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# Rec 14

## Discrete-time Fourier series (DTFS)

We know

$$x[n] = e^{j\Omega_0 n} \longrightarrow \boxed{H(e^{j\Omega})}_{\text{LTI}} \longrightarrow y[n] = H(e^{j\Omega_0})e^{j\Omega_0 n}$$

with $H(e^{j\Omega}) = \sum_m h[m]e^{-j\Omega m}$

But how useful is this? Well, for an LTI system we can immediately invoke superposition to say:

$$(*) \rightarrow \quad x[n] = \sum_k A_k e^{j\Omega_k n} \longrightarrow \boxed{H(e^{j\Omega})} \longrightarrow y[n] = \sum_k A_k H(e^{j\Omega_k})e^{j\Omega_k n}$$

Yes, but it would seem most inputs of interest won't be of the form $x[n] = \sum_k A_k e^{j\Omega_k n}$ — so how useful is this?

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**Key Fact**: Over any interval of length N (take instants 0, 1, 2, ..., N−1 for concreteness), an arbitrary signal x[n] can be written in the form

the DTFS

$$x[n] = \sum_{k=-N/2}^{\frac{N}{2}-1} X[k]e^{j\frac{2\pi k}{N}n} \quad \leftarrow (**)$$

(we are assuming N is even, for convenience).
So comparing with Eq. (*) above, we have
$A_k = X[k]$, $\Omega_k = \frac{2\pi k}{N}$, and $k = [-\frac{N}{2}, \frac{N}{2}-1]$.
(evenly spaced in $[-\pi, \pi)$)

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We'll return to focus on Eq. (**), but let's first see how we would use this fact in studying the input → output behavior of an LTI system.

Can handle other intervals similarly

Assume the input $x[n]$ to an LTI system takes nonzero values only on the interval $[0, n_x]$, and that the unit sample response $h[n]$ of the system takes nonzero values only on the interval $[0, n_h]$. Then $y[n]$, the output, will take nonzero values only on $[0, \underbrace{n_x + n_h}]$.

pick this to be $N-1$,

i.e., $N = n_x + n_h + 1$

can be > too, if convenient (e.g., to get an even $N$ — or a power of 2, as is convenient for the 'Fast Fourier Transform' (FFT) algorithm for DTFS computation)

OK, so here's the strategy:

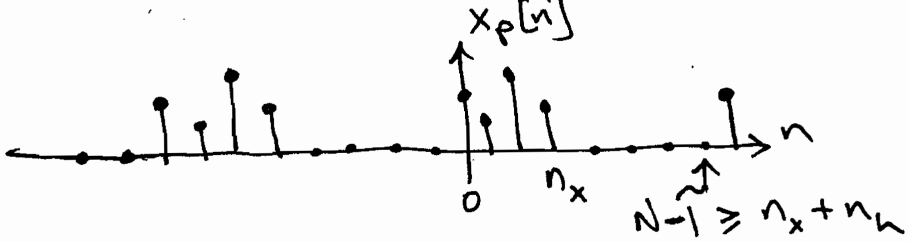
Represent $x[n]$ over the interval $[0, N-1]$ by the expression (**), i.e., its DTFS:

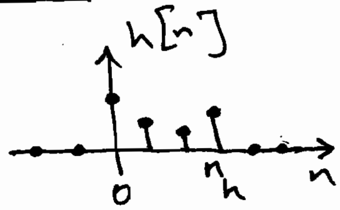
$$x[n] = \sum_{\cancel{-\frac{N}{2}}}^{\cancel{\frac{N}{2}} N-1} \underbrace{X[k] e^{j\frac{2\pi k}{N} n}}$$

referred to as $k$-th frequency component of $x[n]$, or component of $x[n]$ at frequency $\Omega_k = \frac{2\pi k}{N}$.

What is this expression for $n$ outside $[0, N-1]$? $\Longrightarrow$ $N$-periodic repetition of the $x[n]$ we had in $[0, N-1]$, so definitely not the $x[n]$ that we have, but rather its $N$-periodic extension. Let's denote it by $x_p[n]$.

If $x_p[n]$ is the input to $\longrightarrow \boxed{h[n]} \longrightarrow$ ,




the corresponding output is

$$y_p[n] = \sum_{-N/2}^{\frac{N}{2}-1} X[k] H(e^{j\Omega_k}) e^{j\Omega_k n}, \qquad \Omega_k = \frac{2\pi k}{N}$$

also periodic, period N

Is it possible to determine $y[n]$ from $y_p[n]$? — yes! it's just the portion between 0 and $N-1$, i.e.,

$$(***) \longrightarrow y[n] = \sum_{-N/2}^{\frac{N}{2}-1} X[k] H(e^{j\Omega_k}) e^{j\Omega_k n}, \quad \Omega_k = \frac{2\pi k}{N}$$

There's no interference from response to periodic replicates of $x[n]$, because of how N was chosen. $\longrightarrow$ for $n$ in $[0, N-1]$, and is zero outside this!!

Again, why (or when) might this be a more helpful or useful or insightful way of studying input-output behavior, compared to straight convolution?

One situation: $x[n]$ has components at many (or all N) frequencies $\Omega_k = \frac{2\pi k}{N}$, with $k = [-\frac{N}{2}, \frac{N}{2}-1]$, but the frequency response of the system is frequency-selective, i.e., (essentially) magnitude zero in some frequency intervals in $[-\pi, \pi)$. Then $y[n]$ will only have some or a few frequency components, most easily found from (***).

e.g. lowpass or bandpass or highpass