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6.02
11/4/10

Rec. 15

Recapping some earlier material:

Periodic DT sequences

- $\cos \Omega_0 n$ — we refer to $2\pi/\Omega_0$ as the period of the cosine
- So $\cos \frac{\pi n}{6}$ has period 12
- & $\underbrace{\cos \frac{3\pi}{4} n}_{x[n]}$ has period $\frac{8}{3}$ ← only in the sense that $\underbrace{\cos \frac{3\pi}{4} t}_{\text{CT cosine of period } 8/3} \Big|_{t=n, \text{ integer}} = \cos \frac{3\pi}{4} n$

If we are looking for the smallest positive integer N such that $x[n+N] = x[n]$ for all n, then we need to pick $N = 8$ for $\cos\left(\frac{3\pi}{4} n\right)$

— note $\cos\left(\frac{3\pi}{4}(n+8)\right) = \cos\left(\frac{3\pi}{4} n\right)$, all n, and no smaller positive integer will accomplish this.

- More generally, if $\frac{2\pi}{\Omega_0}$ is rational (expressed as ratio of 2 integers that have no common factors) then period N = numerator of the rational
- Period of $\cos \frac{\pi}{6} n + \cos \frac{3\pi}{4} n$? → Least common multiple of 12 & 8, i.e., 24
- Period of $\cos n$? $\Rightarrow \frac{2\pi}{\Omega_0} = \frac{2\pi}{1} \neq$ rational $\Rightarrow$ aperiodic sequence

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* Incorporating suggestions from Prof. Megretski

The DT Fourier series, or 'spectral decomposition' of a signal:

Two ways to think of the DTFS:

1. Given an arbitrary signal $x[n]$ at $n = 0, 1, \ldots, N-1$, we can always write it as

for n in $[0, N-1]$ $\Rightarrow$
$$\boxed{x[n] = \sum_{k=-K}^{K-1} X[k]\, e^{j\Omega_k n} \quad (**)}$$

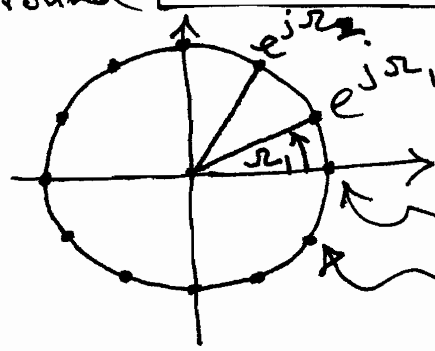
assume N is even

where $K = \frac{N}{2}$, $\Omega_k = k\underbrace{\left(\frac{2\pi}{N}\right)}_{} = k\Omega_1$

Ω_1 — 'fundamental frequency'

i.e., weighted sum of exponentials at frequencies evenly spaced in the interval $\underbrace{-\Omega_K}_{-\frac{N}{2}\cdot\frac{2\pi}{N} = -\pi}$ to $+\underbrace{\Omega_{K-1}}_{\pi - \Omega_1}$,

or equivalently, powers of exponentials $e^{j\Omega_k}$ spaced uniformly around the unit circle:



> The claim in (**) requires a proof, not hard, but we omit it here.

We refer to $X[k]$ in (**) as the k-th Fourier coefficient or k-th spectral coefficient, and $X[k]e^{j\Omega_k n}$ as the k-th spectral component of the signal $x[n]$ over $[0, N-1]$.

The relation (**) is termed the Fourier SYNTHESIS equation, as it 'synthesizes' $x[n]$

in the interval $[0, N-1]$ in terms of a sum of spectral components, at N frequencies evenly spaced in $-\pi$ to π.

A formula for the coefficients in (**) is easy to obtain (see annotated lecture slides for proof):

$$\boxed{(***)\quad X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\Omega_k n}}$$

← note that it uses only the values $x[n]$ for n in $[0, N-1]$.

This is the Fourier ANALYSIS equation.

Note that $X[k]$ is in general complex, but

(*) real →

$$X[0] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$$ → average value of $x[n]$ in the window $[0, N-1]$

$$X[-K] = \frac{1}{N}\sum (-1)^n x[n]$$

Also $X[-k] = X^*[k]$ (complex conjugate of $X[k]$) because changing k to $-k$ in (***) is equivalent to changing j to $-j$, i.e., taking the complex conjugate.

So (if you count up now), the N values of $x[n]$ for n in $[0, N-1]$ determine the 2 real values $X[0]$ and $X[-K]$, and the $N-2$ values that fix the real and imaginary parts of $X[1]$, $X[2], \ldots, X[K-1] \left(= X[\tfrac{N}{2}-1]\right)$, i.e., N values in all — using (***). And (**) goes the other way, using the N values that determine the $X[k]$, for k in $[-K, K-1]$, to fix $x[n]$.

A plot of $X[k]$ as a function of k or Ω_k is called a <u>spectral plot</u>; it displays the <u>spectrum</u> of $x[n]$, n in $[0, N-1]$. Because of (*),

$Re\{X[k]\}$ is an <u>even</u> function of k,
$Im\{X[k]\}$ is an <u>odd</u> function of k
(and $Im\{X[0]\} = Im\{X[-K]\} = 0$).
(We typically include $X[K]$ in the plot too, noting from (***) that $X[K] = X[-K]$.)

<u>e.g.</u> $x[n] = \cos\frac{\pi n}{6} + \sin\frac{3\pi n}{4}$

for n in $[0, N-1]$,

and let's defer picking N for now.

Then $x[n] = \frac{1}{2}\left(e^{j\frac{\pi n}{6}} + e^{-j\frac{\pi n}{6}}\right) + \frac{1}{2j}\left(e^{j\frac{3\pi}{4}n} - e^{-j\frac{3\pi}{4}n}\right)$

If we pick $N=24$, then

$\frac{\pi n}{6} = \frac{2\pi \cdot 2n}{24}$, and $\frac{3\pi n}{4} = \frac{2\pi}{24} \cdot 9n$ } $N=24$ is the smallest integer for which we can do this.

So we have

$$x[n] = \sum_{k=-K}^{K-1} X[k]\, e^{j\Omega_k n}, \quad \text{with } K = 12$$

$N = 24$

$\Omega_1 = \frac{2\pi}{24}$,

$\Omega_k = k\Omega_1$

Comparing these 2 eqns.

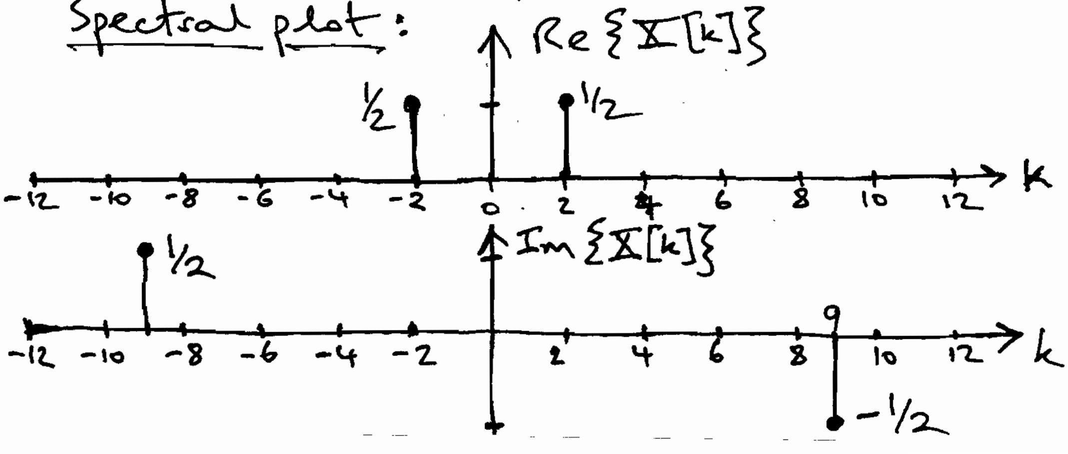
and $X[2] = \frac{1}{2} = X[-2]$

$X[9] = \frac{-j}{2} \Rightarrow X[-9] = X^*[9] = \frac{j}{2}$

all other $X[k] = 0$.

So we have found $X[k]$ without needing to use the analysis formula (***) on p. 3.

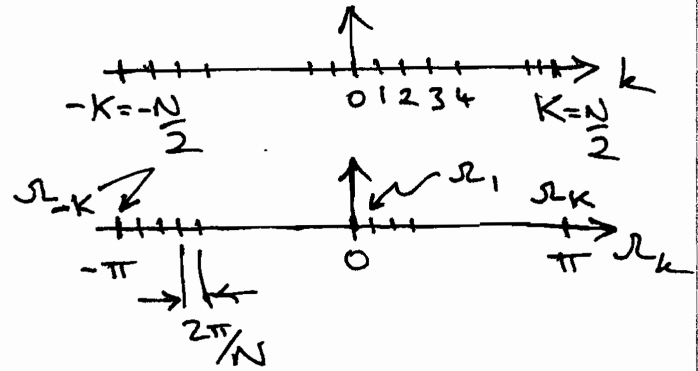
<u>Spectral plot</u>:



Could have plotted magnitude and angle, but Re{ } and Im{ } is better for modulation studies.

Note: $\mathrm{Re}\{X[k]\}$ reflects 'cosine-like' content and $\mathrm{Im}\{X[k]\}$ reflects 'sine-like' content. They play perfectly equal roles, and we use complex algebra to carry the sine and cosine stories along simultaneously (convenient & necessary, because phase shifts turn sines to cosines and vice versa).

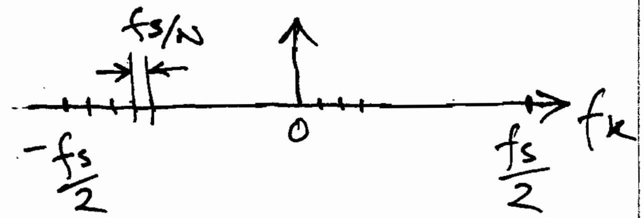
Instead of plotting $X[k]$ versus k, we could also have plotted it versus Ω_k, ranging upward from $-\pi$ in steps of $\frac{2\pi}{N}$, going to $\pi - \frac{2\pi}{N}$ (or include π, no problem):



Another possibility is to plot versus the frequency f_k of a CT sinusoid, e.g., $\sin(2\pi f_k t)$, whose samples taken at rate f_s samples per second will yield a DT sinusoid of angular frequency Ω_k. Noting that

$$\sin(2\pi f_k t)\Big|_{t=\frac{n}{f_s}} = \sin\Big(\underbrace{2\pi \frac{f_k}{f_s}}_{\Omega_k} n\Big)$$

we see f_k ranges from $-f_s/2$ to $f_s/2$, in steps of f_s/N:



In general, a signal $x[n]$ defined in $[0, N-1]$ will have spectral components at all N frequencies Ω_k (from $-\pi$ to $\pi - \frac{2\pi}{N}$, increments of $2\pi/N$).

e.g. $x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$, $N \geq 3$.

$$\Rightarrow X[k] = \frac{1}{3}\left(1 + e^{-j\Omega_k} + e^{-j2\Omega_k}\right)$$

$$= \frac{1}{3} e^{-j\Omega_k}\left(1 + 2\cos\Omega_k\right)$$

So

$$\left.\begin{aligned} \mathrm{Re}\{X[k]\} &= \frac{1}{3}\cos\Omega_k\,(1 + 2\cos\Omega_k) \\ \mathrm{Im}\{X[k]\} &= -\frac{1}{3}\sin\Omega_k\,(1+2\cos\Omega_k) \end{aligned}\right\}\text{Try plotting!}$$

A side comment: The analysis formula in (***), p. 3, is very reminiscent of the formula for the frequency response of an LTI system. Suppose $h[n]$ is 0 outside $[0, N-1]$. Then

$$H(e^{j\Omega}) = \sum_{n=0}^{N-1} h[n] e^{-j\Omega n}$$

If DTFS for $h[n]$ is $\sum_{k=-K}^{K-1} H[k] e^{j\Omega_k n}$,

with $H[k] = \frac{1}{N}\sum_{n=0}^{N-1} h[n] e^{-j\Omega_k n}$,

we see $H(e^{j\Omega_k}) = N\,H[k]$,

i.e., frequency response of the system at frequencies Ω_k is related (by scale factor N) to the frequency content of $h[n]$.

Note that $X[k]$ in (***) is completely defined by $x[n]$ for $n = 0, 1, \ldots, N-1$. The values of $x[n]$ outside $[0, N-1]$ are irrelevant, and can be anything. However, the expression on the right side of the synthesis equation (**), p. 2, is periodic, period N — let's refer to it as $x_p[n]$. So $x_p[n]$ is the period-N replication of the values $x[0], x[1], \ldots, x[N-1]$.

On p. 2 we said there were 2 ways to think of the DTFS. The first way was described there, as Item 1. Here's the second way:

2. A periodic sequence $x_p[n]$ of period N can be represented for all n by the expression

$$x_p[n] = \sum_{k=-K}^{K-1} X[k] e^{j\Omega_k n} \quad \left(\text{exactly as in } (**)\right)$$

where

$$X[k] = \frac{1}{N} \sum_{n} x[n] e^{-j\Omega_k n} \quad \left(\text{exactly as } (***)\right)$$

Sum over any period, e.g. $n = [0, N-1]$
or $n = [-K, K-1]$
or ...

Sometimes the perspective in Item 1, p. 2, is more useful (e.g., see Rec 14 notes), at other times the perspective in Item 2 above.