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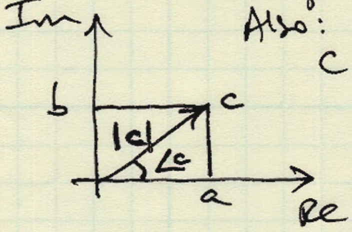
Rec. 12

Discrete-time sinusoids & frequency response of LTI systems

Before getting to DT sinusoids, some facts recalled from __complex numbers__/functions:

Complex number $c = a + jb$ ← real; $j = \sqrt{-1}$ ← electrical engineers save 'i' for current!

Also: $c = |c| e^{j\angle c}$



where $|c| = \sqrt{a^2+b^2}$ ← __Magnitude__ of c

$\angle c = \arctan \frac{b}{a}$ ← in $[-\pi, \pi]$ ← __angle__ of c

arctan is also called atan2

And where did the $e^{j\angle c}$ come from?

→ Euler's formula: ← also Cotes, & deMoivre

$$e^{j\alpha} = \cos\alpha + j\sin\alpha$$

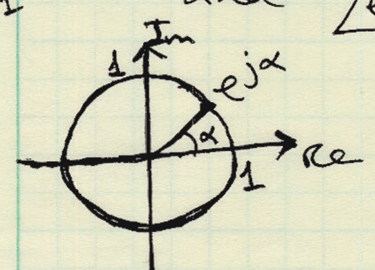
(prove by equating the Taylor series on both sides)

(we'll assume α is real when we see/use this formula, though it holds for complex α too!)

__Note__: $|e^{j\alpha}| = 1$ for any (real) α, ← because $\cos^2\alpha + \sin^2\alpha = 1$

and $\angle e^{j\alpha} = \alpha$ (for α in $[-\pi, \pi]$)

and this yields $c = |c|e^{j\angle c}$



It follows from Euler that

$$\cos\alpha = \frac{e^{j\alpha} + e^{-j\alpha}}{2}; \quad \sin\alpha = \frac{e^{j\alpha} - e^{-j\alpha}}{2j}$$

Also, from $e^{j\alpha} \cdot e^{j\beta} = e^{j(\alpha+\beta)}$ and Euler, we get

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$$

DT sinusoids can be understood in different ways. Useful to think of them as samples of underlying CT (continuous-time) sinusoid taken at regular intervals (the <u>sampling</u> <u>interval</u> T_s) — but some cautions are noted below.

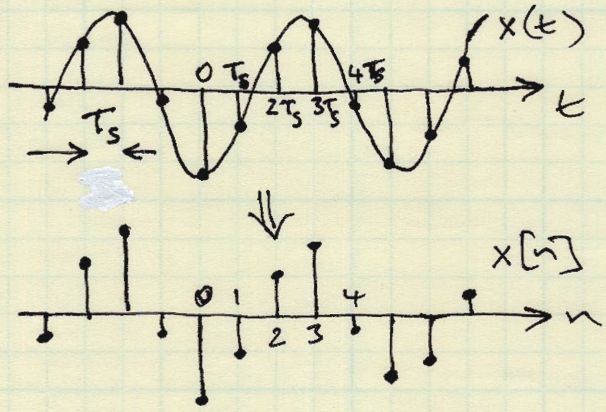
ω_0 is <u>angular</u> frequency, can be +ve or −ve, <u>radians/sec.</u> $x(t)$ has period $\frac{2\pi}{|\omega_0|}$

$$\underbrace{\underbrace{\sin(\omega_0 t + \theta)}_{x(t)}\Big|_{t = nT_s}}_{x(nT_s)} = \sin(\underbrace{\omega_0 T_s}_{\Omega_0} n + \theta)$$

$$= \underbrace{\sin(\Omega_0 n + \theta)}_{} = x[n]$$

DT sinusoid, (angular) <u>frequency</u> Ω_0 (units are <u>radians</u>)

can be +ve or −ve



Note that

$$\sin(\Omega_0 n + \theta)$$
$$= \sin\left(\Omega_0\left(n + \frac{2\pi k}{\Omega_0}\right) + \theta\right)$$

(for $\Omega_0 \neq 0$), but

$\frac{2\pi k}{\Omega_0}$ = integer for some k if & only if

$$\frac{2\pi}{\Omega_0} = \text{rational}$$

So DT sinusoid is periodic <u>as a</u> <u>DT sequence</u> if and only if $\frac{2\pi}{\Omega_0}$ is <u>rational</u> $\Longleftarrow$

i.e., shifting it by some integer gives the same DT signal waveform back

(and the period will be $2\pi k/|\Omega_0|$ for the smallest positive k that makes this an integer, i.e. the numerator of $2\pi/\Omega_0$)

Nevertheless, we refer to $\frac{2\pi}{|\Omega_0|}$ as the <u>period</u> of the DT sinusoid $\sin(\Omega_0 n + \theta)$, no matter what Ω_0 is! This is reasonable if you think of the DT signal as being obtained by sampling an underlying CT signal.

e.g. $\cos\left(\frac{3\pi}{4}n\right)$ has period $\frac{2\pi}{3\pi/4} = \frac{8}{3}$ (but must be shifted by an integer multiple of 8 to look unchanged)

Note that $\sin(\Omega_0 n + \theta) = \sin((\Omega_0 + 2\pi l)n + \theta)$ for any integer l, so DT frequency is only defined to within an integer multiple of 2π. It suffices then to restrict ourselves to Ω_0 in some 2π range. We take $[-\pi, \pi]$. Good idea to always be thinking of Ω_0 in this range — do mod(2π) addition/subtraction to bring it here.

We'll call $[-\pi, \pi]$ the principal range

e.g. $$\cos\left(\frac{3\pi}{2}n + \theta\right) = \cos\left(-\frac{\pi}{2}n + \theta\right)$$

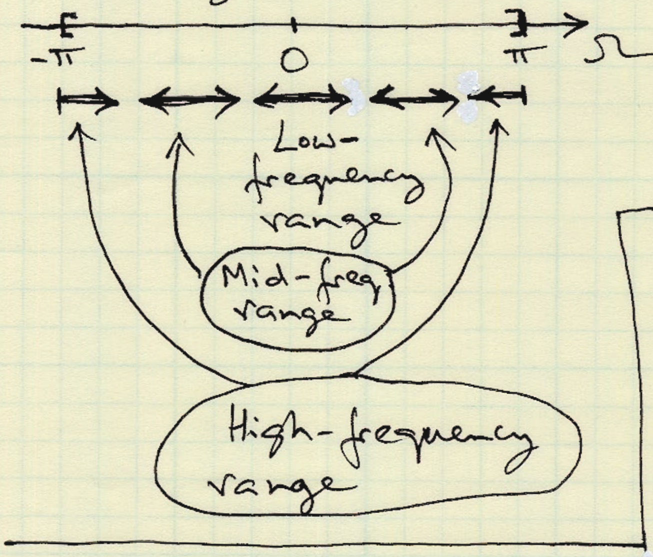
freq $\frac{3\pi}{2}$, equivalent to $\frac{3\pi}{2} + 2\pi l$, pick $l = -1$

Note $\cos(\Omega_0 n)$ for $\Omega_0 = 0$ is 1 for all n, i.e., no variation at all

$\therefore$ slowest frequency $= 0$

but $\cos(\Omega_0 n)$ for $\Omega_0 = \pi$ or $-\pi$, is $(-1)^n$, i.e., varies as fast as a DT signal can vary! So π and $-\pi$ are the highest possible DT frequencies.

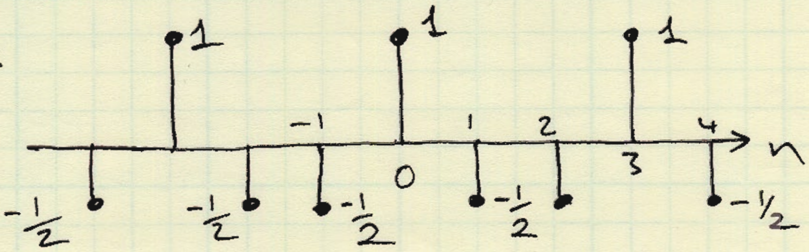
Frequency axis:



Example $\cos\left(\frac{2\pi}{3}n\right) = x[n]$, freq $\frac{2\pi}{3}$ is already in $[-\pi, \pi]$, period $= 3$. Evaluating it for a few values of n, we find $x[0] = 1$, $x[-1] = -\frac{1}{2}$, $x[1] = -\frac{1}{2}$. This is all we need, since we know the signal has period 3, because now we just replicate this every 3 time steps:

→

$\cos\left(\frac{2\pi}{3}n\right)$:



May not seem very sinusoidal! — but you can imagine these as samples of $\cos\left(\frac{2\pi}{3}t\right)$ at integer times (i.e., $T_s = 1$).

One last thing on DT sinusoids: aliasing. Samples of $\sin(\omega_0 t + \theta)$ at $t = nT_s$ will coincide with samples of $\sin\left(\underbrace{\left(\omega_0 + \frac{2\pi}{T_s}\ell\right)}_{\omega_\ell} t + \theta\right)$ at $t = nT_s$. In other words, there's an infinite number of sinusoids in CT whose samples at integer multiples of a given T_s will give the DT sinusoid $\sin(\underset{\omega_0 T_s}{\Omega_0} n + \theta)$ — we say that all these CT sinusoids alias to the same DT signal.

integer ℓ

The lowest-frequency CT sinusoid that will yield these samples corresponds to the one for which $\underbrace{\left(\omega_0 + \frac{2\pi}{T_s}\ell\right)}_{\omega_\ell} T_s$ lies in $[-\pi, \pi]$ i.e. do mod$\left(\frac{2\pi}{T_s}\right)$ addition/subtraction to bring ω_0 to the range $\left[-\frac{\pi}{T_s}, \frac{\pi}{T_s}\right]$.

OK, LTI systems acting on sinusoids, finally!

$$x[n] = \cos(\Omega_o n) \longrightarrow \boxed{\text{LTI, } h[n]} \longrightarrow y[n] = \sum_m h[m]\cos(\Omega_o(n-m))$$

For a slightly different development than in the notes, and a very direct approach, note

$$\cos(\Omega_o(n-m)) = \cos(\Omega_o n)\cos(\Omega_o m) + \sin(\Omega_o n)\sin(\Omega_o m)$$

Substitute in the above convolution, and pull terms not involving m out of the summation:

$$y[n] = \underbrace{\left(\sum_m h[m]\cos(\Omega_o m)\right)}_{\text{Some number, call it } C \text{ (for 'cosine terms')}}\cos(\Omega_o n) + \underbrace{\left(\sum_m h[m]\sin(\Omega_o m)\right)}_{\text{Some number, call it } S \text{ ('sine terms')}}\sin(\Omega_o n)$$

Note, C and S both depend on Ω_o, so should actually write as $C(\Omega_o)$, $S(\Omega_o)$.

You'll come to know & love $H(e^{j\Omega})$ as the DT Fourier transform of $h[n]$

This is the same notation as in 6.003, 6.011

Now recognize

$$y[n] = \sqrt{C^2+S^2}\left(\frac{C}{\sqrt{C^2+S^2}}\cos(\Omega_o n) + \frac{S}{\sqrt{C^2+S^2}}\sin(\Omega_o n)\right)$$

Call this Eq. (*)

To relate this to notation introduced in notes:

$$H(e^{j\Omega}) = \sum_m h[m]e^{-j\Omega m} \quad \text{by definition}$$

Frequency response of system

$$= C(\Omega) - jS(\Omega) \qquad \text{Eq. (**)}$$

$$\Longrightarrow y[n] = |H(e^{j\Omega_o})|\cos(\Omega_o n + \angle H(e^{j\Omega_o}))$$

put (**) in (*)

i.e., output is cosine at same frequency as input, with amplitude scaled by magnitude of frequency response at the frequency of the input, and phase increased by angle (phase) of frequency response at the frequency of the input.

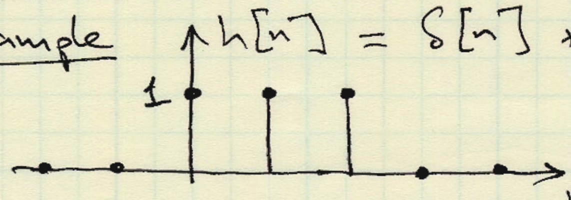
To confirm the preceding expression, note $|H(e^{j\Omega_0})| = \sqrt{C^2+S^2}$

and $\cos\left(\Omega_0 n + \angle H(e^{j\Omega_0})\right) = \cos\angle H(e^{j\Omega_0}) \cos(\Omega_0 n)$

$$- \sin\angle H(e^{j\Omega_0}) \sin(\Omega_0 n)$$

but $\cos\angle H(e^{j\Omega_0}) = \dfrac{C}{\sqrt{C^2+S^2}}$; $\sin\angle H(e^{j\Omega_0}) = \dfrac{-S}{\sqrt{C^2+S^2}}$.

Example $h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$, as shown.



Then $H(e^{j\Omega})$
$= 1 + e^{-j\Omega} + e^{-2j\Omega}$.

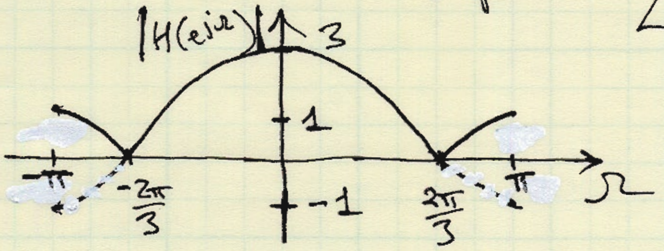
A good way to determine the magnitude and angle of this particular $H(e^{j\Omega})$ is to rewrite it as

$$H(e^{j\Omega}) = e^{-j\Omega}\left(e^{j\Omega} + 1 + e^{-j\Omega}\right)$$

$$= e^{-j\Omega} \cdot (1 + 2\cos\Omega)$$

So $|H(e^{j\Omega})| = \underbrace{|e^{-j\Omega}|}_{=1} \cdot \underbrace{|1 + 2\cos\Omega|}_{\Downarrow \text{ plot}}$, and let's not worry for now about $\angle H(e^{j\Omega})$

If
$c_1 = |c_1| e^{j\angle c_1}$
$c_2 = |c_2| e^{j\angle c_2}$
then
$c_1 c_2 = \underbrace{|c_1| \cdot |c_2|}_{\text{magnitude of product = product of magnitudes}} e^{j\overbrace{(\angle c_1 + \angle c_2)}^{\text{angle of product = sum of angles}}}$



— So if $x[n] = 1$ for all n (i.e., sinusoid at 0 frequency), $y[n] = 3$ (easily checked by convolution).

— If $x[n] = \cos\left(\frac{2\pi}{3} n\right)$, then $y[n] = 0$ (again check by convolution, since we've determined $x[n]$ explicitly on p.4). What if $x[n] = \sin\left(\frac{2\pi}{3} n + \frac{\pi}{17}\right)$? $\rightarrow y[n] = 0$.

— If $x[n] = \cos(\pi n + 0.3)$, then $y[n] = \cos(\pi n + 0.3 + \angle H(e^{j\pi}))$ and $\angle H(e^{j\pi}) = \angle e^{-j\pi} + \angle 1 + 2\cos\pi = -\pi + \pi = 0$, so $y[n] = x[n]$. Check by convolution.