Problem 5.33 Dimensionless form of the well-depth analysis

Even the messiest results are cleaner and have lower entropy in dimensionless form. The four quantities h, g, T, and c_s produce two independent dimensionless groups (Section 2.4.1). An intuitively reasonable pair are

$$\overline{h} \equiv \frac{h}{gT^2} \quad \text{and} \quad \overline{T} \equiv \frac{gT}{c_s}. \tag{5.40}$$

a. What is a physical interpretation of $\overline{T}$?

b. With two groups, the general dimensionless form is $\overline{h} = f(\overline{T})$. What is $\overline{h}$ in the easy case $\overline{T} \to 0$?

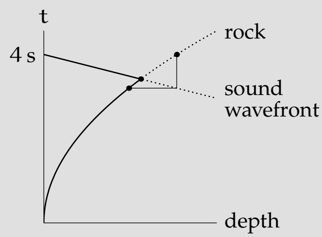
c. Rewrite the quadratic-formula solution

$$h = \left(\frac{-\sqrt{2/g} + \sqrt{2/g + 4T/c_s}}{2/c_s} \right)^2 \tag{5.41}$$

as $\overline{h} = f(\overline{T})$. Then check that $f(\overline{T})$ behaves correctly in the easy case $\overline{T} \to 0$.

Problem 5.34 Spacetime diagram of the well depth

How does the spacetime diagram [44] illustrate the successive approximation of the well depth? On the diagram, mark h_0 (the zeroth approximation to the depth), h_1, and the exact depth h. Mark t_0, the zeroth approximation to the free-fall time. Why are portions of the rock and sound-wavefront curves dotted? How would you redraw the diagram if the speed of sound doubled? If g doubled?



5.5 Daunting trigonometric integral

The final example of taking out the big part is to estimate a daunting trigonometric integral that I learned as an undergraduate. My classmates and I spent many late nights in the physics library solving homework problems; the graduate students, doing the same for their courses, would regale us with their favorite mathematics and physics problems.

The integral appeared on the mathematical-preliminaries exam to enter the Landau Institute for Theoretical Physics in the former USSR. The problem is to evaluate

$$\int_{-\pi/2}^{\pi/2} (\cos t)^{100}\, dt \tag{5.42}$$

to within 5% in less than 5 min without using a calculator or computer!

That $(\cos t)^{100}$ looks frightening. Most trigonometric identities do not help. The usually helpful identity $(\cos t)^2 = (\cos 2t - 1)/2$ produces only

$$(\cos t)^{100} = \left(\frac{\cos 2t - 1}{2}\right)^{50}, \tag{5.43}$$

which becomes a trigonometric monster upon expanding the 50th power.

A clue pointing to a simpler method is that 5% accuracy is sufficient—so, find the big part! The integrand is largest when t is near zero. There, $\cos t \approx 1 - t^2/2$ (Problem 5.20), so the integrand is roughly

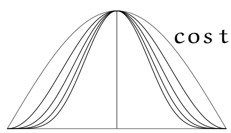
$$(\cos t)^{100} \approx \left(1 - \frac{t^2}{2}\right)^{100}. \tag{5.44}$$

It has the familiar form $(1+z)^n$, with fractional change $z = -t^2/2$ and exponent $n = 100$. When t is small, $z = -t^2/2$ is tiny, so $(1+z)^n$ may be approximated using the results of Section 5.3.4:

$$(1+z)^n \approx \begin{cases} 1 + nz & (z \ll 1 \text{ and } nz \ll 1) \\ e^{nz} & (z \ll 1 \text{ and } nz \text{ unrestricted}). \end{cases} \tag{5.45}$$

Because the exponent n is large, nz can be large even when t and z are small. Therefore, the safest approximation is $(1+z)^n \approx e^{nz}$; then

$$(\cos t)^{100} \approx \left(1 - \frac{t^2}{2}\right)^{100} \approx e^{-50t^2}. \tag{5.46}$$



A cosine raised to a high power becomes a Gaussian! As a check on this surprising conclusion, computer-generated plots of $(\cos t)^n$ for $n = 1 \ldots 5$ show a Gaussian bell shape taking form as n increases.

Even with this graphical evidence, replacing $(\cos t)^{100}$ by a Gaussian is a bit suspicious. In the original integral, t ranges from $-\pi/2$ to $\pi/2$, and these endpoints are far outside the region where $\cos t \approx 1 - t^2/2$ is an accurate approximation. Fortunately, this issue contributes only a tiny error (Problem 5.35). Ignoring this error turns the original integral into a Gaussian integral with finite limits:

$$\int_{-\pi/2}^{\pi/2} (\cos t)^{100}\, dt \approx \int_{-\pi/2}^{\pi/2} e^{-50t^2}\, dt. \tag{5.47}$$

Unfortunately, with finite limits the integral has no closed form. But extending the limits to infinity produces a closed form while contributing almost no error (Problem 5.36). The approximation chain is now

$$\int_{-\pi/2}^{\pi/2} (\cos t)^{100}\, dt \approx \int_{-\pi/2}^{\pi/2} e^{-50t^2}\, dt \approx \int_{-\infty}^{\infty} e^{-50t^2}\, dt. \tag{5.48}$$

Problem 5.35 Using the original limits

The approximation $\cos t \approx 1 - t^2/2$ requires that t be small. Why doesn't using the approximation outside the small-t range contribute a significant error?

Problem 5.36 Extending the limits

Why doesn't extending the integration limits from $\pm\pi/2$ to $\pm\infty$ contribute a significant error?

The last integral is an old friend (Section 2.1): $\int_{-\infty}^{\infty} e^{-\alpha t^2}\, dt = \sqrt{\pi/\alpha}$. With $\alpha = 50$, the integral becomes $\sqrt{\pi/50}$. Conveniently, 50 is roughly 16π, so the square root—and our 5% estimate—is roughly 0.25.

For comparison, the exact integral is (Problem 5.41)

$$\int_{-\pi/2}^{\pi/2} (\cos t)^{n}\, dt = 2^{-n}\binom{n}{n/2}\pi. \tag{5.49}$$

When $n = 100$, the binomial coefficient and power of two produce

$$\frac{12611418068195524166851562157}{158456325028528675187087900672}\,\pi \approx 0.25003696348037. \tag{5.50}$$

Our 5-minute, within-5% estimate of 0.25 is accurate to almost 0.01%!

Problem 5.37 Sketching the approximations

Plot $(\cos t)^{100}$ and its two approximations e^{-50t^2} and $1 - 50t^2$.

Problem 5.38 Simplest approximation

Use the linear fractional-change approximation $(1 - t^2/2)^{100} \approx 1 - 50t^2$ to approximate the integrand; then integrate it over the range where $1 - 50t^2$ is positive. How close is the result of this 1-minute method to the exact value $0.2500\ldots$?

Problem 5.39 Huge exponent

Estimate

$$\int_{-\pi/2}^{\pi/2} (\cos t)^{10000}\, dt. \tag{5.51}$$