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6.02
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## Rec 5 (filling in some further details)

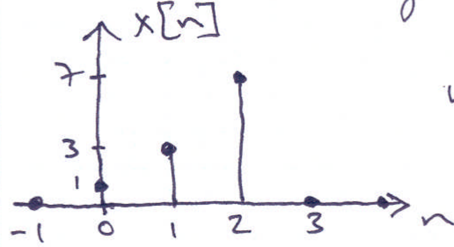
1. A little fun (?!): Find the product of

$X(z) = 1 + 3z^{-1} + 7z^{-2}$ and

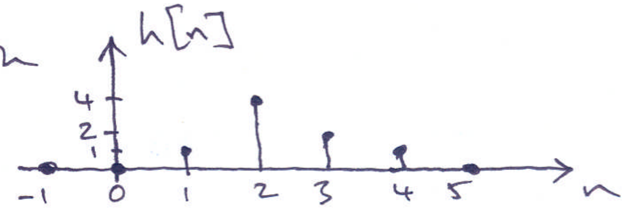
$H(z) = z^{-1} + 4z^{-2} + 2z^{-3} + z^{-4}$

(arranging your answer similarly, in increasingly negative powers of z).

Congratulations! You have just succeeded in convolving




with




using 'Z transforms'! (more in 6.003)

2. If the nonzero values of $x[n]$ are guaranteed to lie in the interval $[\underline{n}_x, \overline{n}_x]$, and those of $h[n]$ in the interval $[\underline{n}_h, \overline{n}_h]$, what smallest interval is guaranteed to contain the signal $y[n]$ that results from convolving $x[\cdot]$ and $h[\cdot]$?

Hint: Think of

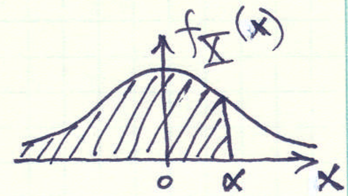
$x[n] \rightarrow \boxed{\text{LTI } h[n]} \rightarrow y[n]$

The first contribution to the output comes from $x[\underline{n}_x]\,\delta[n-\underline{n}_x]$ at input, etc.

This notation generally stands for 'the signal x', i.e., the entire signal, not its value at a specified time. Lecture notes also use $X$ for this.

3. The CDF of the 'standard' Gaussian distribution — i.e., mean 0, variance 1 — may be denoted by $\Phi(\alpha)$:

$$\Phi(\alpha) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$




Write the CDF of a Gaussian of mean $\mu$ and variance $\sigma^2$ in terms of $\Phi(\cdot)$.

<u>Note</u>: The pdf of such a Gaussian distribution is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{\frac{(y-\mu)^2}{2\sigma^2}\right\}.$$

4. Now go through addendum to Lec 5 slides on website, "Example of Bit Error Rate (BER) Calculation with Intersymbol Interference (ISI)"

Important example, integrates many of the things we have been learning.

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Answers to 1, 2, 3 above:

1. With $Y(z) = H(z)X(z)$, you'll discover that the coefficient of $z^{-n}$ in $Y(z)$ is exactly $y[n] = \sum_k x[k]h[n-k]$.

2. $\underline{n}_y = \underline{n}_x + \underline{n}_h$, $\bar{n}_y = \bar{n}_x + \bar{n}_h$ ← check on specific examples

<u>Reason</u>: $x[\underline{n}_x]h[n-\underline{n}_x]$ has its <u>first</u> nonzero value at $n = \underline{n}_x + \underline{n}_h$. Similarly, $x[\bar{n}_x]h[n-\bar{n}_x]$ has its <u>last</u> nonzero value at $n = \bar{n}_x + \bar{n}_h$.

3. $CDF_Y(\beta) = \Phi\left(\frac{\beta-\mu}{\sigma}\right)$.

5. More <u>optional</u> exploration with convolution (related to Item 2 on the first page):

If $x[n]$ and $h[n]$ are <u>nonnegative</u> at all $n$ (as in many of our examples), one can think of them as mass distributions on the time axis. The <u>center of mass</u> then gives an idea of where the signal is <u>concentrated in time</u>. Accordingly:

$$\tau_x = \frac{\sum_n n\,x[n]}{\underbrace{\sum_n x[n]}_{\text{'Mass' of } x[\cdot],\ \text{denote by } M_x}}, \quad \tau_h = \frac{\sum_k k\,h[k]}{\underbrace{\sum_k h[k]}_{M_h}}$$

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h[n]\end{matrix}} \longrightarrow y[n]$$

If $y[n] = \sum_m x[m]\,h[n-m]$,

then $$\tau_y = \frac{\sum_n n \sum_m x[m]\,h[n-m]}{\sum_n \sum_m x[m]\,h[n-m]}$$

<u>Numerator</u> of $\tau_y$ $= \sum_m x[m] \sum_n n\,h[n-m]$

$$= \sum_m x[m] \Big( \underbrace{\sum_n (n-m)\,h[n-m]}_{M_h \tau_h} + \underbrace{\sum_n m\,h[n-m]}_{m M_h} \Big)$$

$$= M_x M_h (\tau_h + \tau_x).$$

<u>Denominator</u> of $\tau_y$ $= \sum_m x[m] \sum_n h[n-m]$

$$= M_x M_h .$$

So $\boxed{\tau_y = \tau_h + \tau_x}$ Feels right!