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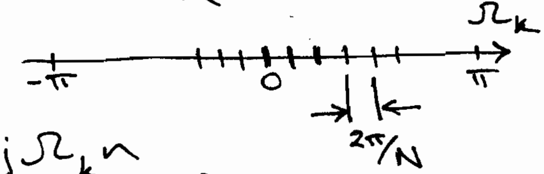
6.02
11/9/10

Rec. 16

- Reminder: Any $x[n]$ on $n = 0, 1, \ldots, N-1$, or any $N$-periodic DT sequence $x_p[n]$ can be written on the interval $[0, N-1]$ as a weighted sum of exponentials $e^{j\Omega_k n}$ at frequencies $\Omega_k$ that are uniformly spaced in $[-\pi, \pi)$ at locations $k\Omega_1$, where $\Omega_1 = \frac{2\pi}{N}$, $k = [-\frac{N}{2}, \underbrace{\frac{N}{2}}_{K} - 1]$:

Assume N is even

$$\left.\begin{array}{c} x[n] \\ \text{or} \\ x_p[n] \end{array}\right\} = \sum_{k=-K}^{K-1} X[k] e^{j\Omega_k n}, \quad (*)$$




where $X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\Omega_k n}$ $(**)$

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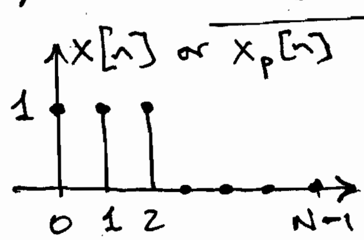
- Small correction to example on p. 6 of Rec. 15 notes:




$N \geq 3$, $X[k] = \frac{1}{N}\left(1 + e^{-j\Omega_k} + e^{-j2\Omega_k}\right)$

using Eq. (**)

(not 3)

$\Omega_k = \frac{2\pi}{N}$

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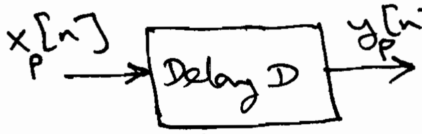
- What does a time shift do to the DTFS?




$y_p[n] = x_p[n-D]$

$$Y[k] = \frac{1}{N} \sum_{n=0}^{N-1} x_p[n-D] e^{-j\Omega_k n}$$

$$= e^{-j\Omega_k D} \frac{1}{N} \sum_{n=0}^{N-1} x_p[n-D] e^{-j\Omega_k (n-D)}$$

$$= e^{-j\Omega_k D} X[k]$$

(as we should have anticipated!)

Freq. resp. $H(e^{j\Omega}) = e^{-j\Omega D}$

More generally:

$$x[n] = \sum_{k=-K}^{K-1} X[k]\, e^{j\Omega_k n} \longrightarrow \boxed{H(e^{j\Omega})} \longrightarrow y[n] = \sum_{k=-K}^{K-1} X[k]\, H(e^{j\Omega_k})\, e^{j\Omega_k n}$$

superposition in action

$$= \sum_{k=-K}^{K-1} Y[k]\, e^{j\Omega_k n}$$

$$\Longrightarrow \boxed{Y[k] = H(e^{j\Omega_k})\, X[k]}$$

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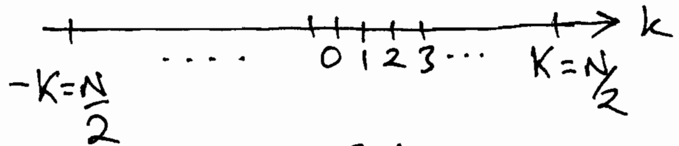
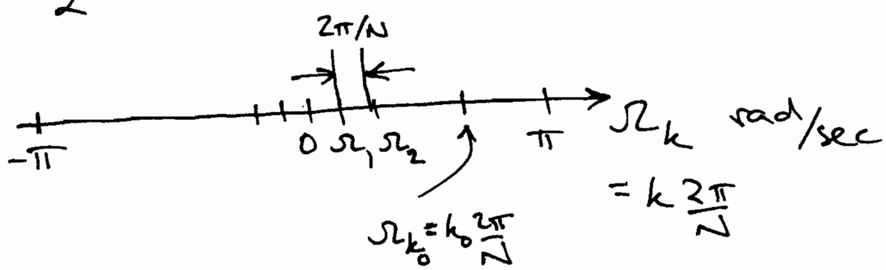
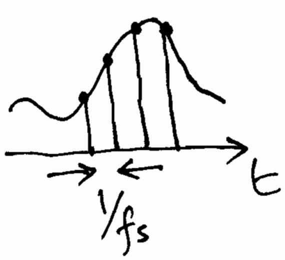
- Reminder on possible choices of abcissa when plotting signal spectrum:

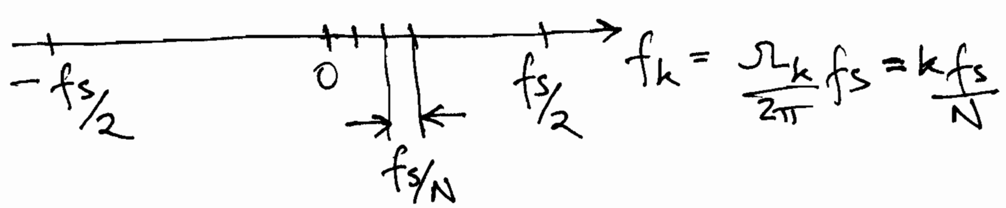



or




or in terms of frequency (in Hz) of underlying CT signal whose samples at frequency $f_s$ samples/sec yield the DT signal under consideration:

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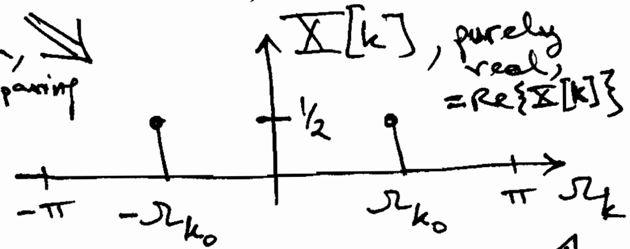
Some more DTFS pairs:

- $x[n] = \cos(\Omega_{k_0} n) = \frac{1}{2}\left(e^{j\Omega_{k_0} n} + e^{-j\Omega_{k_0} n}\right)$

$= k_0 \frac{2\pi}{N}$, <u>integer</u> $k_0$ in $[0, N-1]$

by inspection, comparing with Eq. (*)



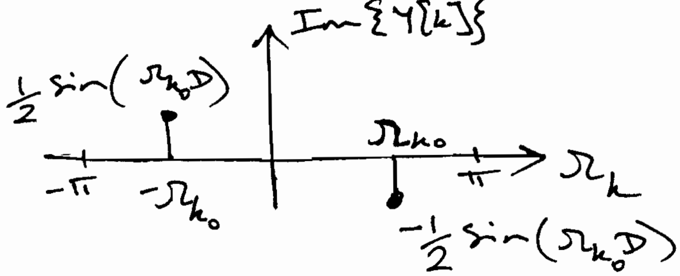

(Could simplify notation and write $\Omega_0$ instead of $\Omega_{k_0}$, but keep this for now)

we have chosen to plot against $\Omega_k = \frac{k2\pi}{N}$ instead of $k$

- $y[n] = \cos(\Omega_{k_0}(n - D)) \longleftrightarrow Y[k] = e^{-j\Omega_k D} X[k]$

So $\mathrm{Re}\{Y[k]\} = \cos(\Omega_k D) X[k]$ from above

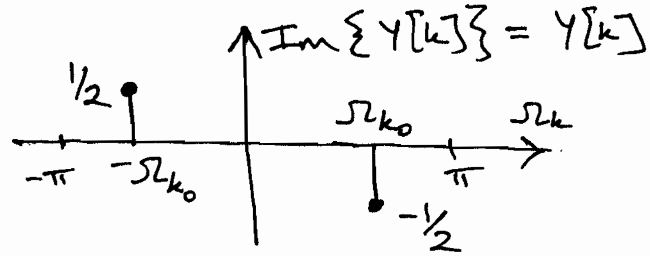
$\mathrm{Im}\{Y[k]\} = -\sin(\Omega_k D) X[k]$


When $\Omega_{k_0} D = \frac{\pi}{2}$, i.e. $\cos \Omega_{k_0} D = 0$ & $\sin \Omega_{k_0} D = 1$, we get:

$y[n] = \cos(\Omega_{k_0} n - \pi/2) = \sin(\Omega_{k_0} n)$

$\longleftrightarrow$ $\mathrm{Re}\{Y[k]\} = 0$ and



# Amplitude Modulation

$y[n] = x[n] \cos(\Omega_c n)$

carrier frequency, assume integer multiple of $\frac{2\pi}{N}$, as before.

in $[0, N-1]$

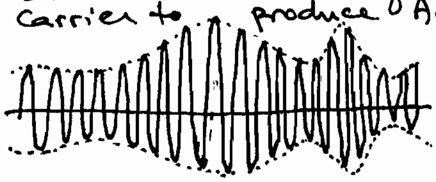
$x[n]$ modulates the amplitude of the carrier, yielding the amplitude-modulated signal $y[n]$.

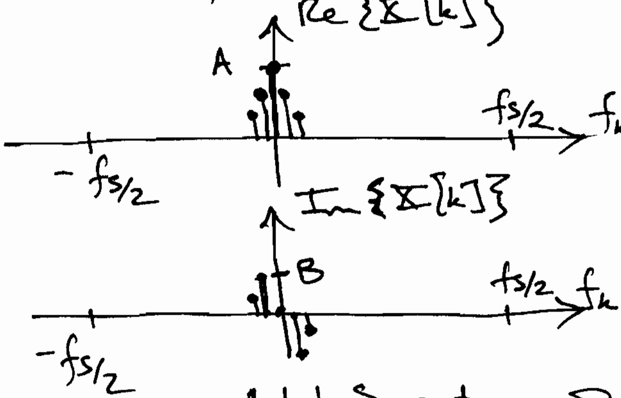
This is (more or less) what goes on in the AM frequency band, 520 kHz – 1710 kHz

Stations (different carrier freqs) spaced 10 kHz apart.

The modulation $x[n]$ is low-frequency. Schematically:

Slow modulation of high-freq. carrier to produce AM signal $y[n]$






Spectrum of $x[n]$ extends $\pm$ 5 kHz in standard AM (adequate for speech, limited for music)

Spectrum of $y[n] = x[n]\cos(\Omega_c n)$ ?

$\Downarrow$

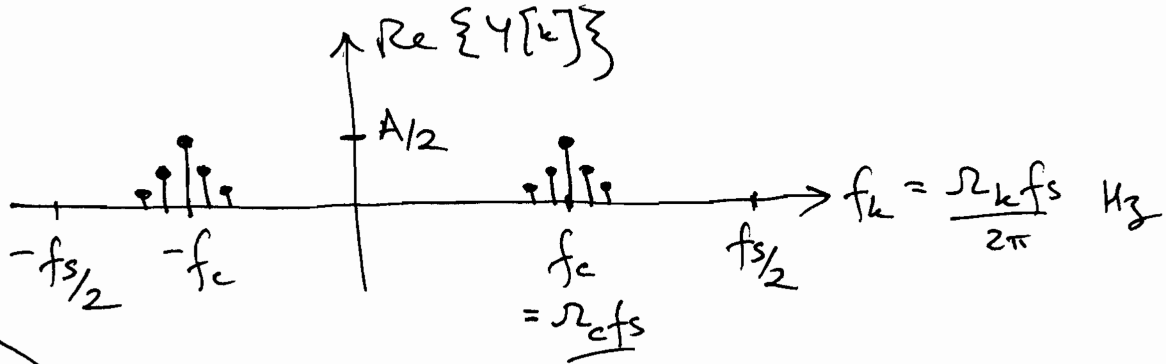
$$x[n] = \sum_{k=-K}^{K-1} X[k] e^{j\Omega_k n},$$

so $$e^{+j\Omega_c n} x[n] = \sum_{k=-K}^{K-1} X[k] e^{j(\Omega_c + \Omega_k)n}$$

so $$y[n] = \frac{1}{2}\left( \text{''} + \sum_{k=-K}^{K-1} X[k] e^{j(-\Omega_c + \Omega_k)n} \right)$$

$\Longrightarrow$

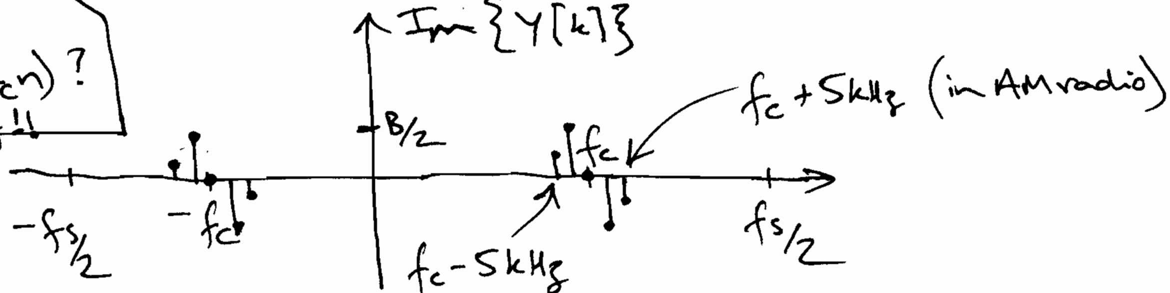
So spectrum of $y[n]$ is same shape as that of $x[n]$, but shifted to be centered at $\pm\Omega_c$, and scaled by $1/2$:




Question

Spectrum of $w[n] = x[n]\sin(\Omega_c n)$?

Try it yourself!!




Question: What is the spectrum of $y[n-D]$?

$$\Longrightarrow e^{-j\Omega_k D}Y[k]$$

, i.e. real part is

$$\cos(\Omega_k D)\,Re\{Y[k]\} + \sin(\Omega_k D)\,Im\{Y[k]\},$$

and imaginary part is

$$\cos(\Omega_k D)\,Im\{Y[k]\} - \sin(\Omega_k D)\,Re\{Y[k]\}$$

So each AM station generates 'baseband' music/speech signal in $\pm 5$kHz range, then modulates it up to be centered on its assigned carrier frequency. Since the carrier frequencies are spaced apart by 10 kHz, stations don't interfere, can share the same airwaves! $\Longrightarrow$ Frequency division multiplexing.

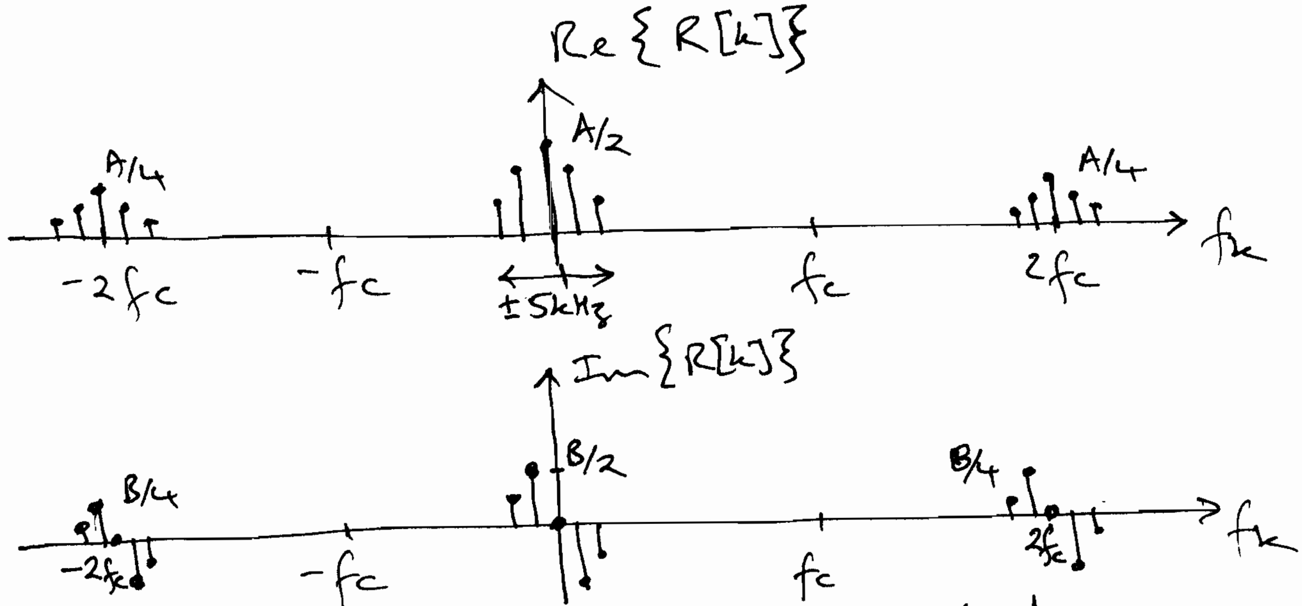
Demodulation:

At the receiver? $r[n] = y[n] \cos(\Omega_c n)$ yields the following spectrum (using the same rule as before: shift the spectrum of $y[n]$ to be centered at $\pm\Omega_c$, scale amplitude by $1/2$):

received signal

from local 'oscillator'

from p 4




i.e., we have managed to get the original baseband spectrum back at baseband again, except for factor of $1/2$, which is easily compensated for by an amplifier (which is anyway needed to make up for attenuation in transmission — much greater than factor of 2!). And the 'stuff' centered at $\pm 2 f_c$ is easily removed by lowpass filtering.

So that's the demodulation story!

(**Question**: What would the spectrum of $y[n] \sin(\Omega_c n)$ look like?
$\Longrightarrow$ zero at baseband!)

So the phase of the local oscillator is important.