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6.02
10/14/10

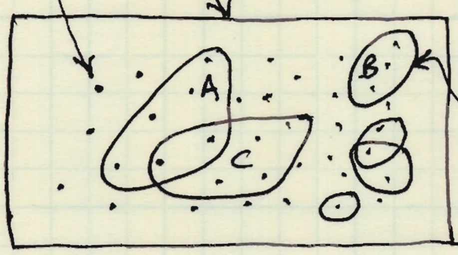
Recs. 10/11 (Medium Access Control)

Since we're back to talking about probabilistic things, it might be worth stating the fundamental ingredients of a probabilistic model more formally:

ψ = 'elementary outcome' of probabilistic model

Ψ = space of elementary outcomes

A, B, C, ... = events, i.e., sets of elementary outcomes.



We say event B 'has occurred' if $\psi \in B$.

Events A and B are called mutually exclusive if they have no outcomes in common, i.e. $\underbrace{A \cap B}_{\text{set intersection}} \neq \underbrace{\phi}_{\text{null (or empty) set}}$.

So the event 'A and B' cannot occur

We require that the collection of events be rich enough that the union of any two events, $\underbrace{A \cup B}_{\text{set union}}$, is itself an event, and similarly the complement $\bar{A}$ of any event A is itself an event.

the event 'A or B'

(It can then be shown that the intersection $A \cap B$ of any two events is an event.)

If these requirements are satisfied, we say we have an event algebra.

Finally, the definition of a probability measure $P(\ )$ on an event algebra: We require

1. $P(A) \geq 0$ for all events A
2. $P(\Psi) = 1$, i.e., the 'certain' event (which always occurs) has probability 1

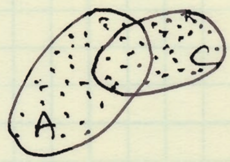
$\Longrightarrow$ and

3. If $A \cap B \neq \phi$, then $P(A \cup B) = P(A) + P(B)$

We use this third defining condition of a probabilistic measure very often! — the probability that one or the other of a pair of mutually exclusive events occurs is just the sum of the individual probabilities. (More generally, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.)

With the fundamental ingredients of a probabilistic model now defined, we can add on some more structure:

Definition of <u>conditional</u> <u>probability</u> $\underbrace{P(C|A)}$



i.e., we know the outcome is in A; what is the probability it is in C too?

Prob. of event C having occurred, given that A has occurred, or "Prob. of C, given A".

Natural definition: $P(C|A) = \dfrac{P(C \cap A)}{P(A)}$ (for $P(A) > 0$).

roughly, the fraction of outcomes in A that are also in C.

Finally, A and C are <u>independent</u> if $P(C|A) = P(C)$, i.e., knowing A has occurred does not change the probability that C has occurred. Equivalently, we require

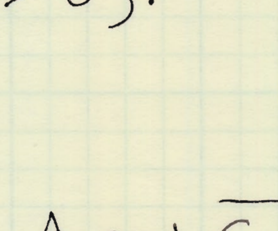
We use this very often, assuming independence

$$P(C \cap A) = P(C)P(A)$$

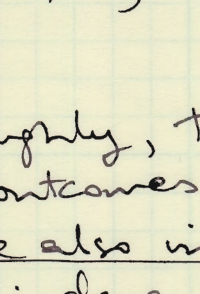
Better definition, allows $P(A)$ or $P(C) = 0$.

i.e. probability of joint event = product of individual probabilities

e.g. take areas of sets as probabilities in



, e.g.



/// = A, $P(A) = 1/2$

\\\ = C, $P(C) = 1/2$

= $A \cap C$, $P(A \cap C) = \frac{1}{4}$

OK, time to apply some of this! Hints for problems:

1. A) Slotted ALOHA, N nodes. Unused slot has probability $(1-p)^N$ —— why product? because slots are making their decisions about whether to transmit independently of each other.

1. B) We saw in lecture that utilization U was maximized at $p = 1/N$. Prob (collision) = 1 − Prob (no collision)

$$(1-p)^N \leftarrow \text{nobody attempts or}$$
$$+ \; Np(1-p)^N \quad \text{only one user attempts}$$

Note that we are invoking both the independence of the users and the mutual exclusivity of successful use, in order to obtain this second term.

Now use $\lim_{N\to\infty}\left(1-\frac{1}{N}\right)^N = e^{-1}$

Recall $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

so $e^{-1/N} = \underbrace{1 - \frac{1}{N}}_{} + \frac{1}{2N^2} - \frac{1}{3!N^3} + \dots$

dominates as $N \to \infty$,

so $\left(1-\frac{1}{N}\right)^N \to \left(e^{-1/N}\right)^N = \frac{1}{e}$

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2. $p_A = 1.5\, p_B$

$U = p_A(1-p_B) + p_B(1-p_A)$

How have we used independence & mutual exclusivity assumptions to write this?

Rewrite U in terms of one of the probs., then maximize.

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3. B has probability $p_b(1-p_a)$ of being successful,

A " " $p_a(1-p_b)$ " " ".

We want the first expression to be twice the second, with $p_a = .25$.

4. The access point is just like one more user, but prob. p_a (rather than $\frac{1}{N}$) of attempting to xmit. So prob of successful transmission is what?

For part C, compare the two terms in your expression for utilization.

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5. A) Slotted ALOHA with 4 users, in effect.

B) In TDMA, each user is given a slot in each cycle of operation, even though only 4 will use a slot in each cycle.

6. In each cycle, what fraction of time is devoted to data transmission, relative to total time on data plus overhead (token passing).

i.e., supporting but not directly useful work.

Let's try Prob 5 of Ch 10 notes → Prob 4 of previous page, with the following modifications:

1 access point, N client nodes

no packets to send

all backlogged, attempting xmit with prob p

Correct transmission if one or two users take a given slot, otherwise interference (failed transmission).

a) Find probability P_2 of exactly 2 of the nodes (any 2) attempting transmission in a slot. Ans: $\binom{N}{2} p^2 (1-p)^{N-2}$

$\frac{N(N-1)}{2}$

b) What is the utilization?

First note that max rate $= 2$ packets/slot.

What is expected number of packets under above protocol?

Another notion from probability!

→ $1 \cdot P_1 + 2 \cdot P_2$

prob. of 1 packet $= Np(1-p)^{N-1}$.

So $U = \dfrac{P_1 + 2P_2}{2}$

For a random variable X that takes values x_i with probability p_i (a discrete random variable), $E[X] = \sum_i p_i x_i$.

For a (continuous) random variable Y with PDF $f_Y(y)$,

$$E[Y] = \int_{-\infty}^{\infty} f_Y(y)\, y \, dy.$$

(Intuition: center of (probability) mass)

c) Find p for max U.

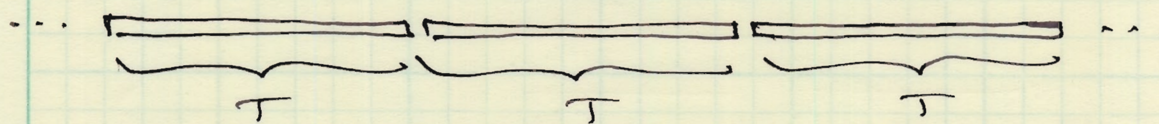
For such problems, it is often easier to max $\ln(U)$. Note

$\frac{d(\ln p)}{dp} = \frac{1}{p}$; $\frac{d \ln(1-p)}{dp} = -\frac{1}{1-p}$

Section 10.6 of notes:

Analysis of slotted ALOHA, but with packet lengths that can be several slot lengths:

Take slot length = 1, packet length = T (integer ≥ 1) and assume the slotted ALOHA protocol, i.e., each of N backlogged users (independently) begins a transmission in a given slot with probability p (assumed same for all users). As a benchmark for comparison, consider what full utilization would mean:



i.e., packets end-to-end. So if we looked at a random slot, the probability of finding the header (first slot) of a T-slot packet would be $1/T$. Now what would be this probability under slotted ALOHA? — it's the probability that either user 1 or user 2 or ... or user N starts a successful transmission in this randomly chosen slot — these are mutually exclusive possibilities, because only one user can successfully use the slot, so we can add their individual probabilities. Therefore, just compute it for user 1, and multiply by N.

Probability that user 1 starts a successful transmission in this randomly selected slot

= Prob (user 1 starts a transmission and encounters no interference from other users with overlapping packets, and no interference from packets user 1 initiated at other overlapping times)

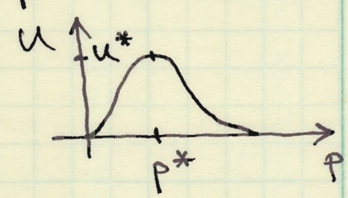
$$= p\left[(1-p)^{N(2T-2)}(1-p)^{N-1}\right] = p\cdot(1-p)^{N(2T-1)-1}$$

This product form is the result of independence assumptions!

Note: Our simple probabilistic model allows <u>self</u>-interference, which any reasonable user will know how to avoid, because it presumably knows it has a packet being transmitted currently! We make this assumption to preserve the analytical tractability of the model (by preserving independence of what happens at the different time slots). The result of our analysis should be a (good) <u>lower bound</u> on actual performance — our model is susceptible to transmission failures (due to self-interference) that a more realistic scheme would not have.

$$\text{So:}\quad U = \frac{Np(1-p)^{N(2T-1)-1}}{1/T} = T\underbrace{Np(1-p)^{N(2T-1)-1}}$$

$= 0$ at $p = 0$ and $p = 1$, smooth and positive in between:



What p maximizes this?

Simplest to set $\frac{d}{dp}(\ln U) = 0$

i.e. $$\frac{1}{p^*} - \frac{N(2T-1)-1}{1-p^*} = 0$$

or $$p^* = \frac{1}{N(2T-1)}$$ (reduces to $\frac{1}{N}$ when $T = 1$, as one would hope)

Corresponding U^*

$$= \frac{T}{2T-1}\underbrace{\left(1 - \frac{1}{N(2T-1)}\right)^{N(2T-1)-1}}$$

As a check, note that we get our familiar expression when $T = 1$.

of the form $$\left(1 - \frac{1}{m}\right)^{m-1} = \frac{\left(1 - 1/m\right)^m}{1 - 1/m}$$

so $\lim_{m\to\infty} = \frac{1}{e}$, as before

For large N,

$$U^* \longrightarrow \frac{T}{2T-1}\,\frac{1}{e}$$

For $T = 1$, $U^* \to 1/e$;

for $T \gg 1$, $U^* \to 2/e$

(like <u>un</u>slotted ALOHA)