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6.02
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Rec. 6

A review (of material relevant to Quiz 1, plus some excursions), using Prob. 2 of Lab 3 problems as the vehicle:

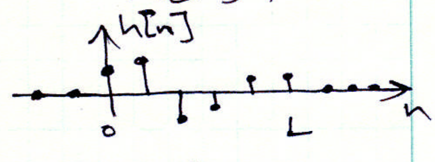
Deconvolution

Assume for now this model perfectly represents x, y relation

$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[0]x[n] + h[1]x[n-1] + \ldots$
$\qquad + h[L]x[n-L] \qquad (*)$

causal LTI,
assume unit sample response (USR) has finite duration, 0 to L
(i.e., $h[k] = 0$ for $k \notin [0, L]$)

Suppose for now that $h[0] \neq 0$:



Rearrange:

$$(**) \quad x[n] = \frac{1}{h[0]}\Big(y[n] - h[1]x[n-1] - \ldots - h[L]x[n-L]\Big)$$

This is a system that takes in the sequence of y[n] values and produces the sequence of x[n] values: $y[n] \longrightarrow \boxed{\text{deconvolver}} \longrightarrow x[n]$

Start from $n=0$,
assuming $x[k] = 0$ for at least $x[-L]$ to $x[-1]$.

Is the deconvolver (i.e., is Eq. (**)) time-invariant? causal? linear?

time-invariant: Yes, rule by which input, y[n] is processed to give x[n] is fixed, no dependence on n.

causal: yes, only uses present and past values of y[k] to generate x[n]

As for <u>linearity</u>, (**) is a linear constraint among the variables, so superposition holds, i.e., if $x[n] = x^{\alpha}[n]$, $y[n] = y^{\alpha}[n]$ is one set of signals satisfying (**), and if $x^{\beta}[n]$, $y^{\beta}[n]$ is another set, then

$$c_1 x^{\alpha}[n] + c_2 x^{\beta}[n], \quad c_1 y^{\alpha}[n] + c_2 y^{\beta}[n]$$

will also satisfy (**), for any scalars c_1, c_2.

So the <u>deconvolver</u> here is a <u>causal LTI</u> system, whose unit sample response we can denote by $g[n]$: $\quad y[n] \longrightarrow \boxed{g[n]} \longrightarrow x[n]$

<u>How to find $g[n]$?</u>

One way: Use (**): set $y[n] = \delta[n]$, then $x[n]$ will be $g[n]$. Specifically,

$$g[k] = 0 \text{ for } k < 0,$$

$n=0$: $\quad g[0] = \dfrac{1}{h[0]}$,

$n=1$: $\quad g[1] = \dfrac{1}{h[0]} \dfrac{-h[1]}{h[0]}$,

$n=2$: $\quad g[2] = \dfrac{1}{h[0]}\left[\left(\dfrac{-h[1]}{h[0]}\right)^2 - \dfrac{h[2]}{h[0]}\right]$

etc., etc.

May not seem very revealing, but already suggests that powers of $\dfrac{-h[m]}{h[0]}$ will appear ($m = 1$ to L). Could mean $g[n]$ will grow in magnitude if $\left|\dfrac{h[m]}{h[0]}\right| > 1$ for some m.

Another way to think of $g[n]$:

$$x[n] \longrightarrow \boxed{h[n]} \xrightarrow{\;y[n]\;} \boxed{g[n]} \xrightarrow{\;x[n]\;}$$

$h[n]$: channel model (assumed perfect, for now)

$g[n]$: deconvolver, constructed using knowledge of $h[n]$.

Reproduces $x[n]$ at output, no matter what the input $x[n]$ is. So if input $x[n] = \delta[n]$, then $y[n] = h[n]$, and $x[n]$ at output again $= \delta[n]$. So we learn that: $h[n] \longrightarrow \boxed{g[n]} \xrightarrow{\;\delta[n]\;}$

$$\boxed{h * g = \delta} \quad (***)$$

Convolving $h[\;]$ with $g[\;]$ gives $\delta[n]$.
This is what a deconvolver does. The deconvolver 'inverts' the convolution (hence also called the <u>inverse system</u>)

<u>A simple example</u>: Suppose

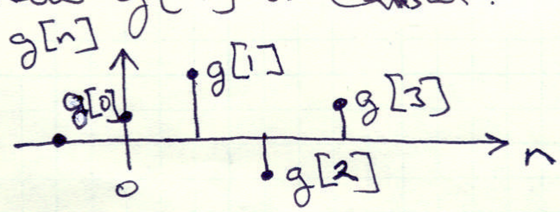
$$h[n] = \delta[n] + 2\delta[n-1]$$

i.e. $y[n] = x[n] + 2x[n-1] \rightarrow (*)$ for this example

and (rearranging) we get the inverse system: $x[n] = y[n] - 2x[n-1] \rightarrow (**)$ for this example.

unit sample response of inverse system

We know $g[n]$ is causal:



Find $g[0]$, $g[1]$, $g[2]$, ... using $(***)$.

Since $h * g = \delta$, and using the 'flip, slide, inner product' operation to compute the convolution, we get:

$$h[0]g[0] = 1 \implies \boxed{g[0] = 1},$$

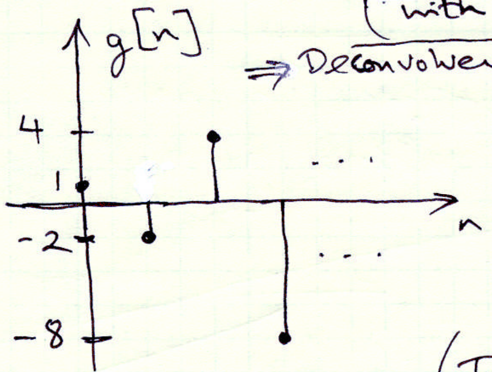
$$h[1]g[0] + h[0]g[1] = 0 \implies \boxed{g[1] = -2}$$

$$\underbrace{h[2]}_{\substack{=0 \text{ for} \\ \text{our example}}} g[0] + h[1]g[1] + h[0]g[2] = 0$$

etc. $\implies \boxed{g[2] = 4}$

$$\boxed{g[n] = (-2)^n u[n]}$$

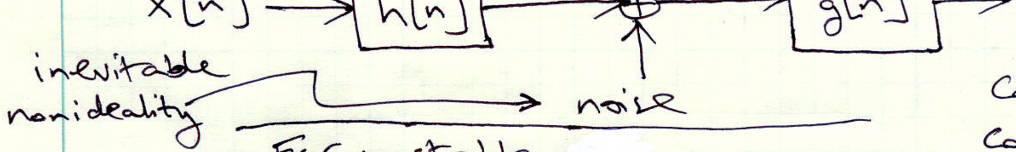
(Another way to get this: use (**) with $y[n] = \delta[n]$, solve forwards.)



$\Rightarrow$ Deconvolver has an exponentially growing unit sample response! Doesn't seem like a good system to work with.

(In the language of 6.003, it's <u>not</u> 'bounded-input bounded-output' stable — a bounded input can produce an unbounded output. The telltale sign is that $\sum_{n=-\infty}^{\infty} |g[n]|$ is <u>not</u> finite.)

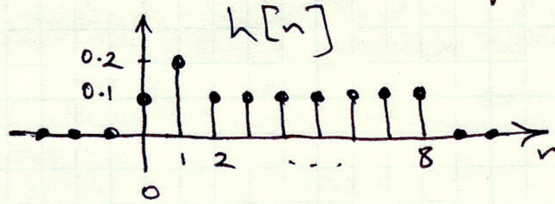
Here's the problem with the deconvolver in this example:



By superposition, output $w[n] = x[n] +$ the component contributed by $\underline{g * \text{noise}}$

For unstable inverse system $g[\cdot]$, this blows up!!

Now consider the specific $h[n]$ from Prob. 2:



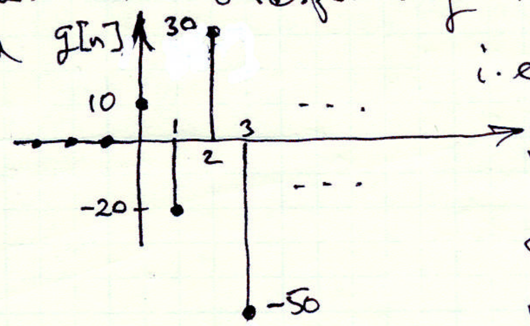
In the noise-free case,

$$y[n] = 0.1x[n] + 0.2x[n-1] + 0.1\sum_{m=2}^{8} x[n-m] \quad (*)$$

$$\Longrightarrow x[n] = 10y[n] - 2x[n-1] - \sum_{m=2}^{8} x[n-m] \quad (**)$$

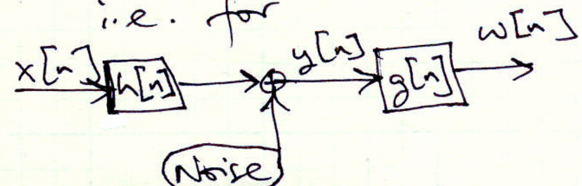
This is the $w_1[n]$ equation in Prob. 2

Is the $g[n]$ that goes with this deconvolver well-behaved (stable)? Using $h * g = \delta$, or alternatively choosing $y[n] = \delta[n]$ in $(**)$ and seeing what the corresponding $x[n] = g[n]$ is, we find



i.e. steadily growing in magnitude, unstable.

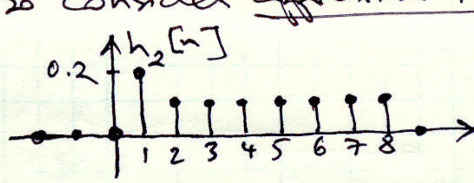
So in the presence of noise, i.e. for



this will do poorly.

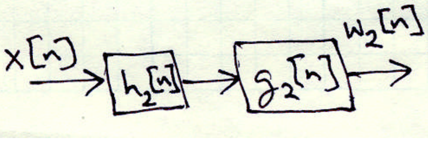
The essence of the problem in $(**)$ is that $\left|\frac{h[1]}{h[0]}\right| = 2 > 1$, so $g[n]$ tends to grow in magnitude by around this factor at each step.

So consider approximating $h[n]$ by the following:



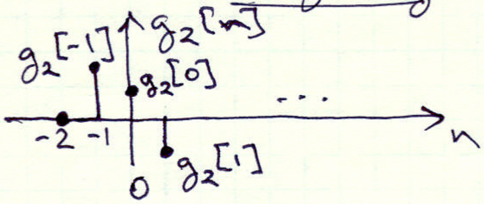
, i.e., $h_2[0] = 0$ instead of 0.1, and $h_2[n] = h[n]$ for $n \neq 0$.

What is the deconvolver $g_2[n]$ (or inverse system) for $h_2[n]$?



Note first that there is no causal $g_2[n]$ such that $h_2 * g_2 = \delta$, because that would require $\underbrace{h_2[0]}_{=0} g_2[0] = 1$, impossible.

But a slightly <u>noncausal</u> $g_2[n]$ will work, i.e., will give $h_2 * g_2 = \delta$.



For this we need

$$h_2[1]\,g_2[-1] = 1$$

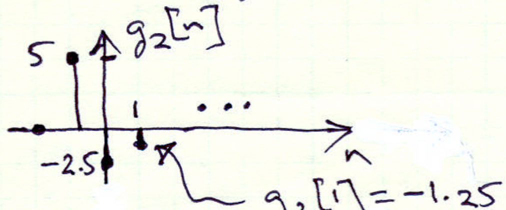
$$h_2[2]\,g_2[-1] + h_2[1]\,g_2[0] = 0$$

$$h_2[3]\,g_2[-1] + h_2[2]\,g_2[0] + h_2[1]\,g_2[1] = 0$$

etc.

$$\Rightarrow \quad g_2[-1] = 5, \quad g_2[0] = -2.5, \quad g_2[1] = -1.25$$

etc. — seems better behaved as a deconvolver.



To recover $x[n]$ from $y[n]$ if we had

$$x[n] \longrightarrow \boxed{h_2[n]} \longrightarrow y[n],$$

we would start with $y[n] = h_2[1]\,x[n-1] + h_2[2]\,x[n-2] + \ldots + h_2[8]\,x[n-8]$ and solve for $x[n-1]$. This is exactly where the $w_2[\;]$ equation comes from for the second deconvolver in Prob 2:

$$w_2[n-1] = \frac{1}{h_2[1]}\Big(y[n] - h_2[2]\,w_2[n-2] - \ldots - h_2[8]\,w_2[n-8]\Big)$$

or

$$\underbrace{w_2[n-1] = 5\,y[n]}_{} - \frac{1}{2}\sum_{m=2}^{8} w_2[n-m] \qquad (**)$$

Note non-causality — we need $y[n]$ to determine $w_2[n-1]$, i.e., we generate our estimate of $x[n]$ with a one-step delay.

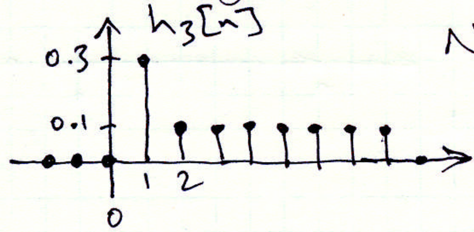
(in Prob 2 statement)

Looking at the response of $w_2[\,]$, we notice that it's a little more 'unsettled' than the response of the next deconvolver, $w_3[\,]$, and also does not quite settle at 1 when $y[\,]$ is 1.

What is the steady-state value of $w_2[\,]$ if $y[\,]$ is kept at 1? Set $w_2[\,] = \overline{w}_2$, constant, and $y[n] \equiv 1$ in (**) on previous page:

$$\overline{w}_2 = 5 - \frac{1}{2} \cdot 7\overline{w}_2 \Longrightarrow \overline{w}_2 = \frac{10}{9}, \text{ i.e.}$$

the values settle slight higher than 1. The reason the steady-state is off is because $\sum_m h_2[m] = 0.9 \neq \sum_m h[m] = 1.0$, i.e., our approximation to $h[n]$ has a unit step response that settles to $\frac{9}{10}$ instead of 1. To fix this, try a different approximation to $h[n]$:



Now $\sum_m h_3[m] = \sum_m h[m] = 1$, so unit step response settles to 1.

Proceeding as before, the associated deconvolver is (as in Prob. 2):

$$w_3[n-1] = \frac{10}{3} y[n] - \frac{1}{3} \sum_{m=2}^{8} w_3[n-m] \qquad (**)$$

Comparing with (**) on p. 6, we see this is more stable, because of the factor $\frac{1}{3}$ that weights past values of $w_3[\,]$, versus $\frac{1}{2}$ for $w_2[\,]$.

— I leave you to find $g_3[n]$.