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6.02
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Rec 2

(Linearity, time-invariance, causality, channel responses, eye diagrams)

Let's talk first about linearity in the static case, i.e., no time aspect, just a function (or mapping)

$$y = f(x)$$

y: output; x: input

For linearity, we require that if

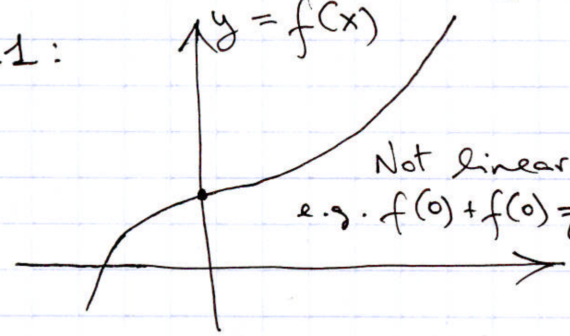
$$f(x_1) = y_1 \quad \text{and} \quad f(x_2) = y_2$$

then

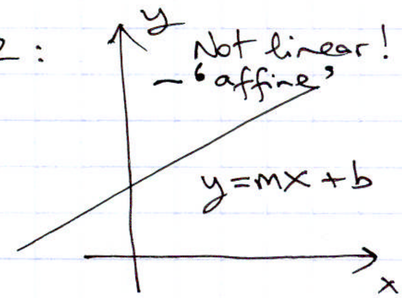
$$f(a_1 x_1 + a_2 x_2) = a_1 y_1 + a_2 y_2, \quad \text{super-position}$$

no matter what values are taken by $\underbrace{x_1, x_2, a_1, a_2}$.

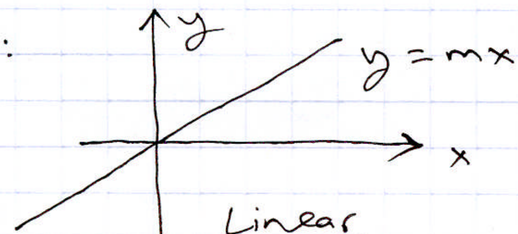
Ex. 1:



Ex. 2:



Ex. 3:

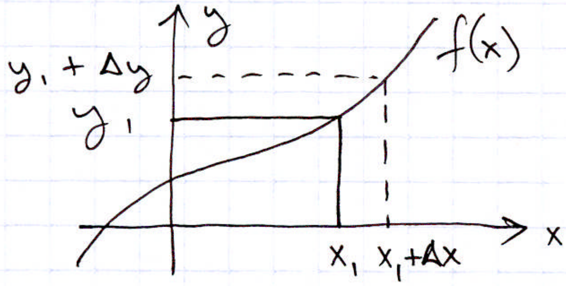


Ex. 4: relation between $\Delta y \ (= y_2 - y_1)$ and $\Delta x \ (= x_2 - x_1$ for corresponding x_1 and $x_2)$

for $y = mx + b$:

$$\Delta y = m \cdot \Delta x, \text{ linear!}$$

Ex. 5: The idea of Ex. 4 (i.e., look at increments from some baseline value) applied to Ex. 1:



$$\Delta y \approx \left.\frac{df(x)}{dx}\right|_{x=x_1} \cdot \Delta x$$

i.e., first-order Taylor series approximation.

This is more generally why we talk so much about linearity: a good representation for small deviations from a fixed baseline. (Plus we can say a lot about linear systems, lots of structure!)

(We could discuss more general mappings similarly, e.g. two variables v, w to two variables y, z:

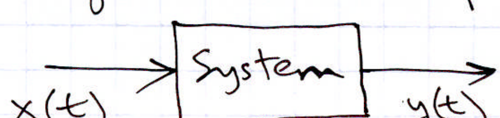
$$y = f_1(v, w), \quad z = f_2(v, w)$$

Again, this pair of mappings is linear if superposition applies. We find that this has to be of the form

$$\begin{bmatrix} y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{matrix } M} \begin{bmatrix} v \\ w \end{bmatrix}$$

for superposition to hold. Compare w. Ex. 3.)

Now consider the <u>dynamic</u> case, of mapping from an input signal to an output signal:



<u>Linearity</u>: $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$

$$\Longrightarrow a_1 x_1(t) + a_2 x_2(t) \longrightarrow a_1 y_1(t) + a_2 y_2(t)$$

for all $\underbrace{x_1(t),\ x_2(t)}_{\text{specific choices of } x(t)}$ and scalars a_1, a_2

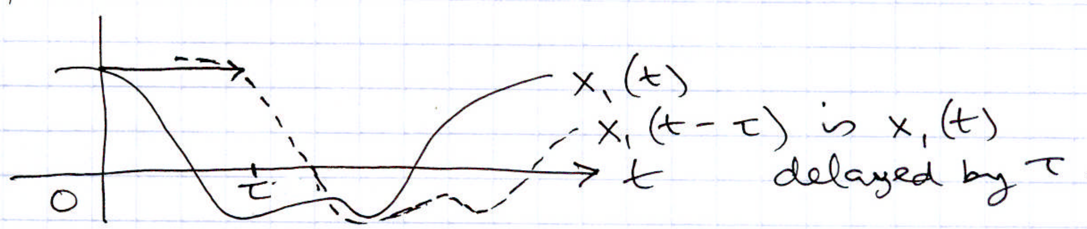
(Can generalize to multiple input signals and output signals.)

Time invariance $\quad x_1(t) \longrightarrow y_1(t)$
(specific $x(t)$) $\quad$ (corresponding $y(t)$)

$\Longrightarrow$

$$x_1(t-\tau) \longrightarrow y_1(t-\tau)$$

for all $x_1(t)$, τ

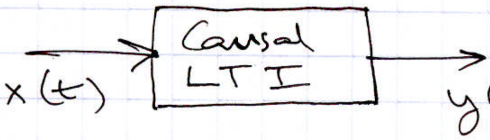


Causality $\quad$ If $x_1(t)$ and $x_2(t)$ are equal for $t < t_0$, then $y_1(t)$ and $y_2(t)$ are equal for $t < t_0$. (Till $t = t_0$, a causal mapping doesn't know that $x_1(t)$ and $x_2(t)$ are not identical.)

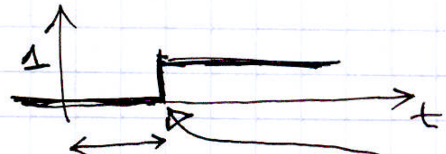
Causality seems like a reasonable assumption to make about a communication channel (but games can be played by buffering to do some quasi-non-causal things!)

Linearity and time-invariance are clearly approximations. e.g. no linearity once your signals are large enough to saturate amplifiers or sensors. But for small perturbations, linearity is reasonable. And for short time horizons, time-invariance may be reasonable.

Causal LTI models for channels:

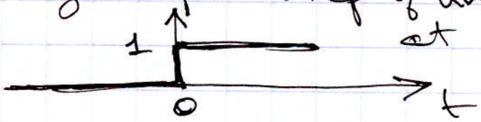


Ideal response:



transmission or propagation delay

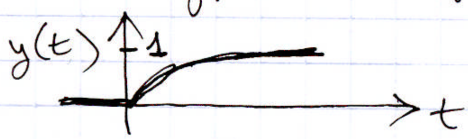
e.g. input step of unit height at $t = 0$.



(Convince yourself that $y(t) = x(t-7)$ is an LTI, causal mapping)

usual to label this as time 0 at receiver, i.e., subtract off transmission delay.

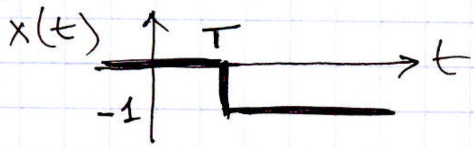
More typical output response



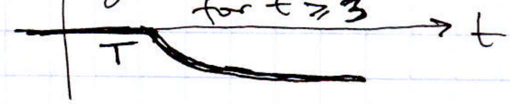
e.g. if channel input and output are related by

$$\frac{dy(t)}{dt} + 3y(t) = 3x(t) \implies y(t) = 1 - e^{-3t} \text{ for } t \geq 0$$

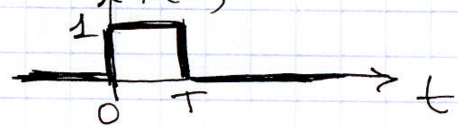
By linearity and time-invariance, response to



is $y(t) = -1 + e^{-3(t-T)}$ for $t \geq 3$

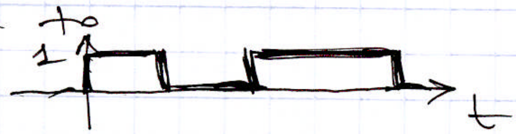


So, by linearity, response to

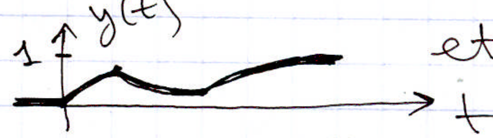


is

and to



is



etc.

__Eye diagram__ helps answer this question: Is there a threshold and a periodic sampling instant such that the rule below works well: sample > threshold $\implies$ 1
sample < threshold $\implies$ 0 ?