6.055J/2.038J (Spring 2010)

Solution set 3

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That's roughly 55 mph.

Here is the solution set. Apologies that i posted it late. Do the memo by Friday at 9am.

Could you list these on the main list of "things we need to do"? I keep forgetting that these are due.

Is there someway that the system could be set up so we can see our answers that we submitted, sometimes I forget what I had come up with.

Wow. I completely misinterpreted this "passenger-miles per gallon."

Isn't the point just to have a baseline energy comparison between different modes of transportation? Like if I drive my car I get 25 miles per gallon but if I fly I get 50 miles mpg. So if I'm traveling alone, flying is about 2x as efficient.

Oh.. I guess I was very off from guessing the price of the ticket. I used a one way ticket price because I think round trip tickets are very discounted.

I live in California, and I've never been able to find a roundtrip ticket home for this cheap!

most of the time I pay 300 - 360 to get to sacramento.

For a while I was paying 600 for a roundtrip ticket to california, but now it's a lot closer to 250-300, I was also really confused with my answer because I remember hearing somewhere that flying is much less efficient than cars, but I guess there's more people in a plane too

For estimating this price, is it best to use the maximum, minimum, or average value these tickets generally go for?

Wow, I did not think to use the price of a plane ticket!

I did, the problem stated to. But I felt that using the price of a plane ticket lacked precision because of so many more approximations that it would open up, like the fraction of cost to fuel, and there seems to be so many other factors. It seems to me that fuel efficiency more intrinsically a physical estimation, with size of airplane?

I guess I had overlooked that detail.

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I definitely forgot to count the backwards trip, but I still had a similar number. What is the "actual" answer to this question?

Yep, an actual answer would be nice. I ended up with 30 passenger-miles per gallon, which is roughly the same fuel efficiency as a car, and I got confused...

You need to multiply the fuel efficiency for a car by the number of passengers it can carry.

Oh gosh, I forgot to account for the return trip as well! It's always in the assumptions at the very beginning of the problem that I trip up

I think I'm confused – I thought the only point of considering the round trip was just to make the math easier?

I also forgot the return trip in the cost, so maybe I overestimated. I also didn't bother thinking about costs other than fuel, which in hindsight probably wasn't logical.

I didn't account for the return trip either but I got about the same answer just using a one way trip. I think as long as your cost and mileage accounted for just one way it works out to be about the same.

Me too... I also underestimated the price. I guess most people aren't poor college students going for the cheapest flight.

Oops... I forgot the each way part! But my ticket price was lower and journey distance higher, so everything evens out in the end.

I liked this question because I had real world experience with all the estimations I had to make.

That is a very cool way to solve this problem. I tried to estimate the drag on a plane and on a car and used that ratio. Unfortunately, I was way off.

i dont see why you would use this considering it said use the cost of a plane ticket.

I did it the same way as the solution but I had trouble deciding how much of your plane ticket went to fuel. How did you come up with this number?

i didn't think that not all the ticket went to fuel, and my answer was close

i used a much lower fraction considering there are a lot of other business expenses to account for.

Yeah I'd have to disagree with that fraction. It seems a bit high. There are many other costs incorporated into plane ticket prices: inflation, business, profit, etc.

And this was hard to estimate because there's not much in my life that has given me experience to how much it costs to run an airline. I did this by estimating how much the fuel costs, and based my estimate on my knowledge of gas prices for the kind of gas for my car. Don't know if this was more or less accurate.

maybe leave it in dollars?

oops, I completely forgot about this part, but ended up with the same answer...

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That's roughly 55 mph.

I started this the same but halfway through did it differently by estimating the cost of jetfuel. I got a very different answer, which makes sense because I remember fudging the units in mine once.

Yeah me too. My strategy was pretty much the same but I just estimated the fuel cost differently.

I think I ended up close; same process. It took me a bit to connect the ticket price to the fuel cost.

doing a problem like this made me realize how little real world knowledge i have

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Yeah me too. But with a little help from Google you get to learn a few stuff. I also like this class because you learn real world stuff like that

This is a great question, It would be interesting to compare it to trains or boats or something as well.

yeah! this is similar to my thought process!

I just cut out all the services and went with pure fuel cost - but my plane ticket estimate was on 300 dollars

Yeah I didn't even think of the money going anywhere but fuel...

I didn't either, but in retrospect of course other things are being paid for with our money than just gas.

Is fuel actually two-thirds of the price? I was wondering this when solving the problem and figured it couldn't be that much bc then they would hardly make money.

While this definitely works for order of mag estimates, I have heard that jet fuel is more expensive. perhaps even up to 3 times more expensive.

This can be true, as it fluctuates with the oil market (similar to automobile fuel fluctuation). \$3-\$5 per gallon is a pretty good estimate (like you said, it wouldn't get much more than \$9)

I said they get fuel closer to \$2 a gallon. I know that airlines often will buy options for their fuel consumption but I'd also assume they can get it at a somewhat discounted price as compared to car drivers because of the large quantity they buy.

while they do get "breaks" on the cost for buying large quantities, the gas itself is much more expensive because it has to be processed more. [at least that's what I've heard]

i was surprised to find that jet A (gas for aircraft is not that much more expensive than regular unleaded and \$3/gallon is actually a fairly accurate assessment.

crap I missed a decimal in the answer box, at least my estimate was only off by a factor of two

what exactly is a passenger-mile per gallon. I think a got a similar number for gas mileage however, I assumed that you must multiply this by the number of passengers flying. You didn't really talk about passengers in the solution so I am just wondering what these units actually represent. Nevertheless, I made almost all the same assumption so I think I at least learned the lesson here.

I had this same question as to how the passenger part comes into play. Based on this I think the final solution I came up with was actually a mpg rating for the aircraft...

Oh I misunderstood and ended up divided by the number of people on the flight at some point.

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I feel like this was such an easy problem for most people, but coming up with the relevant parameters is still difficult for me.

Isn't m 9.8? So I would think we would round down to 600 here.

That's true - that's what I would have done as well.

I jsut completely forgot to figure in gravity for some odd reason

You're not the only one... oops!

Forgetting gravity! Your high school physics teacher should be ashamed...

Maybe he rounded to 700 to account for the obesity epidemic (65 kg = 143 lbs)?

Random fact: the FAA sets the average weight for American passengers on planes to be 170 pounds...

Nah, it's not really significant and easier just to say it's 10 and thus 700.

i thought that if you used pounds, that the pounds themselves are units of force, so you wouldnt multiply by gravity.

oops by mistake I had a v^3 term in my answer since I took the Power equation from the notes on drag (in the cycling section) instead of the Force equation!

Are we just supposed to know this off the top of our head?

The equation was derived in the reading for this week, but I agree that it isn't something I would have known off the top of my head otherwise.

Isn't it also true that if the wind speed is equal to the weight of a person, it is the same as if the person were falling from the sky at terminal velocity.

Yes, it is the same thing as terminal velocity. That's how I figured out this problem.

hm–I was thinking about using terminal velocity, because I thought it made sense mathematically, but I just couldn't wrap my head around it conceptually...

It's really annoying that they don't teach this in 8.01. ignoring air resistance is so silly, knowing and internalizing this equation would be so much more helpful for everyday questions!

I would love if equations like this were provided, I for sure don't know many off the top of my head, and they come up in a ton of problems for this class.

yeah used this, called on memory of values from 2.005, got similar answer

The problem I had here was that this equation states that v is "proportional" to that expression. Shouldnt there be a constant somewhere?

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A list of useful equations with the useful numbers would be helpful.

this isn't really a standard equation. This particular equation was discussed in the reading and basically is just a force balance. m*g=ro*(v^2)*A. then you solve for v.

the equation that relates F proportionally to the pv^2A was in the reading, and his equation here is just that same equation, but with v isolated

I was going to use Newton's third law to calculate it but it was a lot harder! this is much better!

got to stand in a 45 mph wind in the wind tunnel sophomore year. was a lot of fun to lean into the wind and and being buoyed up by it...

I wasn't sure what to do with the cross sectional area here so I just dismissed it by using 1... Luckily that worked out great!

an A of 1 seems like a pretty large person... I used smaller area and a mass that was a little larger and got a wind speed of 55 m/s... I kinda doubt that your terminal velocity in free fall would be this small....? ... haha I just searched it and it's -56 m/s

I also said that the cross-sectional area of a person is about 1. it's an overestimate for the whole population, but if you think about it, take a (built, and slightly taller than) 6-foot man, cut him in half and put the two halves together, it's about a square meter right...

This seems like a very tall and skinny person. But I guess its all close enough. Its closer to the cross sectional area of any person than it is to the cross sectional area of something else, like an elephant.

Ha! I considered the density of a person for some reason. That was dumb. And also explains my incorrect answer.

I sometimes forget which density to use in these problems, but what helps me is to look at the equation and think about what should happen to the force when you increase/decrease the density.

i did the same thing (used the density of the person!)

sick. I got the exact same answer. First time.

Unfortunately, there is no "exact" answer in an approximation class...

But congratulations!

I got that answer too. Woot! This is just the same as terminal velocity right?

Nice! My test person was a bit lighter (I used 600 so it would work better)

haha... i messed up the math and got the way too low answer of 0.6. so i made that the low range of my answer and used something that i just thought made sense for the high end. i guessed 30, which is just right for "a little higher than correct"

eh same order of magnitude counts right?

Lol, I got something close to the speed of light.... I used the formula for impulse instead of using the one for terminal velocity.

It's nice to come close; i also got a very close answer.

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So for this problem we are assuming that the wind is picking the body up such that feet don't create friction with the ground?

I had a very similar approach but for some reason came to a very different answer. This answer makes more sense from experience though.

I did this differently. The terminal velocity of a human being is about 120 mph. At the terminal velocity, the force of wind balances the force of gravity, so I estimated about 54 m/s.

this makes _so_ much more sense to me! Thanks.

This is also how I did it. Although this does require that you know the answer, fortunately the answers from both methods are very similar.

This problem was inspired by the high winds (and rain) a couple weeks ago. As I was walking home in that miserable weather, I leaned sharply into the wind in order not to get toppled over – indicating that the drag force was comparable to my weight.

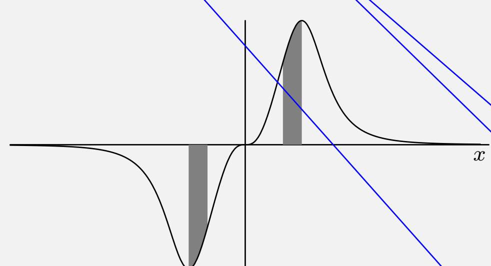
3. Daunting integral

Evaluate

$$\int_{-\infty}^{\infty} \frac{x^3}{1+7x^2+18x^8}\,dx. \quad (3)$$

☐ ± ☐ *or* ☐ . . . ☐

The integrand, $x^3/(1+7x^2+18x^8)$, is antisymmetric: When x becomes −x, the integrand changes sign. So, for every sliver of rectangle in the negative-x region, there's a corresponding sliver with the opposite sign in the positive-x region. The net sum is therefore zero.



This problem was inspired by my days as a physics undergraduate. Physics problem sets often meant doing tons of complicated integrals. Our bible was Gradshteyn and Ryzhik's *Table of Integrals, Series, and Products*, now in its 7th edition. Often when we couldn't find an integral in Gradshteyn, we later realized, after much painful integration gymnastics, that the integral had to be zero by symmetry. So, don't miss those chances to use symmetry.

Problems

4. Solitaire

You start with the numbers 3, 4, and 5. At each move, you choose any two of the three numbers – call the choices a and b – and replace them with $0.8a - 0.6b$ and $0.6a + 0.8b$. The goal is to reach 4, 4, 4. Can you do it? If yes, give a move sequence; if no, show that you cannot.

To see whether solitaire games are solvable, look for an invariant. Alas there is no algorithm for finding invariants; you have to use clues and make lucky guesses.

Speaking of clues, is it a happy coincidence that $0.8^2 + 0.6^2 = 1$? That convenient sum suggests looking at sums of squares, and how those are changed by making a move. Replacing a and b by $a' = 0.8a - 0.6b$ and $b' = 0.6a + 0.8b$ makes the sum of squares $a^2 + b^2$ into $a'^2 + b'^2$. Expand that expression:

$$\begin{aligned} a'^2 + b'^2 &= (0.8a - 0.6b)^2 + (0.6a + 0.8b)^2 \\ &= 0.64a^2 - 0.96ab + 0.36b^2 + 0.36a^2 + 0.96ab + 0.64b^2 \\ &= a^2 + b^2. \end{aligned}$$

Comments on page 2

How fast were the winds that day?

Weather underground shows max wind gusts last month at 57 mph at the top of the Green building, and 36 mph in Central Square. The canyon effects on campus could definitely produce wind forces equal to your weight.

Haha yes, I remember this. I couldn't close the door in front of my dorm....

The winds here are terrible, and on top of that, half of MIT is a wind tunnel... especially dorm row!

This problem was so cool! How would the equation change if you were running rather than walking?

You'd add your increased speed to the relative velocity between you and the air, but you'd be providing forward force with your legs.

I think the force with your legs would be irrelevant to the problem if you're just trying to understand the force of the wind.

I kind of guessed it would be same amount negative as positive - my calculus is all fuzzy these days, but how could we identify that from just looking at the function?

yeah me too...I just kind of guessed that it had to be antisymmetric so that it would all cancel out, otherwise there wouldn't be a way to solve it without calculating it out..what's the correct way to get this type of intuition though?

The intuition is that the integrand is symmetric about the origin. This means that it is a mirror image using the origin as the axis. If we are then integrating over the same distance in both negative and positive x (in this case, negative infinity to positive infinity), then the integration before x = 0 will cancel out the integration after x = 0.

aw shoot, odd of course! I didnt notice that and just thought that is was symetric about the y axies, and then used a triangle to approximate.

I definitely stared at this for 10 to 15 minutes before even thinking about even or odd functions...I agree that it brings up a really nice symmetry that makes you feel silly afterward

I pretty much figured that this would be symmetric since there was no other way of solving this that we had learned, although before I realized it I did try to calculate it by estimating it as the integral of x^-4

i read your whole explanation as if this were "asymmetric." needless to say i was entirely confused (but understand now).

though while it makes sense, i don't know how i would have known it was symmetric without my ti-84

think about it at large values of x. at large values, the equation pretty much becomes 1/x^5, which is symmetric about the origin

you could look at the equation and realize that if you plugged in a number or that number times -1, the denominator did not change, but the numerator turned negative. this would mean that the curve is symmetric and positive to the right of x=0 and negative to left of x=0

how do you know this by looking at it?

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Evaluate

$$\int_{-\infty}^{\infty} \frac{x^3}{1+7x^2+18x^8}\,dx. \qquad (3)$$

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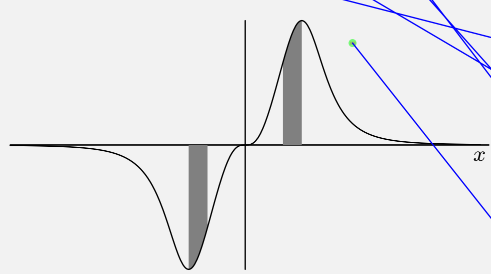

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As soon as I graphed this, it made sense. My question is, would you like us to remember how to tell a symmetric integrand from an antisymmetric integrand without being able to graph it?

Look for odd or even functions.

Exactly. Like the professor, I typically start looking for symmetries whenever things don't fall into a typical 18.01 formula.

Does an odd function = antysymmetric function?

yes.

think of y=x [an odd function] – antisymmetric.

think of y=x^2 [an even function] – symmetric.

The way to determine if the integrand is to determine, as mentioned earlier, if the integrand is odd or even. Let f(x) = the integrand. If f(-x) = -f(x), then the integrand is odd and thus antisymmetric.

With multiple powers in this equation, sometimes determining this is a bit difficult. This one works out nicely though, seeing as the denominator has all even powers, it will never be negative, and in fact, never 0, with symmetry across the y axis. With this said, the numerator is clearly odd, making the function odd.

but this would have been hard if you don't have a calculator to graph it

I definitely stared at this for a solid 20 min without writing anything before realizing how simple the problem becomes.

It's simple, but I still don't understand, mathematically, why the integrand is antisymmetric (without the even odd function stuff).

I'd like to get more into mathy solutions in class as well as the approx.

This is the most useful thing I have learned about integrals since 18.01.

this seems like a _really_ complicated way of explaining odd functions...without actually doing any reasoning as to why is is an odd function.

does an odd function have to be perfectly antisymmetric to come out to 0 or does it go without saying that an odd function integrated from -inf to +inf is 0

You should check that the integral converges, though, before canceling sides (because -inf + inf !=0...). In this case, the denominator is bounded below by 1, and as x->+inf the function is bounded above by 1/(18*x^5), which converges (and it is similarly bounded as x-> -inf)

I was certain that there was an ever so small displacement from 0 for being anti-symmetric, although as an estimate it comes out to the same

I don't understand the graphs as much. Make later in the course or via office hours can you explain approximates in graphing in greater details? I just assume the answer would be zero based on what I know about integrals.

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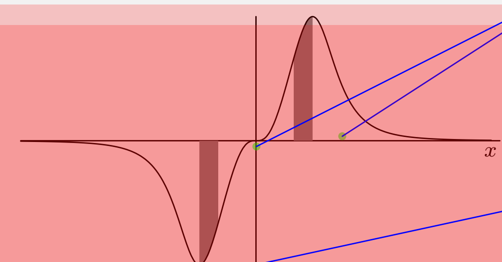

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I thought about this kind of the same way, but just used the rule of calculus that you can split the integral into two with different limits (-inf-> 0 and 0-> inf) and saw the symmetry there

I'm not sure how valid this is, but I basically looked at the denominator, saw that it would get large much faster than the numerator (as the function went to infinity) and assumed that the function would go to zero.

I think the most sure way would be to just double check by plugging in numbers. right off the bat i plugged in 1 and -1 and realized that the magnitude of the function stayed the same but just changed signs

I wouldn't have realized the integral was antisymmetric just from looking at the equation.

Were we supposed to graph it?

I solved it without graphing by approximating the integrand to be x^-5 and evaluating.

I graphed the first few values. I really liked this problem, while it looks very complicated, the solution is quite simple and can be solved after a little bit.

I guess I will think about problems more visually from now on.

Graphing it was helpful, but I think the idea is to be able to recognize its zero from looking at it. Integrals with odd integrands and symmetric limits are always zero. this fits that description.

As a physics undergraduate, I saw this right away. Such symmetries are crucial to finishing p-sets in under 24 hours.

As an engineering major, I was pissed that I wasn't allowed to use my calculator.

hahaha nice

Ah yes, this did smack of many a physics pset.

This class is dragging up a lot of things I didn't think I would use again.

So when you solve this, do you need to take anything into consideration besides the x^8 in the denominator. Because I just saw that and figured you'd have 0-0 anyways.

This book should be required...yet I've never heard of it

i think i'm going to start using this phrase at 4am, when i'm still up doing math.

Even now, this is one of the things in the course that gives me the most trouble. Finding these little invariants in the game are always strange to me.

I really hate invariants. Is there anyway to get better at them? this whole seeing more of them thing is just making me hate them more and more.

I remember this in lecture...!

why are my guesses in this class never lucky?

I never would have thought about it like that

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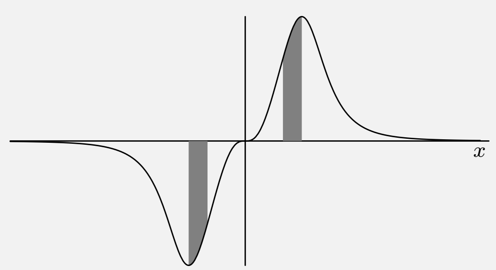

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interesting, Ii completely missed this

I did not see this coming at all.

i feel like this would have been easier to see if they'd been written as fractions, just because we've trained to spot 3-4-5 triangles, etc.

I like to think I'd notice these clues if I weren't so tired when I did my work, but I think that's wishful thinking.

d'oh. totally missed that equation.

I never thought to do this...I wish I had, now the rest of the problem isn't bad!

Agreed. I couldn't figure out what the invariant was either...

Yeah..reading this makes it seem so much easier.

Is there any way to derive this invariant, or is it just guess and check?

I also never would have thought to do this, is there a good way for seeing these types of things other than practice?

wow, I used straight up sums–and didn't really feel too confident about it, but I didn't think of sums of squares...wow that makes this problem make a whole lot more sense

I also used sums and agree that this was way more clever. I don't think using sums was a bad idea though, it still brought out a similar relationship between a and b that shows the final sum would be impossible to reach.

this was really hard, and how was i supposed to look for this- any hints?

Interesting. I guess it just takes a lot of practice in order to see these sorts of details readily.

yeah i didn't get this, but tried a few variations to come to the proper conclusion instead.

It took me a while to find the invariant, but once I found this the answer to the problem came right along behind it.

Agreed - these problems are all so simple once you solve them... but so tricky to get there!

Yeah Theres a lot of tricky hidden tricks that are hard to get even with some knowledge of symmetry.

Definitely agree with this. Took forever to try to find an invariant...

Yea I never found it. What do you do then?

I tried all sorts of sums and products and subtractions but never found the invariant :(I can't believe I didn't try sum of squares!

Great! Each move leaves the sum of squares unchanged. That sum started out with the invariant at $3^2+4^2+5^2=50$, so it remains 50. The goal state, however, requires that the invariant become $4^2+4^2+4^2=48$. It's not possible to reach the goal.

The invariant has a nice geometric interpretation (a picture). To see it, let $P=(a,b,c)$ be the coordinates of a point in three-dimensional space. Then each move leaves unchanged the distance to the origin, which is $\sqrt{a^2+b^2+c^2}$. So each move shifts P to another location equally distant from the origin, meaning that it moves P on the surface of a sphere. But it cannot escape the surface.

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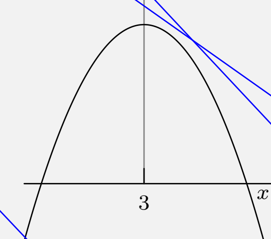

6. Tiling a mouse-eaten chessboard

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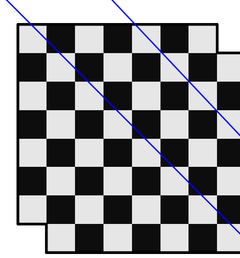

☐ yes

☐ no

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Comments on page 3

I didn't catch this before.

I liked this problem a lot - it required a lot of thinking about invariants, and we had to find one that was nontrivial. You should use this problem (or some version of it) again!

I had trouble finding the invarient at the start, but once I found it the problem was clear and a lot of fun to prove

I agree, I think this was a good problem. Although I definitely spent the most time on this problem compared to the others because it took me a long time to find the invariant. Obviously there aren't any rules about how to find the invariant, but are there any helpful tips aside from "just look for what doesn't change"?

Yeah the invariant was a really tricky find...

I feel like this problem would perhaps have been enhanced by some easier warm up problems to practice finding invariants. I pretty much ended up solving this problem by just trying it out until I had decided it wasn't possible.

I think I'm a bit confused about where the c^2 comes from...?

Yeah, I think that it's not necessarily obvious why we can generalize from a^2+b^2 invariant to a^2+b^2+c^2 also invariant.

This is so much easier once you see the invariant! I spent so much time on this problem until it hit me!

This problem makes more sense now (I had no idea how to do it)...doing another problem similar to this might be useful to imprint the technique on solving it

I used a similar idea for my invariant, although it is not as concrete. I saw that the sum of 3,4,5 and 4,4,4 was 12 and that each move seemed to decrement the sum (at least move it away from 12) so the sum of 12 would be really hard to achieve

I completely missed this but once you see it along with the visualization everything becomes so much easier.

I believe we have the invariant that preserves "rational points", so I don't think we could reach any of the irrational ones on the sphere. Not sure if we could reach all the rational ones though. My guess is no.

I got curious and tried doing something like a change of variables or another invariant to see if it could be done but I couldn't find another one (I tried a couple of geometric things) and can't see any change that makes it more obvious...

Do you mean reach every point, using only that move option?

I realize now that I found the value of x that maximizes 6x-x^2, but forgot to plug that number back into the function to get the actual value.

I actually didn't use symmetry for this, and just calculated it using the derivative...

I did too. This seemed like too trivial of an example to really use symmetry.

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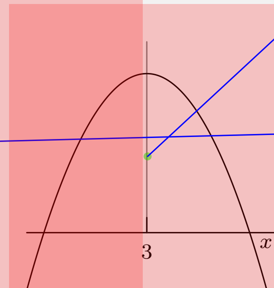

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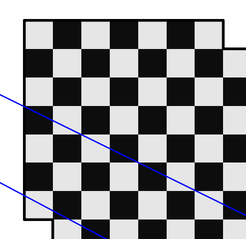

☐ yes

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I found it much easier to use guess and check. Maybe the example problem for symmetry should be more difficult to solve unless you implement a symmetry strategy.

Yea, I just plugged in values 1-5 to the equation and it became obvious where the max was.

This is one thing that I just had a hard time with on this Pset – trying to some type of operation (or invariant point) that works! It always makes sense afterwords, but I'm not so good at coming up with them.

Yeah... this one took me a lot longer than the others. I would stare at a problem for a while and not see the answer. Some of them came to me later though while I was doing other things... funny how the brain works!

this one took me soooo long! I felt to stupid...the answer is just so simple

wow I didn't think about it like that; I looked for the zeros.

So when this problem said "use symmetry", I didn't think about finding an invariant that doesn't change, I thought about the graphically symmetric nature of parabolas (if it intercepts the x-axis at x=0 and x=6, then the max value must be at x=3). By "use symmetry", are you actually saying "find an invariant"?

I thought the same thing. I solved this and in my explanation tried to convince myself it was symmetry although I still don't completely understand.

I explained the symmetry by seeing that the curve was symmetric about the line x=3 and that meant that x=3 was the global max

I ended up using a different kind of symmetry for this - I know a parabola is symmetric so the max would be directly between the zeros. I was having issues finding an invariant.

I feel like the zeros method is a lot less confusing than the method here.

Agreed. Although the method used here may be more generally applicable. But I also just used previous knowledge of parabolas to solve this problem.

yeah, i used that same general symmetry of parabolas, but if you look at it, that's basically what the solution is doing too, but saying that the local extrema on a parabola remains unchanged with reflection through a vertical line through it, while everything else is symmetrical...basically the same thing–both sides of a max/min are symmetrical

I think I'd need this process if the polynomial were more complex; for this one I just pictured it.

i agree. this seemed like a really easy problem. something you couldn't graph in your mind would have proved a point a little better.

If I had thought to factor this i would have been able to see the symmetry much quicker.

I just solved it by using simple differentiation, I will admit that your approach applies more to the class, but with such an easy problem it is hard to not just do it the simple way, maybe next example could be made more difficult to solve based on math, and therefore would require the application of symmetry.

did you mean maximum here?

I think this was one of the trickiest problems. It's funny because the explanation seems so simple. However, I think that the problems without a lot of math often require the most thought.

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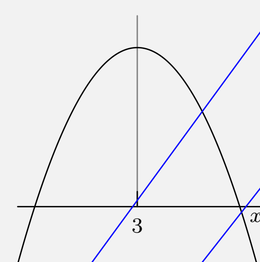

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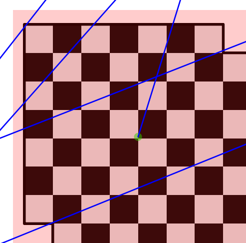

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This is one of those that if you have seen before you can solve it immediately, and will never forget it.

Yeah, I definitely have seen this problem before so I remembered how to prove it

I like this problem. It seems like a strange IQ test question, and indeed I think I remember it in a calendar of puzzles that I have, but it is fairly reasonable to work out.

I wonder if you were not given a picture of the chessboard if you would actually think in colors...I've tried it on my friends verbally and they could not get it

Well, the professor did say in class how people can understand things better with a visual of the problem.

I actually couldn't see or didn't think about looking at this problem as finding an invariant. But seeing it now, its a pretty nifty method of doing it.

If it wasn't for the theme of the homework set, I wouldn't have thought of it either... and it took me a while to realize how to think of it.

I am still a little shaky on the idea of invariants. I thought that the idea was more for the entire board not for the color of the squares.

It could be for the whole board but using squares greatly simplifies the complexity

ahhhh. that's it

very nice solution

Wow, I never would have found an invariant in this problem. I just realized that in order to fill this up with a 2x1 domino, you had to eventually make a piece whose area was 3, which isn't possible because we have 2x1 pieces...

Yea but I think thats only half the battle- you still need to prove it.

We've seen this problem before (well, a slight variation, but essentially the same). It would have been nice to see something a bit more different.

I wasn't able to find/use this invariant when solving the problem...instead i looked for visual "invariants;" that is, combos of white/black square in different shapes, trying to fit them into the board.

I went ahead and calculated all the pairs via counting and matching up. I did your method also but I assumed that counting would be more reliable. Perhaps I made a error in counting. Make approximating is more reliable.

instead of this i divided the board into sections and there eventually were sections that had an odd number of squares and couldn't be covered with a domino.

yea i did something different also, i made a smaller version of the board 3x3 and made it larger each time by scaling up

Great! Each move leaves the sum of squares unchanged. That sum started out with the invariant at $3^2+4^2+5^2=50$, so it remains 50. The goal state, however, requires that the invariant become $4^2+4^2+4^2=48$. It's not possible to reach the goal.

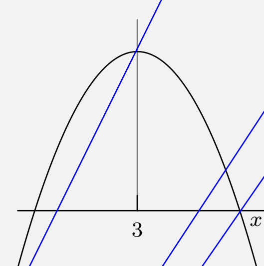
The invariant has a nice geometric interpretation (a picture). To see it, let $P=(a,b,c)$ be the coordinates of a point in three-dimensional space. Then each move leaves unchanged the distance to the origin, which is $\sqrt{a^2+b^2+c^2}$. So each move shifts P to another location equally distant from the origin, meaning that it moves P on the surface of a sphere. But it cannot escape the surface.

An interesting question to which I don't know the answer: Can you reach every point on the surface of the sphere? The distance invariant does not forbid it, but maybe other constraints do?

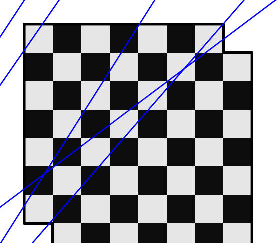
5. Maximizing a polynomial
Use symmetry to find the maximum value of $6x-x^2$.

☐ ± ☐ *or* ☐ . . . ☐

The polynomial factors as $P=x(6-x)$. As a symmetry operation, try replacing x with $6-x$. That operation is a reflection through the vertical line $x=3$. It turns P into $(6-x)x$, which is again P just with the factors swapped. Let's call x_0 the value of x that maximizes P. Because changing x to $6-x$ doesn't change the curve, it doesn't change the location of the minimum, which is at $(x_0, P(x_0))$. Thus x_0 turns into x_0 under the symmetry operation $x \mapsto 6-x$. The only value of x that is unchanged by a reflection through the vertical line $x=3$ is 3 itself, so $x_0=3$ and $P(x_0)=9$.



6. Tiling a mouse-eaten chessboard
An 8×8 chessboard gets two diagonally opposite corners eaten away by a mouse. You have dominoes, each 2×1 in shape – i.e. each covers two adjacent squares. Can you tile the mouse-eaten chessboard with these dominoes? In other words, can you lay down the dominoes to cover every square exactly once (no empty squares and no overlaps)?



☐ yes

☐ no

Placing a domino on the board is one move in this solitaire game. For each move, you choose where to place the domino – which means you might have many choices for each move. Can you cover the whole board? The space of possible moves grows rapidly. Hence, look for an invariant: a quantity unchanged by any move of the game.

Because each domino covers one white square and one black square, the following quantity is invariant (unchanged):

$$I = \text{number of uncovered black squares} - \text{number of uncovered white squares}. \quad (4)$$

With a regular chess board, the initial position would have $I=0$, from 32 white squares and 32 black squares. With this modified board, two black squares have vanished, so I is $30-32=-2$. However, in the winning position, all squares are covered; therefore $I=0$. Because I is invariant,

Oh this is very clever - I just kept trying to tile the board until I convinced myself it couldn't be done - I didn't even think about solving the problem this way.

I realized that it would have been possible if both missing corners had been on one side (one black and one white), but didn't attribute it to the color differences.

Same. I never even thought to look for an invarient, i just applied what I thought of as symmetry

Yeah, I had the same conclusion - I didn't even realize it was 2 black squares that were missing! If it was one white and one black, then it would have to work. huh, that's cool.

I'm curious to know whether or not all boards with the same number of white and black tiles could be tiled with dominoes. I feel like almost all the invariant questions we are dealing with are done to prove something is not true,

This explanation helped me a lot.

took me a while to see this. I approached this problem with recursion trying to break down the board into similar shapes, until I reached a base case.

I was pretty sure the answer to this was no, but I couldn't find a way to prove it...seeing that two 'black' spaces were removed makes it much clearer.

I also wanted to say no, but without being able to prove it and knowing the way this class can be, I expected to be shocked by some crazy simplification that solved the problem.

It really helps if you can visualize. I got this one much quicker then the integral.

this is really smart

So I don't understand why I not equaling 0 means it's impossible to cover the board. I had to cover this whole board. Also, I feel like a better way to solve this problem is to minimize it. Change it into a 2x2 board and you see how it's impossible.

Actually, when I=0 it means that it is possible to cover the board. The explanation says that with a regular chess board, the initial position would have I=0. This makes sense because the board is square with an even number of squares on each side. This explanation is saying that in this scenario, I=-2, and that's why it's impossible.

no sequence of domino moves can turn the initial uncovered board into the winning board (with all squares covered).

Optional!

7. Symmetry for second-order systems

This problem analyzes the frequency of maximum gain for an LRC circuit or, equivalently, for a damped spring–mass system. The gain of such a system is the ratio of the input amplitude to the output amplitude as a function of frequency.

If the output voltage is measured across the resistor, and you drive the circuit with a voltage oscillating at frequency ω, the gain is (in a suitable system of units):

$$G(\omega) = \frac{j\omega}{1 + j\omega/Q - \omega^2},$$

where $j = \sqrt{-1}$ and Q is quality factor, a dimensionless measure of the damping. Do not worry if you do not know where that gain formula comes from. The purpose of this problem is not its origin, but rather using symmetry to maximize its magnitude.

The magnitude of the gain is

$$|G(\omega)| = \frac{\omega}{\sqrt{(1-\omega^2)^2 + \omega^2/Q^2}}.$$

Find a variable substitution (a symmetry operation) $\omega_{new} = f(\omega)$ that turns $|G(\omega)|$ into $|H(\omega_{new})|$ such that G and H are the same function (i.e. they have the same structure but with ω in G replaced by ω_{new} in H). Use the form of that symmetry operation to maximize $|G(\omega)|$ without using calculus.

When maximizing a parabolic function such as $y = x(6 - x)$, the symmetry is reflection about the line $x = 3$. In symbols, the transformation is $x_{new} = 6 - x$.

Let's transfer a few lessons from the parabola example to the problem of maximizing the gain. In the parabola example, the symmetry is a reflection about an interesting point (there, the point halfway between the two roots $x = 0$ and $x = 6$). Analogously, an interesting frequency is $\omega = 1$ because it makes the real part of the denominator in $G(\omega)$ go to zero, and making the real part go to zero helps minimize the denominator.

Therefore reflecting about $\omega = 1$ is worth trying, perhaps $\omega_{new} = 1 - \omega$. For frequencies, however, differences are not as important as ratios. For example, a musical octave is a factor of 2 in frequency, rather than a difference. So reflect in a multiplicative way: $\omega_{new} = w^{-1}$.

This transformation works either in $G(\omega)$ or in the magnitude $|G(\omega)|$. It's slightly easier in $G(\omega)$:

$$G(\omega) = \frac{j\omega}{1 + j\omega/Q - \omega^2} \mapsto H(\omega_{new}) = \frac{j/\omega_{new}}{1 + j/Q\omega_{new} - 1/\omega^2_{new}}.$$

Multiply numerator and denominator by ω^2_{new}:

$$H(\omega_{new}) = \frac{j\omega_{new}}{\omega^2_{new} + j\omega_{new}/Q - 1},$$

which is the same function as $G(\omega)$, except for negating the real part in the denominator. Negating the real part in the denominator doesn't affect the magnitude of the denominator, so $|H(\omega_{new})|$ has the same form as $|G(\omega)|$.

Comments on page 4

This is very interesting... I wish I could think of it this way...

I thought about this in the same way but I couldn't find an argument strong enough to prove it. The proofs are always so subtle that I can't come up with them but they make perfect sense when I see it.

Are you ever going to give us a question where there is a solution?

Haha that's true...most of the time now I begin with the assumption that the problem can't be solved.

Well he had us solve for f_1000 in the last class.

This is a solution. there is one exact answer -"you can't do it". ha.

Oh this is interesting. I looked at it with a completely different method that now I think of it, is rather uncouth.

And what different method was that?

What I did was I thought about the 4X4 case and I reasoned that if it can't work for the 4X4 case, then it won't work for the 8X8 case

I did something similar - I realized that the board was symmetrical along the midline, and realized that in each half there were an odd number of squares so it wasn't possible.

I did something similar as well - I looked at just the first two columns and identified that it wouldn't work when applied to the whole board.

I did it your way at first and then the notion of the number of black and white squares hit me.

See I got this answer, but not through any of the means you've taught us. I'm worried that I'm not learning how to apply them well enough.

Since $\omega_{new} = 1/\omega$, the maximum value of ω_{new} will be ω_{max}^{-1}. That's one equation.

Since the two magnitudes $|G(\omega)|$ and $|H(\omega_{new})|$ are the same function, the maximum value of ω_{new} is also the maximum value of ω. That's the second equation.

Together they produce $\omega = \omega_{new} = 1$ (ignoring the negative-frequency solution $\omega = -1$). At that frequency, $|G(\omega)|$ is Q. For the electrical and mechanical engineers: The quality factor Q is also the gain at resonance.

8. Inertia tensor

[For those who know about inertia tensors.] Here is the inertia tensor (the generalization of moment of inertia) of a particular object, calculated in a lousy coordinate system:

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix}$$

Change coordinate systems to a set of principal axes. In other words, write the inertia tensor as

$$\begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

and give the values of I_{xx}, I_{yy}, and I_{zz}. *Hint:* What properties of a matrix are invariant when changing coordinate systems?

Whatever coordinate change I make, I will leave the x axis alone because the I_{xx} component is already separated from the y- and z submatrix. That submatrix is

$$\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

I have to figure out how changing the coordinate system changes this submatrix. Rather than find the coordinate change explicitly, I use invariants to avoid that computation.

One invariant of any matrix, not just of this 2×2 matrix, is its determinant. Another invariant is its trace (the sum of the diagonal elements). In the nasty coordinate system, the trace of the y- and z submatrix is $5 + 5 = 10$. So the trace is 10 in the nice coordinate system. The determinant is $5 \times 5 - 4 \times 4 = 9$, so it the determinant is 9 in the nice coordinate system.

Those facts are sufficient to deduce the submatrix in the nice coordinate system (without needing to figure out what the nice coordinate system is). In the nice coordinate system, the 2×2 submatrix looks like

$$\begin{pmatrix} I_{yy} & 0 \\ 0 & I_{zz} \end{pmatrix}$$

So I need to find I_{yy} and I_{zz} such that

$$I_{yy} + I_{zz} = 10 \quad \text{(from the trace invariant)}$$

and

$$I_{yy} I_{zz} = 9 \quad \text{(from the determinant invariant)}$$

The solution is $I_{yy} = 1$ and $I_{zz} = 9$ (or vice versa). So the inertia tensor becomes

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

Comments on page 5

I didn't do the optional questions, but I like reading their solutions. Thanks for taking the time to post them!

Agreed. They are fun to review.

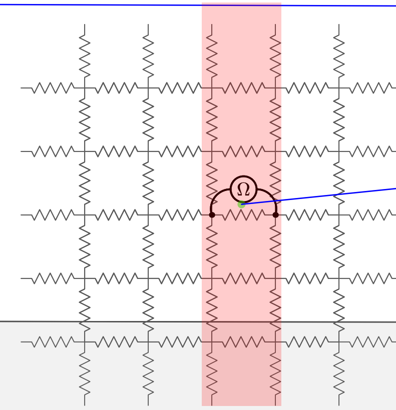
I like the idea of applying abstraction to 2.001. This really honed in the concept for me.

I did this by eigenvalues, which are also invariant under basis set transformation. The eigenvalues of the 2x2 submatrix are 1 and 9, so just write out the diagonalized matrix with those values.

That's true, so the method outlined in the solution set gives you a way to compute eigenvalues. That's the method I use to compute eigenvalues in my head.

9. Resistive grid

In an infinite grid of 1-ohm resistors, what is the resistance measured across one resistor?



To measure resistance, an ohmmeter injects a current I at one terminal (for simplicity, say $I = 1\,\text{Å}$), removes the same current from the other terminal, and measures the resulting voltage difference V between the terminals. The resistance is $R = V/I$.

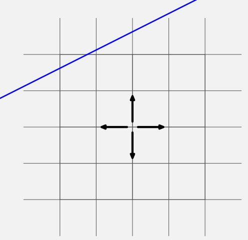
Hint: Use symmetry. But it's still a hard problem!

I'd like to find the current flowing through the resistor when 1 Å is sent into one terminal of the ohmmeter and removed from its other terminal. The solution has two steps, each subtle:

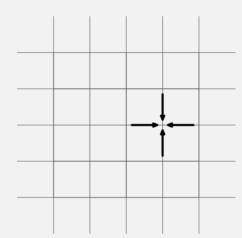
1. Break the resistance-measuring experiment into two parts, each having a lot of symmetry.

2. Analyze those parts using symmetry.

The current distribution that results from the full resistance-measuring experiment is not sufficiently symmetric because it has a preferred direction along the selected resistor. However, if I break the experiment into two parts – inserting current and removing current – then each part produces a symmetric current distribution.



By symmetry – because all four coordinate directions are equivalent – inserting 1 Å produces 1/4 Å flowing in each coordinate direction away from the terminal. Let's call this terminal the positive terminal. So inserting the 1 Å at the positive terminal produces 1/4 Å through the selected resistor, and this current flows away from the positive terminal.



By symmetry, removing 1 Å produces 1/4 Å in each coordinate direction, flowing toward the terminal. Let's call this terminal the negative terminal. So removing 1 Å produces 1/4 Å through the selected resistor, flowing toward the negative terminal. Equivalently, it produces 1/4 Å flowing away from the positive terminal.

Now superimpose the two pictures to reproduce the experiment of measuring the resistance. The experiment produces 1/2 Å through the resistor, flowing from the positive to the negative terminal. The voltage across the resistor is the current times its resistance, so the voltage is 1/2 V. Since a 1 Å test current produces a 1/2 V drop, the effective resistance is 1/2 Ω.

If you want an even more difficult problem: Find the resistance measured across a diagonal!

Comments on page 6

Do you think you could explain this problem in class?

This problem was awesome! i'd love to see it make its way into a lecture or even the textbook, provided its extremely well explained. I think there is so little E&M involved in the problem that it would be fair.

That's interesting. I did the resistive grid on just this part, i.e., infinite in one dimension, and got the same answer.

That result worries me. It means that the entire rest of the grid isn't participating in current flow – i.e. isn't providing alternative paths between the two ends of the ohmmeter.

Whereas I would expect the resistance of the outlined ladder to be slightly higher than 1/2. When I calculated the resistance of the outlined ladder and got a number slightly larger than 0.5.

I checked my answer again. I got (1+sqrt(3))/(3+sqrt(3)) approx 0.577

Sorry about the confusion.

In thinking about this, I was always worried that it didn't superimpose. That is, what if the fact that inserting current at one node and removing current at a second node affected the way the current flows because different directions have differing lengths between the nodes. That's pretty cool. I realize that I have to consciously fight against that worry in order to use superposition.