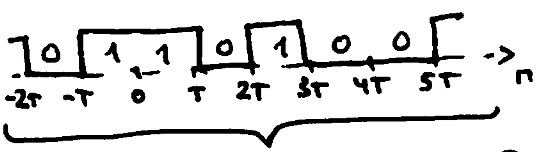
PROBABILISTIC MODELS / BER

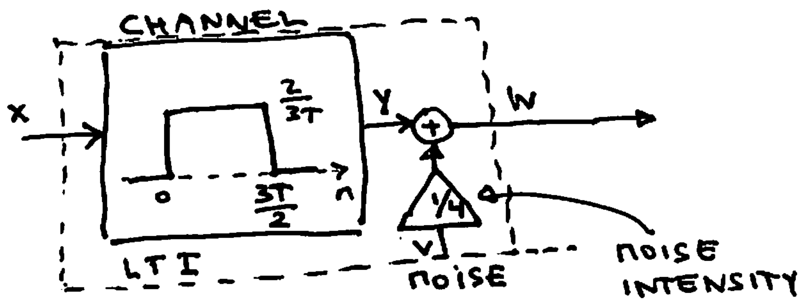
PROB. MODELS ARE DEFINED BY
- $\rightarrow$ SPECIFYING SOME PROBABILITIES
- $\rightarrow$ ASSUMING INDEPENDENCE OF SOME GROUPS OF EVENTS/VARIABLES

"ERGODICITY" $\longrightarrow$ ASSOCIATION BETWEEN EVENT'S PROBABILITY AND ITS FREQUENCY IN MULTIPLE WEAKLY DEPENDENT TRIALS

EXAMPLE:



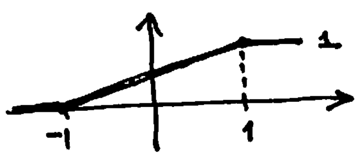
ENCODED BIT SEQUENCE,
BIT PERIOD = T
(T - even, T > 3)



PROB. MODEL:

(a) $\mathbb{P}(b[n]=0) = \frac{1}{3}$ FOR ALL n

(b) $v[n]$ has CDF



(c) $\ldots, b[-1], b[0], b[1], \ldots, v[-1], v[0], v[1], \ldots$ — INDEPENDENT

$\boxed{1}$ $\mathbb{P}(x[5]=1) = ?$

USE: $\mathbb{P}(A)+\mathbb{P}(B)=1$ WHEN $A\cup B$ = EVERYTHING, $A\cap B$ = EMPTY

$\boxed{2}$ $\mathbb{P}(x[0]=1,\ x[2T+7]=1) = ?$

USE: INDEPENDENCE OF $b[0]$ AND $b[n]$ FOR $n\neq 0$

$\boxed{3}$ $\mathbb{P}(x[0]=1,\ x[1]=0) = ?$

USE: COMMON SENSE

$\boxed{4}$ $\mathbb{P}(x[0]=0,\ v[0]<0) = ?$

USE: INDEPENDENCE OF $x[0], v[0]$

$\boxed{5}$ FREQUENCY OF "$x[n] = 1$" ?

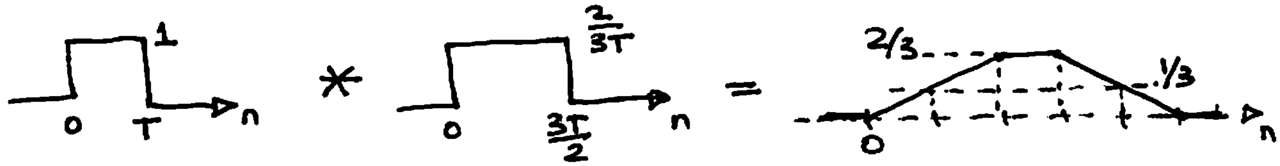
$\boxed{6}$ FREQUENCY OF "$x[n] = 0, x[n+T] = 1$" ?

$\boxed{7}$ FREQUENCY OF "$x[n]=0, x[n+1] = 0$" ?

$\boxed{8}$ FREQUENCY OF "$v[n] > 1/3$" ?

$\boxed{9}$ FIND $\mathbb{P}(Y[\cancel{n}\,k] = a)$ FOR ALL k, a ?

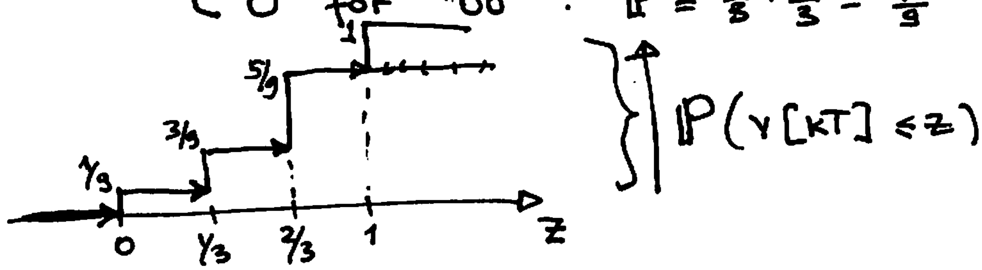
(a)



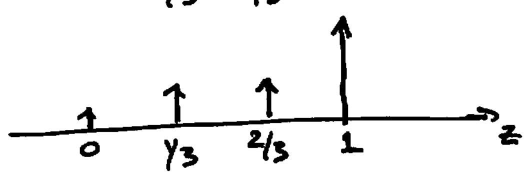
(b) HENCE $Y[kT] = \frac{2}{3} b[k-1] + \frac{1}{3} b[k-2]$

(c)
$$Y[kT] = \begin{cases} 1 \text{ for "11"} : & \mathbb{P} = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9} \\ 2/3 \text{ for "01"} : & \mathbb{P} = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9} \\ 1/3 \text{ for "10"} : & \mathbb{P} = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9} \\ 0 \text{ for "00"} : & \mathbb{P} = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9} \end{cases}$$

CDF:



PDF



"delta-functions"

$\boxed{10}$ RATE "FREQ." OF $Y[kT] = 2/3$?

USE $\mathbb{P}(Y[kT] = 2/3)$!

BER FOR THRESHOLD = $1/2$?

$$\mathbb{P}(\text{bit}=\text{"1"} \,\&\, w<0.5) + \mathbb{P}(\text{bit}=\text{"0"} \,\&\, w\geq 0.5) = ?$$

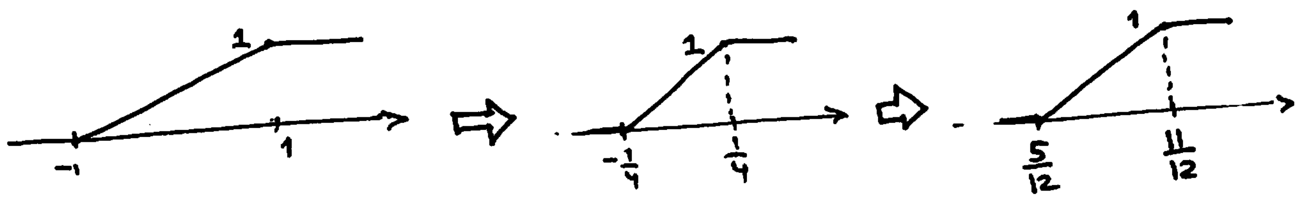
$\downarrow$ $\mathbb{P}(\text{bits}=\text{"01"} \,\&\, w<0.5)$, $\mathbb{P}(\text{bits}=\text{"11"} \,\&\, w<0.5)$

$\downarrow$ $\mathbb{P}(\text{bits}=\text{"00"}, w\geq 0.5)$, $\mathbb{P}(\text{bits}=\text{"10"}, w\geq 0.5)$

bits = "0,0" & $w \geq 0.5$ **IMPOSSIBLE** ($\mathbb{P}=0$)

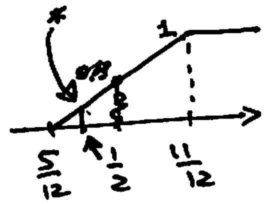
bits = "1,1" & $w < 0.5$ **IMPOSSIBLE** ($\mathbb{P}=0$)

Bits = "0,1" : $w = \frac{2}{3} + \frac{1}{4}\cdot v$



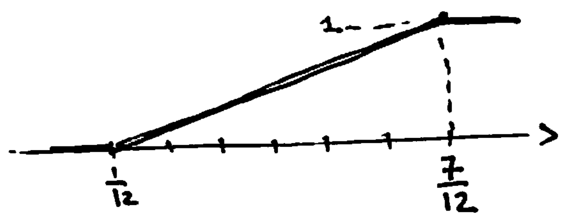
CDF of $v[n]$ CDF of $\frac{1}{4}v[n]$ CDF of $\frac{2}{3}+\frac{1}{4}v[n]$

~~CONDITIONED ON bits = "0,1"~~

$\mathbb{P}(\text{bits}=\text{"0,1"}, w<0.5) = \frac{2}{9}\cdot$  $= \frac{2}{9}\cdot\frac{1}{6} = \cancel{\frac{1}{18}}\ \frac{1}{27}$

CDF of $\frac{1}{3}+\frac{1}{4}v[n]$:

~~CONDITIONED~~

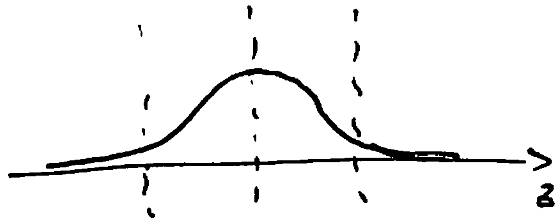


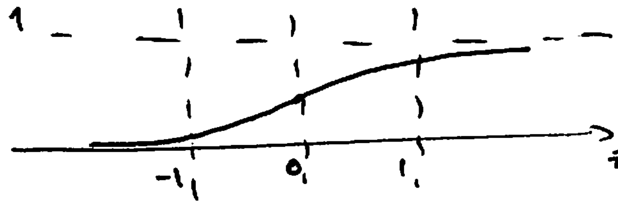
$\mathbb{P}(\text{bits}=\text{"1,0"}, w\geq 0.5)$

$$= \frac{2}{9}\cdot\frac{1}{6} = \frac{1}{27}$$

$$\boxed{\text{BER} = 2/27}$$

WHAT CHANGES WHEN $V[n]$ – normal (GAUSSIAN), ZERO MEAN, UNIT VARIANCE $\Big\}\ V[n] \sim N(0,1)$





PDF: $f_V(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

CDF: $F_V(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{z^2}{2}} dz$

$$\mathbb{P}(\text{bits}=\text{"01"}, w<r) = \frac{2}{9}\mathbb{P}\left(\tfrac{1}{4}v < r-\tfrac{2}{3}\right) = \frac{2}{9}F_V\left(4r-\tfrac{8}{3}\right)$$

$$\mathbb{P}(\text{bits}=\text{"11"}, w<r) = \frac{4}{9}\mathbb{P}\left(\tfrac{v}{4} < r-1\right) = \frac{4}{9}F_V(4r-4)$$

$$\mathbb{P}(\text{bits}=\text{"00"}, w\geq r) = \frac{1}{9}\mathbb{P}\left(\tfrac{v}{4} \geq r\right) = \frac{1}{9}\left(1-F_V(4r)\right)$$

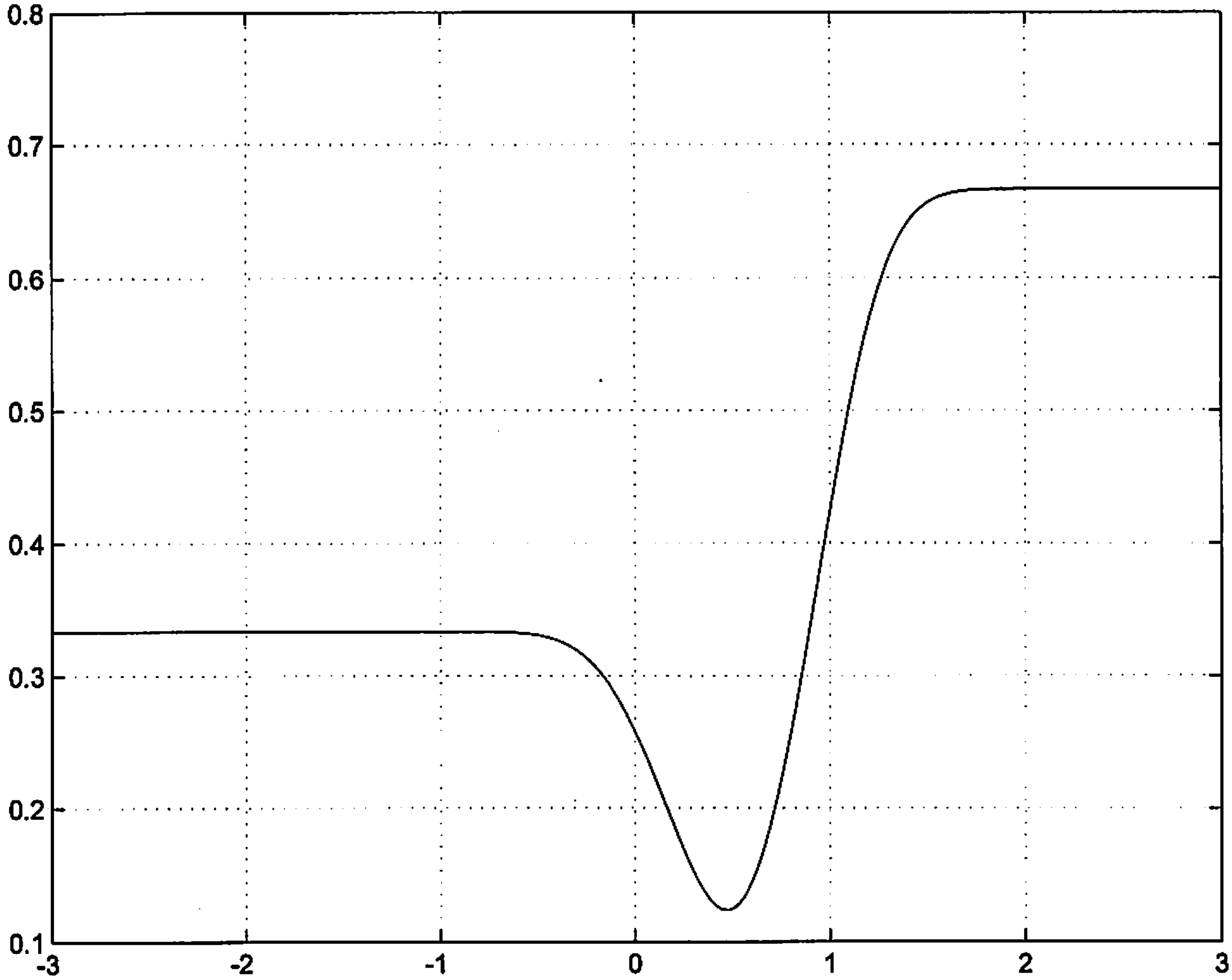
$$\mathbb{P}(\text{bits}=\text{"10"}, w\geq r) = \frac{2}{9}\mathbb{P}\left(v/4 \geq r-\tfrac{1}{3}\right) = \frac{2}{9}\left(1-F_V(4r-4/3)\right)$$

To MINIMIZE BER, MINIMIZE

$$\frac{2}{9}F_V(4r-8/3) + \frac{4}{9}F_V(4r-4) - \frac{1}{9}F_V(4r) - \frac{2}{9}F_V(4r-4/3) + 2/9$$

Min. of ≈ 0.1238 at $r \approx 0.4715$

(SEE THE PLOT ON THE NEXT PAGE)


0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
-3
-2
-1
0
1
2
3