FREQUENCY RESPONSE / FILTERS

6.02 / Oct. 26, 2010

Sinusoid $\longrightarrow$ $\boxed{\text{LTI}}$ $\longrightarrow$ sinusoid (scaled, time-shifted)

$A\cos(\Omega n + \varphi)$

$\mathrm{Re}\left[X e^{j\Omega n}\right]$

$X e^{j\Omega n}$

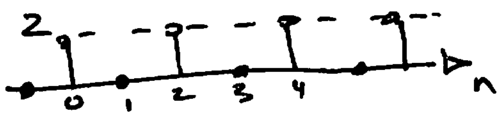
$|H(e^{j\Omega})|\, A \cos(\Omega n + \varphi + \angle H(e^{j\Omega}))$

$\mathrm{Re}\left[H(e^{j\Omega}) X e^{j\Omega n}\right]$

$H(e^{j\Omega}) X e^{j\Omega n}$

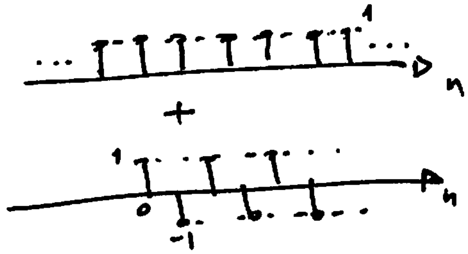
WHERE $H(e^{j\Omega}) = \sum_n h[n] e^{-j\Omega n}$: THE FREQUENCY RESPONSE

Ex. 1



is THIS A "SINUSOID"?

ITS A SUM OF TWO:

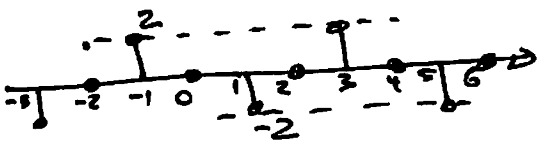


$1 \cdot \cos(0 \cdot n)$

$+$

$1 \cdot \cos(\pi \cdot n + 0)$

Ex. 2

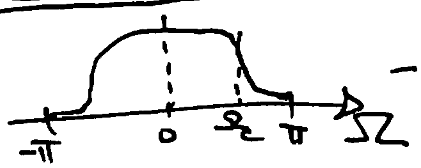


is THIS A "SINUSOID"?

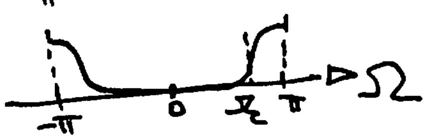
$$x[n] = 2 \cdot \cos\left(\frac{\pi}{2} \cdot n + \frac{\pi}{2}\right) = \mathrm{Re}\Big[\underbrace{2 e^{j\frac{\pi}{2}}}_{X}\; e^{j\frac{\pi}{2} n}\Big]$$

$\Omega = \frac{\pi}{2}$

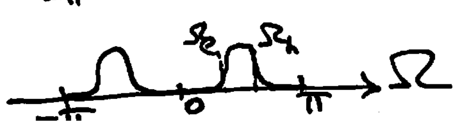
"FILTERS" : judged by $|H(e^{j\Omega})|$



– LOW-PASS (LPF)



– HIGH-PASS (HPF)



– BAND-PASS (BPF)

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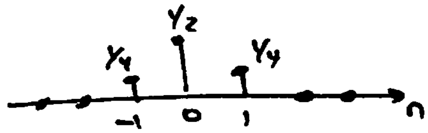
Ex. 3

$h[n] =$

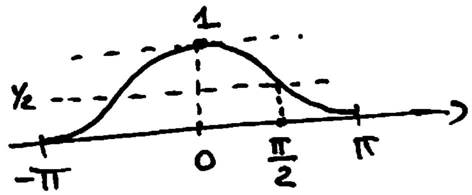


$$H(e^{j\Omega}) = \tfrac{1}{2}\cdot 1 + \tfrac{1}{4}\cdot e^{-j\Omega} + \tfrac{1}{4} e^{j\Omega}$$

$$= \frac{1+\cos(\Omega)}{2}$$

What is this? $\quad H(e^{j\Omega})$ small for $\Omega \to \pm\pi$

— " — largest for $\Omega \to 0$

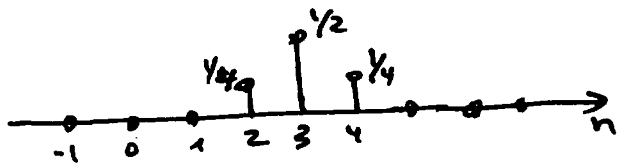


What is Ω_c ?

Roughly, $\frac{\pi}{2}$

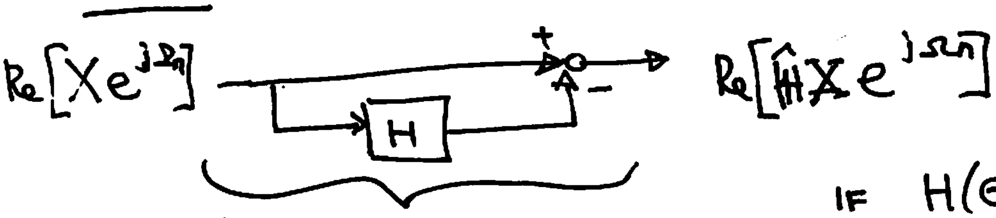
IF ONE WANTS A CAUSAL FILTER OUT OF THIS, JUST SHIFT $h[\cdot]$ IN TIME (BY $K>0$ STEPS): THIS MULTIPLIES $H(e^{j\Omega})$ BY $e^{-j\Omega K}$, HENCE DOES NOT AFFECT $|H(e^{j\Omega})|$!

$h[n] =$



$$\Rightarrow H(e^{j\Omega}) = e^{-3j\Omega}\,\frac{1+\cos(\Omega)}{2}$$

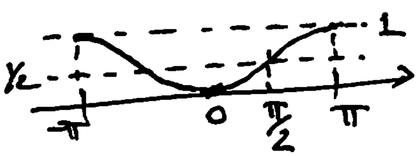
Ex. 4 MAKING A HIGH-PASS FILTER OUT OF Ex. 3 ?



$\hat{H} = 1 - H(e^{j\Omega})$

IF $H(e^{j\Omega}) = \dfrac{1+\cos(\Omega)}{2}$

THEN $1 - H(e^{j\Omega}) = \dfrac{1-\cos(\Omega)}{2}$



Ex. 5 CAUTION: FILTER INTERCONNECTION OPERATIONS ARE ON THE FREQUENCY RESPONSE $H(e^{j\Omega})$ NOT ON $|H(e^{j\Omega})|$!

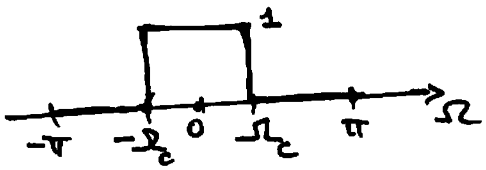
IF $H(e^{j\Omega}) = e^{-3j\Omega}\,\dfrac{1+\cos(\Omega)}{2}$

THEN $1-H(e^{j\Omega})$ IS A MUCH WORSE HPF!

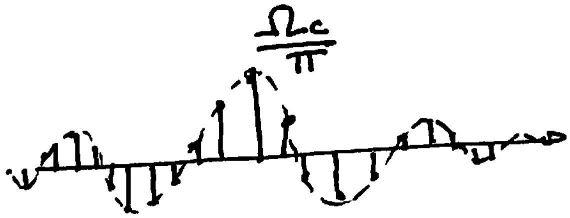
FOR EXAMPLE, $1-H(e^{j\frac{\pi}{3}}) = 1 + \dfrac{1+\cos(\pi/3)}{2} = \dfrac{7}{4} \gg \dfrac{1}{4}$

OTHER CUT-OFF FREQUENCIES?

(A)



$\longleftrightarrow$

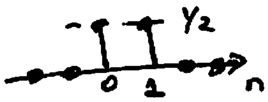


$$H(e^{j\Omega}) = \begin{cases} 1, & \Omega \in [-\Omega_c, \Omega_c] \\ 0, & \text{otherwise} \end{cases} \qquad h[n] = \begin{cases} \dfrac{\Omega_c}{\pi}, & n=0 \\ \dfrac{\sin(\Omega_c n)}{\pi n}, & n \neq 0 \end{cases}$$

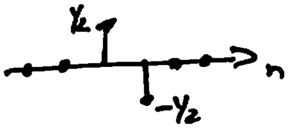
"IDEAL LPF"

THIS IS WHY YOU SEE THOSE "sinc" WAVEFORMS IN UNIT STEP RESPONCES OF GOOD FILTERS!

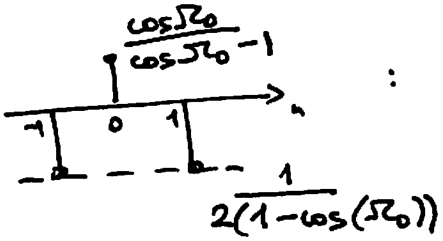
(B) COMBINE FILTERS USING INTERCONNECTIONS



: KILLS $\Omega=\pi \rightarrow H = \dfrac{1+e^{-j\Omega}}{2}$



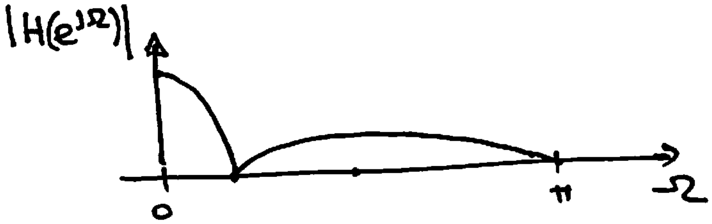
: KILLS $\Omega=-\pi \rightarrow H = \dfrac{1-e^{-j\Omega}}{2}$

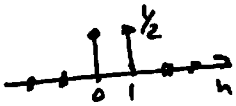


: KILLS $\Omega=\Omega_0 \longrightarrow H = \dfrac{\cos(\Omega)-\cos(\Omega_0)}{1-\cos(\Omega_0)}$

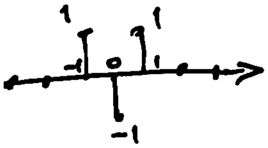
Ex. 6 DESIGN A USR TO SATISFY

$$H(e^{j\cdot 0}) = 1, \quad H(e^{j\cdot\pi}) = 0, \quad H(e^{j\frac{\pi}{6}}) = 0$$

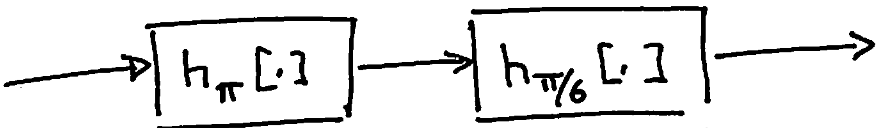




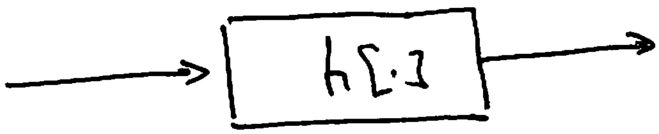
$$\underbrace{h_\pi[n] = \frac{\delta[n] + \delta[n-1]}{2} \quad \text{YIELDS} \quad H_\pi(e^{j0}) = 1,\ H_\pi(e^{j\pi}) = 0}_{H_\pi(e^{j\Omega}) = \frac{1+e^{-j\Omega}}{2}}$$

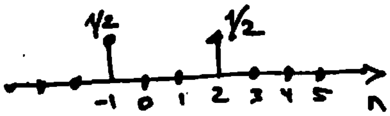


$$\underbrace{h_{\pi/6}[n] = \delta[n+1] + \delta[n-1] - \delta[n] \quad \text{YIELDS} \quad H_{\pi/6}(e^{j0}) = 1,\ H_{\pi/6}(e^{j\pi/6}) = 0}_{H_{\pi/6}(e^{j\Omega}) = \frac{2\cos(\Omega) - 1}{2}}$$



convolution





$$\underbrace{h = h_\pi * h_{\pi/6} = \frac{\delta[n+1] + \delta[n-2]}{2}}_{H(e^{j\Omega}) = \frac{e^{j\Omega} + e^{-2j\Omega}}{2}}$$