FREQUENCY RESPONSE

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$$x[n] \longrightarrow \boxed{\text{ch.}} \longrightarrow y[n]$$

$\ast$LTI

ALSO NOT COMPLETELY NECESSARY SOMETIMES BUT THAT'S A TRICKY BUSINESS $\rightarrow$ $\ast$ STABLE (unit SAMPLE RESPONSE DIMINISHES: $h[n] \rightarrow 0$ AS $n \rightarrow \pm\infty$)

$\ast$ NOT NECESSARILY CAUSAL

STABLE LTI

$$\underbrace{x[n] = A\cos(\Omega n + \varphi)}_{\text{pure everlasting sinusoidal input}} \longrightarrow \boxed{\text{ch}} \longrightarrow \underbrace{y[n] = |H(e^{j\Omega})| \cdot A \cdot \cos(\Omega n + \varphi + \angle H(e^{j\Omega}))}_{\text{pure everlasting sinusoidal output}}$$

ALWAYS GENERATES

where $H(e^{j\Omega}) = \sum_n h[n] e^{-j\Omega n}$

$\mathcal{H}(z) = \sum_n h[n] z^{-n}$ FOR $z = e^{j\Omega}$

EQUIVALENT "COMPLEX EXPONENTIAL" FORMULA:

$$x_c[n] = X \cdot e^{j\Omega n} \longrightarrow \boxed{\text{ch}} \longrightarrow y_c[n] = H(e^{j\Omega}) X e^{j\Omega n}$$

CONNECTION: $A\cos(\Omega n + \varphi) = \mathrm{Re}[X e^{j\Omega n}]$

where $X = A e^{j\varphi}$

$$x[n] = \mathrm{Re}(X e^{j\Omega n}) \longrightarrow \boxed{\text{ch}} \longrightarrow y[n] = \mathrm{Re}(H(e^{j\Omega}) X e^{j\Omega n})$$

WHY IS THIS USEFUL FOR OTHER INPUTS?

– BECAUSE "EVERY" [REASONABLE] INPUT CAN BE APPROXIMATED BY A SUM OF SINUSOIDS!

IF THE SINUSOIDS HAVE A "START" TIME, THE RESPONSE WILL APPROACH THE PRESCRIBED SINUSOID

Ex. 1 AUDIO RECORDING, WHICH SAMPLES SOUND AT 40 KHz, IS CORRUPTED BY 60 Hz SINUSOIDAL NOISE

DESIGN A FILTER

$$y[n] = x[n] + a\left(x[n-1] + x[n+1]\right) \qquad (a = ?)$$

TO COMPLETELY ELIMINATE THE NOISE

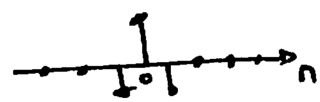
$$x[n] \longrightarrow \boxed{\text{LTI FILTER}} \longrightarrow y[n]$$

$x[n]$: ORIGINAL AUDIO

$y[n]$: "CLEANED" AUDIO

STEP 1 LTI + SINUSOIDS $\longrightarrow$ THINK FREQUENCY RESPONSE

$$h[n] = \delta[n] + a\left(\delta[n-1] + \delta[n+1]\right)$$



$$H(e^{j\Omega}) = 1 + a\left(e^{-j\Omega} + e^{j\Omega}\right)$$

$$= 1 + 2a\cdot\cos(\Omega)$$ $\longleftarrow$ SANITY CHECK: MUST BE 2π-PERIODIC

$$H(z) = 1 + \frac{a}{z} + az = \frac{a + z + az^2}{z} = \frac{1 + z/a + z^2}{z/a}$$

STEP 2 WHAT KIND OF DT FREQUENCY IS 60 Hz WHEN SAMPLED AT 40 KHz ?

CT SINUSOID AT 60 Hz: $\quad w(t) = A\cos(2\pi\cdot 60\cdot t + \varphi)$

40 KHz SAMPLING $= 4\cdot 10^4$ SAMPLES/SEC

$= 1$ SAMPLE IN $T = 0.25\cdot 10^{-4}$ SEC

SAMPLED SINUSOID:

$$v[n] = w(nT) = A\cos\left(\underbrace{2\pi\cdot 60\cdot \tfrac{1}{4}\cdot 10^{-4}}_{\Omega_0 = 3\pi\cdot 10^{-3}}\, n + \varphi\right)$$

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STEP 3 TO CANCEL THE SINUSOIDS AT FREQUENCY Ω_0

THE MOST IMPORTANT PART! $\rightarrow$ $\boxed{\text{NEED } H(e^{j\Omega_0}) = 0}$

STEP 4 ALGEBRA: FIND a SUCH THAT $H(e^{j\Omega_0}) = 0$

i.e. $1 + \frac{z}{a} + z^2$ HAS ROOT $z = e^{j\Omega_0}$

THEN $z = e^{-j\Omega_0}$ IS THE OTHER ROOT

HENCE $1 + \frac{z}{a} + z^2 = \underbrace{(e^{j\Omega_0} e^{-j\Omega_0})}_{1} + \underbrace{(-e^{j\Omega_0} - e^{-j\Omega_0})}_{-2\cos\Omega_0} z + z^2$

$$\frac{1}{a} = -2\cos\Omega_0$$

$$\boxed{a = -\frac{1}{2\cos(3\pi 10^{-3})}}$$

DISCUSSION: WHILE ELIMINATING THE NOISE, BOTH PHASE AND AMPLITUDE OF "OTHER SINUSOIDS" WILL BE MODIFIED

IS THIS BAD?

CHANGE IN AMPLITUDE: HOW LOUD A TONE IS...

* EAR SENSITIVITY IS "LOGARITHMIC", I THINK

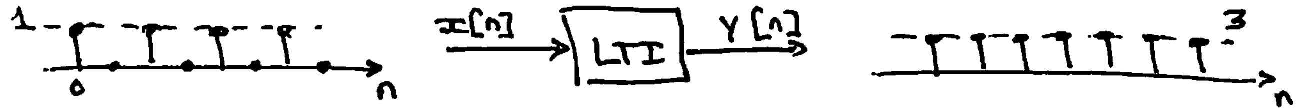
$\cancel{1+\frac{z}{a}+z^2 \approx}$ $\cancel{1+2a\cos\Omega}$ $|1 + 2a\cos\Omega|$ does not change dramatically, except at $\Omega \approx 0$

CHANGE IN PHASE: (THE RELATIVE PHASE IS WHAT'S IMPORTANT)

HUMAN EAR APPEARS TO BE QUITE INSENSITIVE TO PHASE ERRORS

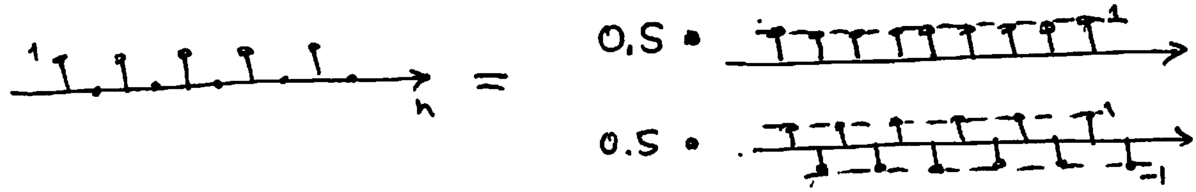
PERHAPS BECAUSE THE SENSING IS BASED ON EXCITING RESONATORS

Ex. 2



WHAT DOES THIS TELL ABOUT THE FREQUENCY RESPONSE?

STEP 1 INPUT AS A SUM OF SINUSOIDS?



$$x[n] = 0.5\cos[0\cdot n] + 0.5\cos(\pi n)$$

$$y[n] = 3\cdot\cos(0\cdot n)$$

STEP 2 IN TERMS OF THE (UNKNOWN) FREQUENCY RESPONSE:

$$Y[n] = |H(e^{j\cdot 0})|\cdot 0.5\cdot\cos(0\cdot n + \angle H(e^{j\cdot 0}))$$
$$+ |H(e^{j\pi})|\cdot 0.5\cdot\cos(\pi n + \angle H(e^{j\pi}))$$

HENCE $H(e^{j\cdot 0}) = 6$

$H(e^{j\pi}) = 0$

Ex. 3

WHICH VALUES OF a, b ~~MATCHES~~ IN $h[n] = a\delta[n] + b\delta[n-1]$ ~~[illegible]~~ MATCH THE FREQUENCY RESPONSE PLOT?

$H(z) = a + b/z$

$H(e^{+j0}) = 0 \Rightarrow b = -a$

$H(e^{j\pi}) = 1 \Rightarrow a - b = 1$

$a = 0.5, \ b = -0.5$