-bit uncorrectable error.

D1	D2	D3	R1
D4	D5	D6	R2
D7	D8	D9	R3
C1	C2	C3	—

1	1	1	1
1	1	0	0
0	1	1	0
0	1	1	–

Error:______

0	0	1	0
0	1	1	1
0	1	1	0
0	0	1	–

Error:______

1	1	0	0
1	0	0	1
0	0	1	0
0	0	1	–

Error:______

1	0	1	0
1	0	0	1
1	1	0	0
1	1	1	–

Error:______

1	1	1	0
1	1	1	1
1	0	0	0
0	0	1	–

Error:______

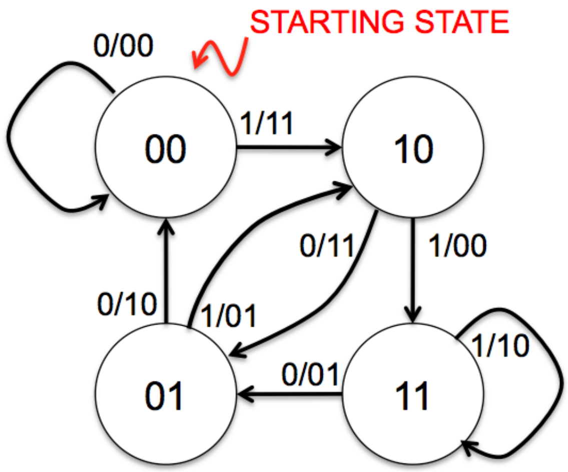
Solution: 1) Error in column 3 parity bit. 2) M. 3) Error in D8. 4) No errors. 5) M.

II Convolutional Codes

Please also study the problems in the online pset for Lab 4.

5. Consider the following state transition diagram for a $K = 3$, $rate = \frac{1}{2}$ convolutional encoder. The message stream is the bit sequence $X = x[0]x[1]\ldots x[n]$; the states are labeled with $x[n-1]x[n-2]$

and the transition arcs are labeled at the source end with $x[n]/p_0p_1$. Assume that the output of the encoder is created by concatenating the parity bits generated from processing each message bit in turn, i.e., $p_0[0]p_1[0]p_0[1]p_1[1] \ldots p_0[n]p_1[n]$.

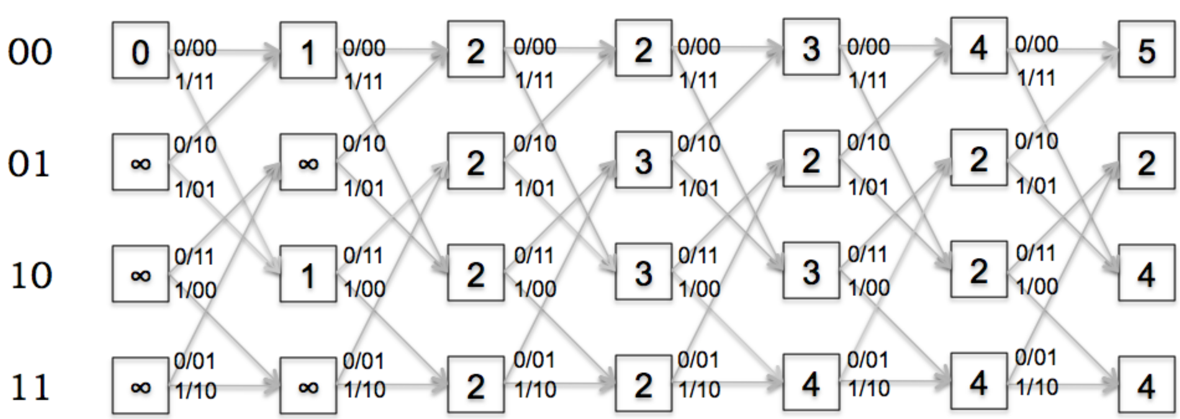


(A) If $X = 01011\ldots$, what is the sequence of bits produced by the convolutional encoder?

sequence produced by encoder: 00 11 11 01 00.

The receiver determines the most-likely transmitted message by using the Viterbi algorithm to process the (possibly corrupted) received parity bits. The path metric trellis generated from a particular set of received parity bits is shown below. The boxes in the trellis contain the minimum path metric as computed by the Viterbi algorithm.

Time step	1	2	3	4	5	6
Received	01	01	00	01	01	11



(B) Referring to the trellis above, what is the receiver's estimate of the most-likely transmitter state after processing the bits received at time step 6?

most-likely final state: ________________

(C) Referring to the trellis above, show the most-likely path through the trellis by placing a circle around the appropriate state box at each time step and darkening the appropriate arcs. What is the receiver's estimate of the most-likely transmitted message?

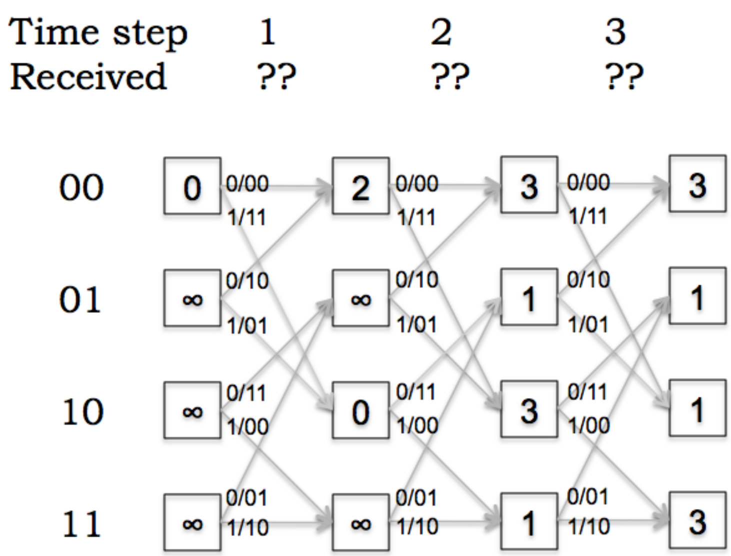
most-likely transmitted message: ________________

(D) Referring to the trellis above, and given the receiver's estimate of the most-likely transmitted message, at what time step(s) were errors detected by the receiver? Briefly explain your reasoning.

time steps at which errors detected: ________________

Solution: (B): 01. (C): 011010. (D): 1 and 2; where the path metric increments along the most likely path.

Now consider the path metric trellis generated from a *different* set of received parity bits.



(E) Referring to the trellis above, determine which pair(s) of parity bits *could* have been been received at time steps 1, 2 and 3. Briefly explain your reasoning.

Possible pairs of parity bits time 1: 11

Possible pairs of parity bits time 2: 01, 10

Possible pairs of parity bits time 3: 01

6. Consider two convolutional coding schemes, I and II. The generator polynomials for the two schemes are:

Scheme I: G0 = 1101, G1 = 1110
Scheme II: G0 = 110101, G1 = 111011

The notation is follows: if the generator polynomial is, say, 1101, then the corresponding parity bit for message bit n is $x[n] + x[n-1] + x[n-3]$. (All arithmetic in $\mathbb{F}_2$.)

A. Indicate True or False for each choice below:

(a) **True / False** Code rate of Scheme I is 1/4.
(b) **True / False** Constraint length of Scheme II is 4.
(c) **True / False** Code rate of Scheme II is equal to code rate of Scheme I.
(d) **True / False** Constraint length of Scheme I is 4.

Solution: The code rate (r) and constraint length (K) for the two schemes are
I: r = 1/2, K = 4.
II: r = 1/2, K = 6.
So, the answers are:
(a) false
(b) false
(c) true
(d) true

B. How many states will there be in the state diagram for Scheme I? For Scheme II?

Solution: The number of states is given by 2^{K-1} where K is the constraint length. Following the convention of state machines as outlined in Lecture 8, the number of states in Scheme I is 8 and in Scheme II, 32.

7. Consider a binary convolutional code specified by the generators (1011, 1101, 1111).

A. Give the values of:

(a) The constraint length of the code.
(b) The rate of the code.
(c) The number of states at each time step of the trellis.
(d) The number of branches transitioning into each state.
(e) The number of branches transitioning out of each state.
(f) The number of expected parity bits on each branch.

Solution: (a) 4 (b) 1/3 (c) $2^{4-1} = 8$ (d) 2 (e) 2 (f) 3.

A 10000-bit message is encoded with the above code and transmitted over a noisy channel. During Viterbi decoding at the receiver, the state 010 had the lowest path metric (a value of 621) in the final time step, and the survivor path from that state was traced back to recover the original message.

B. What is the most likely number of bit errors that are corrected by the Viterbi decoder?

Solution: 621 errors were likely corrected by the decoder to produce the final decoded message.

C. If you are told that the decoded message had no uncorrected errors, can you guess the approximate number of bit errors that would have occured had the 10000 bit message been transmitted without any coding on the same channel?

Solution: 3×10000 bits were transmitted over the channel and the received message had 621 bit errors (all corrected by the convolutional code). Therefore, if the 10000-bit message had been transmitted without coding, it would have had approximately 621/3 = 207 errors.

D. From knowing the final state of the trellis (010, as given above), can you infer what the last bit of the original message was? What about the last-but-one bit? The last 4 bits?

Solution: The state gives the last 3 three bits of the original message. In general, for a convolutional code with a constraint length k, the state indicates the final k-1 bits of the original message. To determine more bits we would need to know the states along the most-likely path as we trace back through the trellis.

E. Consider a transition branch between two states on the trellis that has 000 as the expected set of parity bits. Assume that 0V and 1V are used as the signaling voltages to transmit a 0 and 1 respectively, and 0.5V is used as the digitization threshold.

Assuming hard decision decoding, which of the two set of received voltages will be considered more likely to correspond to the expected parity bits on the transition: (0V, 0.501V, 0.501V) or (0V, 0V, 0.9V)? What if one is using soft decision decoding?

Solution: With hard decision decoding: (0V, 0.501V, 0.501V) is inferred to be 011 and it has a Hamming distance of 2 from expected parity bits. (0V, 0V, 0.9V) is inferred to be 001 and has a Hamming distance of 1. Therefore, (0V, 0V, 0.9V) is considered more likely.

With soft decision decoding, (0V, 0.501V, 0.501V) will have a branch metric of approximately 0.5. (0V, 0V, 0.9V) will have a metric of approximmately 0.8. Therefore, (0V, 0.501V, 0.501V) will be considered more likely.

8. Indicate whether each of the statements below is true or false, with a brief reason.

A. True / False If the number states in the trellis of a convolutional code is S, then the number of survivor paths at any point of time is S. Remember that if there is a tie between two incoming branches (i.e., they both result in the same path metric), we arbitrarilly choose only one as the predecessor.

True. There is one survivor per state.

B. True / False With hard decision decoding, the path metric of a state s_1 in the trellis indicates the number of residual uncorrected errors left along the trellis path from the start state to s_1.

False. It indicates the number of likely corrected errors.

C. True / False Among the survivor paths left at any point during the decoding, no two can be leaving the same state at any stage of the trellis.

False. In fact, the survivor paths will likely merge at a certain stage in the past, at which point all of then will emerge from the same state.

D. True / False Among the survivor paths left at any point during the decoding, no two can be entering the same state at any stage of the trellis. Remember that if there is tie between to incoming branches (i.e., they both result in the same path metric), we arbitrarilly choose only one as the predecessor.

True. When two paths merge at any state, only one of them will ever be chosen as a survivor path.

E. True / False For a given state machine of a convolutional code, a particular input message bit stream always produces the same output parity bits.

False. The same input stream with different start states will produce different output parity bits depending on the channel noise.

III MAC Protocols

In addition to the problems listed here, please study the problems at the end of the Lecture 10 notes, the problems and examples in the recitation notes by Professors Megretski and Verghese, and the online problem set for Lab 5.

9. Which of these statements are true for correctly implemented versions of stabilized unslotted Aloha, stabilized slotted Aloha, and Time Division Multiple Access (TDMA)? Assume that the slotted and unslotted versions of Aloha use the same stabilization method and parameters.

A. True / False When the number of backlogged nodes is large, unslotted Aloha has a lower maximum throughput than slotted Aloha.

True. By a factor of 2: $1/2e$ instead of $1/e$.

B. True / False When the number of backlogged nodes is large, there exists some offered load for which the throughput of unslotted Aloha is higher than the throughput of slotted Aloha.

False.

C. True / False TDMA has no packet collisions.

True. TDMA eliminates collisions by explicitly allotting time slots.

D. True / False There exists some offered load pattern for which TDMA has lower throughput than slotted Aloha.

True. For example, a skewed workload in which some nodes have much more traffic to send than others.

10. Binary exponential backoff is a mechanism used in some MAC protocols. Which of the following statements is correct?

A. True / False It ensures that two nodes that experience a collision in a time slot will never collide with each other when they each retry that packet.

B. True / False It ensures that two or more nodes that experience a collision in a time slot will experience a lower probability of colliding with each ther when they each retry that packet.

C. **True / False** It can be used with slotted Aloha but not with carrier sense multiple access.

D. **True / False** Over short time scales, it improves the fairness of the throughput achieved by different nodes compared to not using the mechanism.

Solution: B is true. The others are false.

11. Consider a shared medium with N backlogged nodes running the slotted Aloha MAC protocol without any backoffs. A "wasted slot" is one in which no node sends data. The fraction of time during which no node uses the medium is the "waste" of the protocol.

A. If the sending probability is p, what is the waste? What are the smallest and largest possible values of the waste?

Solution: $(1-p)^N$. 0 and 1.

B. If the Aloha sending probability, p, for each node is picked so as to maximize the utilization, what is the corresponding waste when N is large?

Solution: 1/e — same as the utlization!

12. Assume that the shared medium has N backlogged nodes. Which of these statements is true?

A. **True / False** In a slotted Aloha MAC protocol using binary exponential backoff for stabilization, the probability of transmission will always eventually converge to some value p, and all nodes will eventually transmit with probability p.

False. In a binary exponential backoff, the probability of transmission constantly changes. If the a transmission succeeds, then the probability goes up. If the transmission fails, the probability goes down.

B. **True / False** Using carrier sense multiple access (CSMA), suppose that a node "hears" that the channel is busy at time slot t. To maximize utilization, the node must not transmit in slot t and instead must transmit the packet in the next time slot with probability 1.

False. The node should not transmit at time t+1 with 100% probability. Other nodes may have also "heard" that the channel is busy and would want to send a packet at time t+1. So the node should send with a probability less than 100% to reduce the possibility of a collision.

C. **True / False** There is some workload for which an unslotted Aloha with perfect CSMA will not achieve 100% utilization.

True. Multiple nodes may think that the channel is idle and send a packet at the same time, resulting in a collision. Thus, the utilization can never reach 100%.

13. There are shared medium networks inside computers too! Eight Cell Processor cores are connected together with a shared bus. To simplify bus arbitration, Ben Bittdiddle, a young IBM engineer, suggested time division multiple access (TDMA) as an arbitration mechanism. Using TDMA, each of the processors is allocated a equal-sized time slots on the common bus in a round-robin fashion. Ben has been asked to evaluate the proposed scheme on two types of applications: 1) core-to-core streaming, 2) random loads.

Core-to-core streaming setup: Assume each core has the same stream bandwidth requirement.

Random loads setup: core 1 load = 20%, core 2 load = 30%, core 3 load = 10%, core 4 = 5%, core 5 = 1%, core 6 = 3%, core 7 = 1%, core 8 load = 30%.

Help Ben out by evaluating the effectiveness (bus utilization) of TDMA under these two traffic scenarios.

Solution: Core-to-core streaming: All cores have same bandwidth requirements during streaming, so, bus utilization is 100%.

Random loads: Each core is given 12.5% of bus bandwidth, but not all cores can use it, and some need more than that. So bus utilization is: 12.5% + 12.5% + 10% + 5% + 1% + 3% + 1% + 12.5% = 57.5%

14. Token-passing is a variant of a TDMA MAC protocol. Here, the N nodes sharing the medium are numbered $0, 1, \ldots N-1$. The token starts at node 0. A node can send a packet if, and only if, it has the token. When node i with the token has a packet to send, it sends the packet and then passes the token to node $(i+1)$ mod N. If node i with the token does not have a packet to send, it passes the token to node $(i+1)$ mod N. To pass the token, a node broadcasts a token packet on the medium and all other nodes hear it correctly.

A data packet occupies the medium for time T_d. A token packet occupies the medium for time T_k. If s of the N nodes in the network have data to send when they get the token, what is the utilization of the medium? Note that the bandwidth used to send tokens is pure overhead; the throughput we want corresponds to the rate at which data packets are sent.

Solution: Define an epoch to be the time between when node 0 gets its turn to send, until when it gets its turn again. The duration of an epoch is equal to $NT_k + sT_d$. In this duration, the amount of time spent usefully, delivering data packets, is equal to sT_d. The utilization is therefore equal to $\frac{sT_d}{NT_k + sT_d}$.

IV Filters

In addition to the problems listed below, please solve the problems at `http://web.mit.edu/6.02/www/f2010/handouts/q2review/html/filter-review.html` and in the Lab 6 online problem set.

15. Consider an LTI system characterized by the unit-sample response $h[n]$.

A. Give an expression for the frequency response of the system $H(e^{j\Omega})$ in terms of $h[n]$.
Solution: $H(e^{j\Omega}) = \sum_m h[m]e^{-j\Omega m}$.

B. If $h[0] = 1, h[1] = 0, h[2] = 1$, and $h[n] = 0$ for all other n, what is $H(e^{j\Omega})$?
Solution: $H(e^{j\Omega}) = 1 + e^{\,-2j\Omega}$.

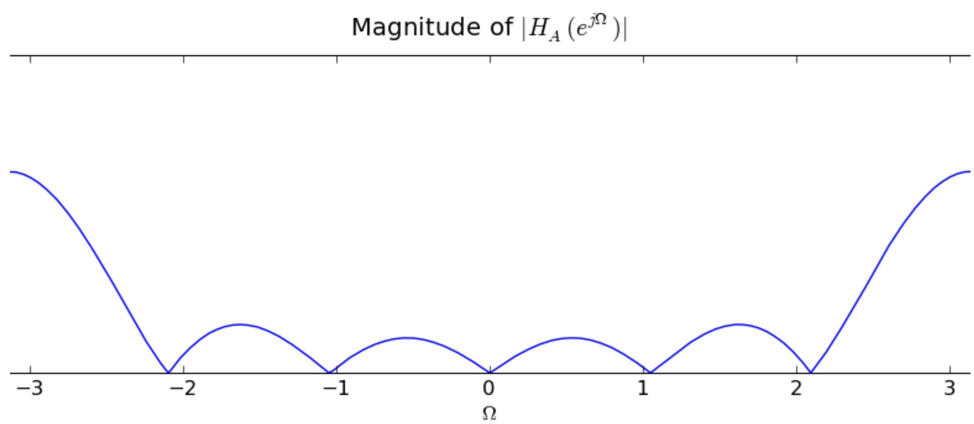
C. Let $h[n]$ be defined as in part B right above and $x[n] = cos(\Omega_1 n)$. Is there a value of Ω_1 such that $y[n] = 0$ for all n and $0 \leq \Omega_1 \leq \pi$?
Solution: $\Omega_1 = \pi/2$.

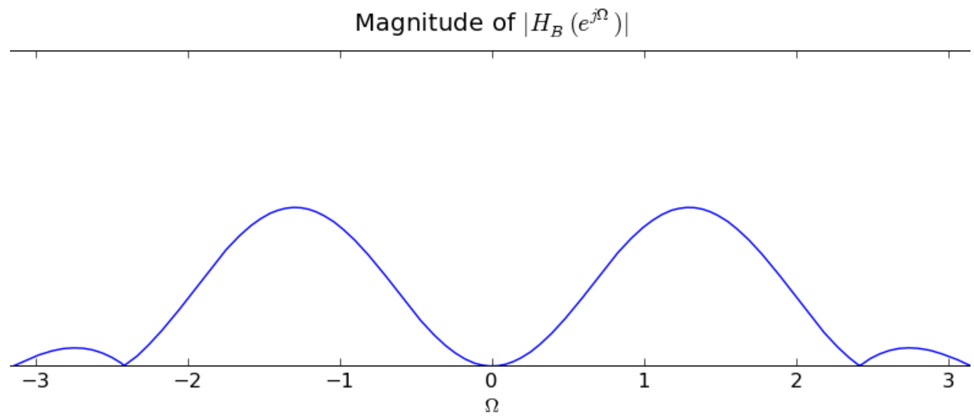
D. Find the maximum magnitude of $y[n]$ if $x[n] = cos(\pi n/4)$.
Solution: Maximum magnitude of $y[n]$ is $\sqrt{2}$.

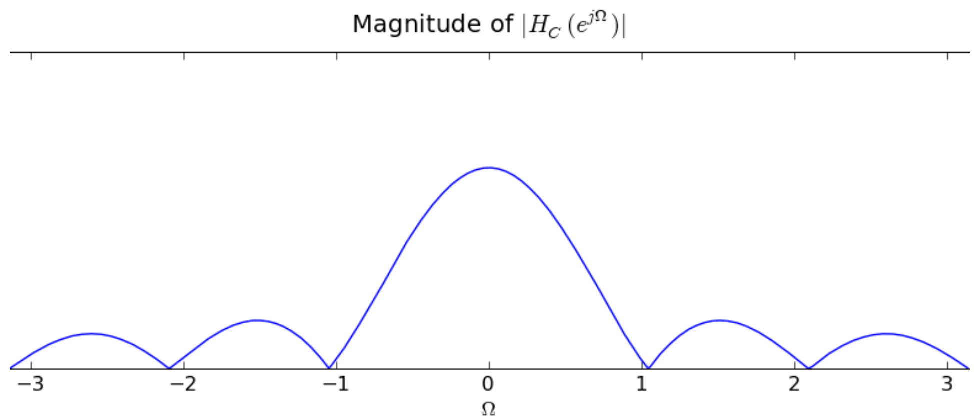
E. Find the maximum magnitude of $y[n]$ if $x[n] = cos(-\pi n/2)$.
Solution: Maximum magnitude of $y[n]$ is 0.

16. In answering the several parts of this question, consider three linear time-invariant filters, denoted A, B, and C, each characterized by the magnitude of their frequency responses, $|H_A(e^{j\Omega})|$, $|H_B(e^{j\Omega})|$, and $|H_C(e^{j\Omega})|$, respectively, as given in the plots below.

For the solutions to these problems (in #16), please see the solutions to Problem 1 on Quiz II from Spring 2010, at `http://web.mit.edu/6.02/www/s2010/handouts/q2solutions.pdf`

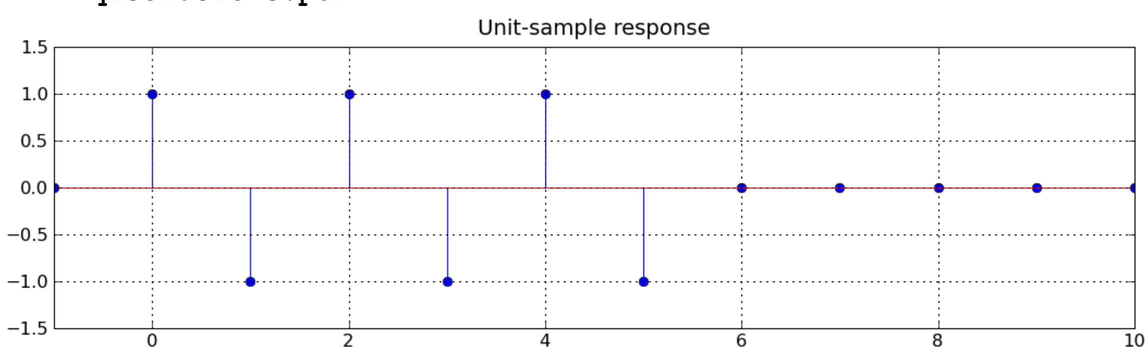






(A) Which frequency response (A, B, or C) corresponds to the following unit sample response, and what is $\max_\Omega |H(e^{j\Omega})|$ for your selected filter? Please justify your selection.

For the solutions to these problems (in #16), please see the solutions to Problem 1 on Quiz II from Spring 2010, at `http://web.mit.edu/6.02/www/s2010/handouts/q2solutions.pdf`

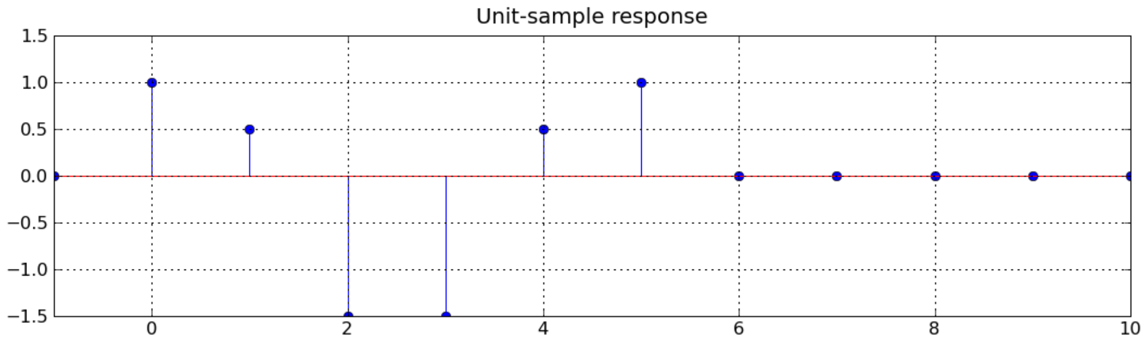


Frequency response plot (A, B, or C) (Be sure to justify your answer) = __________

$\max_\Omega |H(e^{j\Omega})| =$ ______________

(B) Which frequency response (A, B, or C) corresponds to the following unit sample response, and is the $\max_\Omega |H(e^{j\Omega})| > 6$ for your selected filter? Please justify your answers.

For the solutions to these problems (in #16), please see the solutions to Problem 1 on Quiz II from Spring 2010, at `http://web.mit.edu/6.02/www/s2010/handouts/q2solutions.pdf`



Frequency response plot (A, B, or C) (Be sure to justify your answer) = ______________

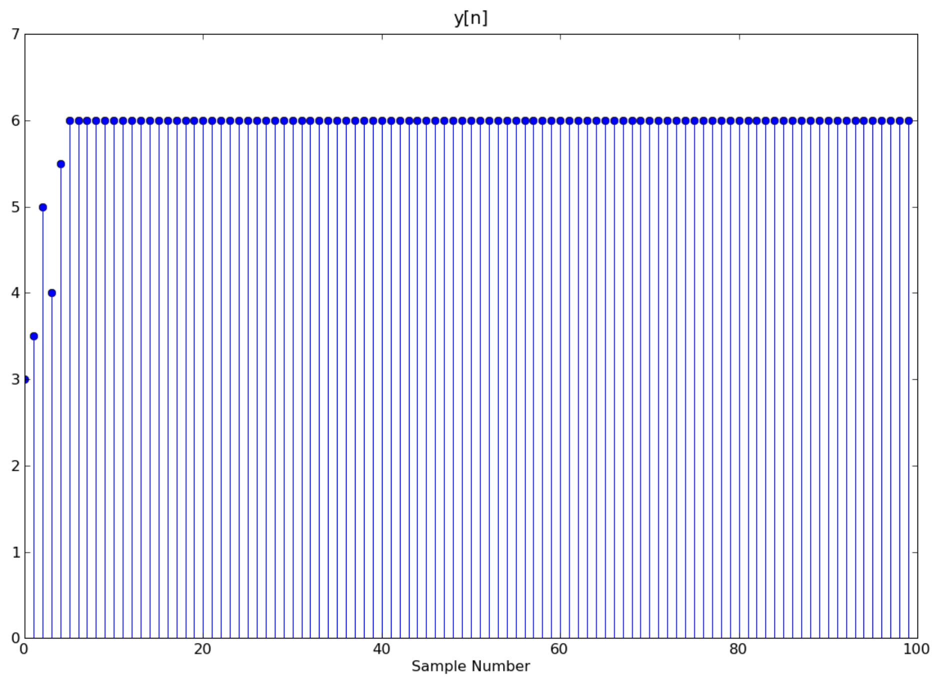
Is $\max_\Omega |H(e^{j\Omega})| > 6$ (yes or no)? ______________

You must provide some justification to receive credit.

(C) Suppose the input to each of the above three filters is $x[n] = 0$ for $n < 0$ and for $n \geq 0$ is

$$x[n] = \cos\frac{\pi}{3.0}n + \cos\pi n + 1.0.$$

Which filter (A, B,or C) produced the output, $y[n]$ below, and what is $\max_\Omega |H(e^{j\Omega})|$ for your selected system?



For the solutions to these problems (in #16), please see the solutions to Problem 1 on Quiz II from Spring 2010, at `http://web.mit.edu/6.02/www/s2010/handouts/q2solutions.pdf`

Frequency response plot (A, B, or C) (Be sure to justify your answer) = ______________

$\max_\Omega |H(e^{j\Omega})| =$ ______________

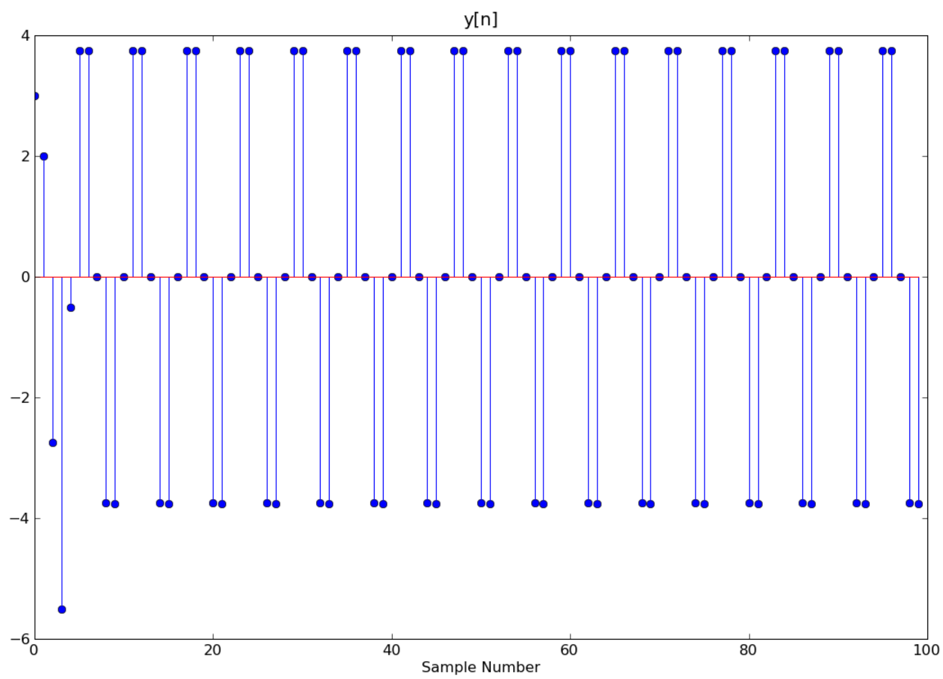
(D) Suppose the input to each of the above three filters is $x[n] = 0$ for $n < 0$ and for $n \geq 0$ is

$$x[n] = \cos \frac{\pi}{3.0} n + \cos \pi n + 1.0.$$

(same input as part C).

Which filter, (A, B,or C), produced the output, $y[n]$ below, and is $\max_\Omega |H(e^{j\Omega})| < 4.0$ for your selected filter?

For the solutions to these problems (in #16), please see the solutions to Problem 1 on Quiz II from Spring 2010, at `http://web.mit.edu/6.02/www/s2010/handouts/q2solutions.pdf`



Frequency response plot (A, B, or C) (Be sure to justify your answer) = ______________

Is $\max_\Omega |H(e^{j\Omega})| < 4$ (yes or no)? ______________

You must provide some justification to receive credit.

(E) Six new filters were generated using the unit sample responses of filters A, B and C, denoted $h_A[n]$, $h_B[n]$, and $h_C[n]$ respectively. The unit sample responses of the new filters were generated in the following way:

$$h_1[n] = h_A[n] + h_B[n]$$

$$h_2[n] = h_A[n] + h_C[n]$$

$$h_3[n] = h_B[n] + h_C[n]$$

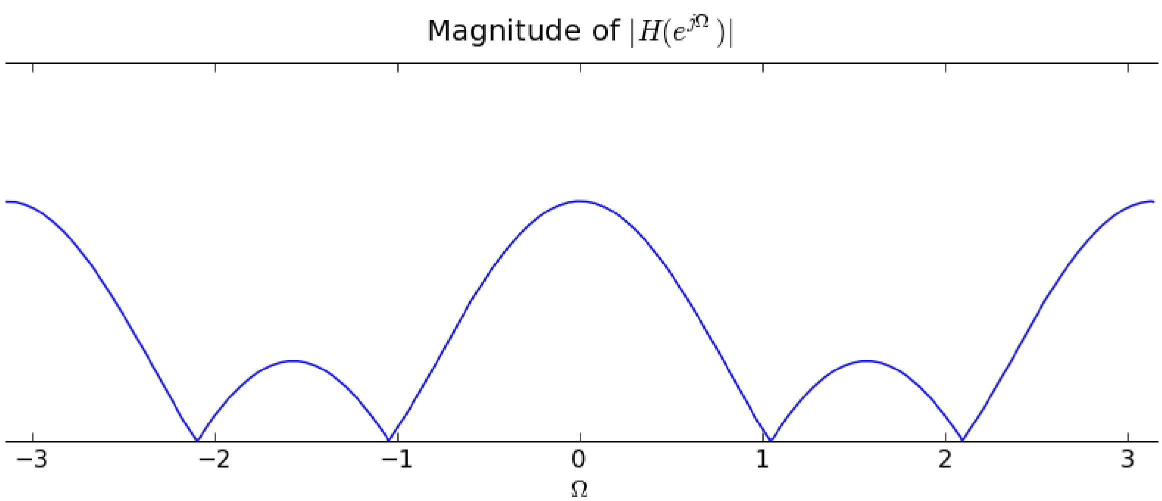
$$h_4[n] = \sum_{m=0}^{m=n} h_A[m]h_B[n-m].$$

$$h_5[n] = \sum_{m=0}^{m=n} h_A[m]h_C[n-m].$$

$$h_6[n] = \sum_{m=0}^{m=n} h_B[m]h_C[n-m].$$

Which of the six new filters has a frequency response as plotted below?

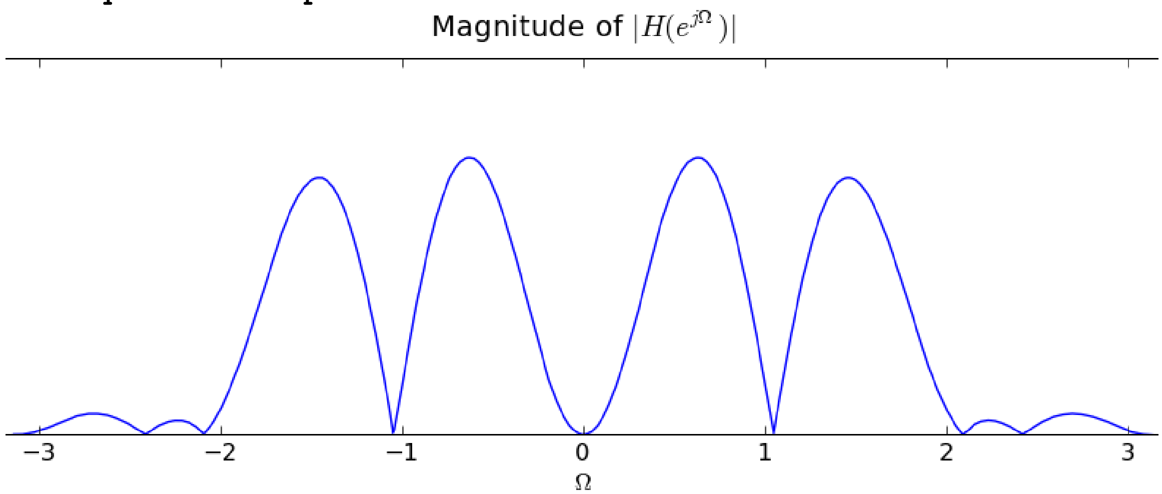
For the solutions to these problems (in #16), please see the solutions to Problem 1 on Quiz II from Spring 2010, at `http://web.mit.edu/6.02/www/s2010/handouts/q2solutions.pdf`



Which filter (1,2,3,4,5 or 6) and you must justify your answer: ______________

(F) Which of the six new filters described in part E has a frequency response as plotted below?

For the solutions to these problems (in #16), please see the solutions to Problem 1 on Quiz II from Spring 2010, at `http://web.mit.edu/6.02/www/s2010/handouts/q2solutions.pdf`



Which filter (1,2,3,4,5 or 6) and you must justify your answer: _______________

In answering the questions below, please consider the unit sample response and frequency response of two filters, H_1 and H_2, plotted below.

Note: the only nonzero values of unit sample response for H_1 are: $h_1[0] = 1, h_1[1] = 0, h_1[2] = 1$.