6.055J/2.038J (Spring 2010)

Solution set 6

*Submit your answers and explanations online by **10pm on Wednesday, 14 Apr 2010**.*

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Homework is graded with a light touch: P (made a decent effort), D (made an indecent effort), or F (did not make an effort).

Problem 1 Guessing an integral using easy cases

Use easy cases to choose the correct value of the integral

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx. \qquad (1)$$

☐ $\sqrt{\pi\alpha}$

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The most useful special cases here are $\alpha \to 0$ and $\alpha \to \infty$. When α is zero, the Gaussian becomes the flat line $y = 1$, which has infinite area. The first choice, $\sqrt{\pi\alpha}$, goes to zero in this limit, so it cannot be right. The second choice, $\sqrt{\pi/\alpha}$, has the correct behavior.

The limit $\alpha \to \infty$ gives the same conclusion: The first choice cannot be right, and the second one might be right.

Comments on page 1

Here it is, for the memo due Thursday.

I definitely enjoy starting with some easier problems like this one. Sometimes the week's readings get me a little confused, but problems like this ease me into the work and assure me that I am learning and do remember important facts.

True I also like the warm up.

Oh, I think I used dimensional analysis here too.

I did as well, it just seemed more intuitive.

I thought this was a good problem to introduce this problem set. Some people were complaining in the comments about how they don't think they could do it without choices, but in real life easy cases isn't really used to get an answer. It's much more useful for checking to see if an answer is incorrect which is very useful in real life.

Yeah I agree with you because this question is meant to test use of easy cases. Therefore being able to rule out an answer as incorrect would be the best you can do with easy cases, as seen in this problem.

I tried doing this with easy cases but i was biased from having seen this integral multiple times in the recent past - made just using atan easier.

I really like this easy cases method- It comes very easily to me, compared to the other methods we've learned. I found this pset much easier than previous ones.

can you do this with the FWHM concept you showed in class today? I guess I do not know what this graph looks like without my calculator.

in problems such as these, would anyone ever split their "answer coins" in any way but all or nothing or half and half? in other words, what would be the justification for something like 7:3?

to clarify my comment: i feel like if you know it, it's 0:1, or if you don't, it'd be like 1:1. i don't know how you can "kinda" know it.

Hahahahahah. Wow. I had the right method, but was totally deficient about what e^0 was. Oops.

so this is useful when you have choices, but was there a way of getting an approximate answer without choices?

We did something like this in class (I think before spring break). It dealt with creating units for x.

It might be helpful to note in the problem that this is the Gaussian too.

I approached this just substituting values for a... I used a=1 and then picked an arbitrary value of a=5... a little different but still came out to find the same answer.

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The limit $\alpha \to \infty$ gives the same conclusion: The first choice cannot be right, and the second one might be right.

How do we know this?

You don't have to know it, but you might recognize it as the form of the Gaussian distribution (PDF).

I'm not familiar with the PDF. Does Gaussian just refer to the graph of the integral?

The function is a form of a Gaussian function with particular parameters. You would graph the function, not the integral.

I would not have thought of this, although I think I could have figured out that it should be a line at 1 and not 0.

This is useful, I didn't think of changing a all the way to 0

This is the only case I did. You don't really need to do both, since there are only 2 choices. If you can cancel out one, using even just one method, you know you're home free. And for me, it was pretty easy to realize the area under a straight horizontal line is infinite.

Oh I didn't even consider a = inf, a=0 just came much more naturally.

Using a = infinity would be a second check if you used a = 0 first. You come to the same conclusion, but it's nice to see that it works twice.

what do you mean by this?

Sorry, "Gaussian" is just jargon for the e^-something*x^2 function.

I first thought about this problem as "a increasing" and "a decreasing", and noted that a increasing should decrease the integral (because it makes the exponent more negative everywhere), and vice versa. It's essentially the same concept, but sometimes infinite and everywhere-zero integrals can be tricky.

I got the answer, but not that way. I depended too much on multiple choice, and pattern matching.

Yeah me too

Ahh. Goodness...so simple!

wow I didn't think of this before, i thought this is a very cool approach !

That's exactly how I did it! (Though I did cheat with mathematica first...)

I did it exactly this way too–but I had a brain fart, so when I thought to myself that as a approches 0, I somehow said that area approached 0 too. I used the easy cases method right though

This makes complete sense and I think it is a very good easy cases problem.

Doesn't the second choice go to infinity as a-> 0?

yes, which is what we are looking for...from the line above.

that's basically the reasoning used. I guess i should have considered the a -> 0 case too so that I could have been some certain about my answer.

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I did this a similar way, but I thought just in graphical terms. I know what exp(-x^2) looks like, and I know what adding a positive constant in front of the x^2 does, so I reasoned that the integral value had to get smaller.

I always forget to check the other case. I usually just stop when I have one easy case result. I guess I should get in the habit of always trying mins AND maxs.

I considered a->0, but not a->inf. I got the right answer anyway.

I was able to arrive at this same conclusion, but if I didn't have the two options I think I'd be stuck.

I came to the same conclusion but I pieced together equations that I knew like e^cx and e^x^2. I didn't even think about taking the cases where a went to zero or infinity because I was so focused and so used to dealing with the x.

I think in general its usually easier to look at the solutions, and what is variable. Noticing that a was either on the denominator or numerator, you can immediately go back to the integral and see which one corresponds to the right behaviour of the integral.

I think this example was very simple for that reason.

I agree. I also started to focus on the exponent rather than on a by itself. It becomes clearer that this is the thing to do later on in the pset though.

Is there a way to derive the pi^1/2 in the answers using easy cases as well?

I looked at the cases where a is 1 and x is not squared. I guessed that the answer would take a similar form and since the integral of e^(-ax) is of the form 1/a I guessed that the answer here had to have a fraction in it.

I used dimensional analysis for this problem (even though it said to use easy cases). If I assume the dimensions of x is length, then the dimensions of a have to be l^-2. The dimension of the integral is l, so the dimension of the answer has to be x^-1/2. This matches the solution obtained using easy cases.

but since we already did this in class, I think we were trying to use different methods to check our answer here...

would "rough graphing in my head" be considered a use of the "easy cases" method?

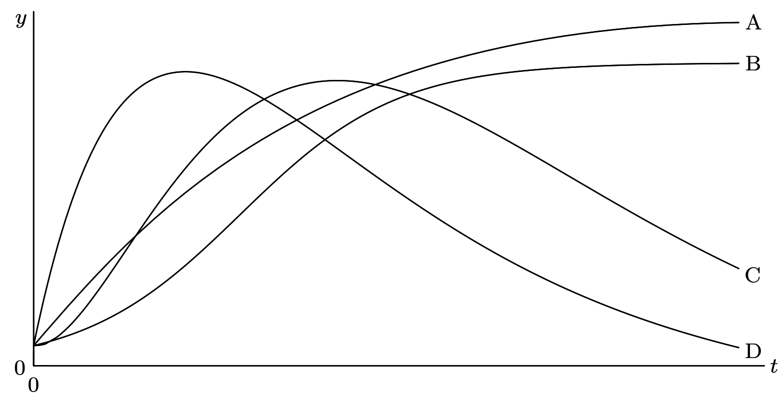
I think it would be considered estimation broadly, but I don't know about calling that "easy cases"!

Problem 2 Differential-equation solution

Which sketch shows a solution of the differential equation

$$\frac{dy}{dt} = Ay(M-y),$$

where A and M are positive constants?



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$$\frac{dy}{dt} = AMy \propto y.$$

It is the equation for exponential growth (since AM is positive). Therefore, for small t, the curve should follow an exponential, which is concave upwards ('holds water'). Only curves B and C satisfy this test.

In the large-t extreme case, y approaches M. Then $dy/dt = 0$, which makes y constant (consistent with the assumption $y \to M$). Among curves B and C, the only curve that becomes flat is curve B.

As a further piece of evidence in favor of curve B, the derivative dy/dt must always be positive. Why? For it to be negative, y would have to exceed M. But when y reaches M, then dy/dt becomes 0 and y stops changing. Therefore, y can never exceed M. Contradiction! Therefore, the derivative cannot be negative. Curve C, however, has a region of negative slope.

This is just an autonomous equation from 18.03. They are really easy to solve if you can sketch out the behavior of y' vs y.

I've never taken 18.03, so some of this class has actually been quite difficult for me.

how would it change if they were negative

I think it would depend on the signs of A and M relative to each other. For instance, if A was positive but M was negative, then dy/dt would always be negative (when y > 0). But if A was also negative, then I think dy/dt would always be positive (again when y > 0).

I guess this is a pretty standard differential equation problem. I thought about it the wrong way though - I should have seen M as a stable point (y greater, and it goes back down, y less and it goes up), and thus B would be obvious. Oh well.

I didn't quite do this, but I wound up setting the slope to zero and finding which values of y had a flat slope... in particular, y=0 was one of those values so I also came to the same conclusion that it must be concave upwards (or rather, flattish at y=0).

Are we assuming that y is also approaching 0 in this small case?

Just that y is smaller than M such that in M-y, the y can be ignored

How do we know y is small when t is small?

We know that y is small when t is small because A and M are constants. Therefore, if you let t be small, y is on the other side of the equation, and has to fluctuate accordingly.

The two cases I looked at were where dy/dt=0. looking at the equation, this happens either as y approaches 0 or y approaches M. so it has to be choice B. also, this is the logistic growth equation, which has an s shape.

I dont quite get this. Why is it proportional to y if y is really small and insignificant .

well since A and M are constants, the only variable is y. therefore it is proportional to y.

That is an interesting way of solving this. I tried to use easy cases, but it never occurred to me to use proportional reasoning as well.

When things are small it means we can cut out second order terms (hence proportional to y), but we still need some sort of independent variable to relate everything together.

agreed. if y is tiny, why is it not "porportional to zero?"

This alludes me; I don't quite understand the proportionality.

This is the same method I used to analyze it. I definitely think it's the simplest and fastest way to do it.

I made a stupid mistake here and forgot for a second that we were looking at dy/dt so I said the slope for small y was linear, which it clearly is not. I really liked this problem though.

I made that mistake at first also, but then I had to go back and change it.

I am not sure I even remembered this equation, but I agree now that this is a simple way to negate some options.

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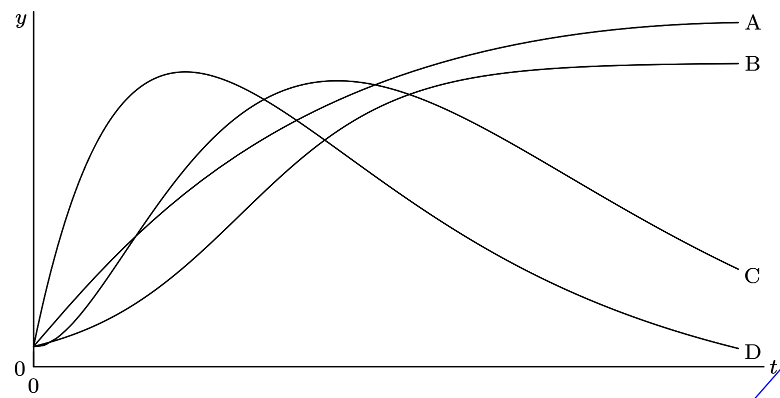

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I really liked this explanation. I missed the fact that at a small t the curve will show exponential growth. But it makes sense as it is rewritten here.

Didn't think of this.

I just figured that it should be a positive growth.

how about D, it also shows some exponential growth in the beginning

This is an interesting solution! However, I'm still confused as to how this is an example of the "easy cases" we've learned.

It's easy if you let M and A = 1 and look at your solution. You still arrive at the same solution but it might be a bit easier to see.

Its an example of easy cases because we are looking at smal t and large t.

I feel like easy cases is y=0 and y=M, and don't worry about limits... isn't that technically calculus? :)

please don't highlight the entire solution box in the future.

thanks.

agreed. it makes it difficult to mark other areas of the solution. That being said I agreed with the solution pretty easily.

I used similar thinking, but it was clearly faulty, since I was deciding between curves A and B.

So did I, I still don't really understand how they were choosing between B and C, I was choosing between curves C and D, oops.

How do you know what happens for large t. I basically looked at the the effects on the slope of the curve from small and large y. I still got the right answer though.

I agree. I thought that large y meant that the m term disappeared completely. I don't know why it can only approach M.

Could you explain why y approaches M in the large-t case?

Hmm i forgot about extreme cases here... i ended up looking at the zeros. I was a little off though.

Once I narrowed down to curves B and C (using a slightly different method from above), I mistakenly looked at the curve for large y and not large t. Thus, my answer was curve C based on that wrong assumption. However, could someone please explain why y approaches M for large t.

I did the same thing, apparently I've forgotten how to think about this type of differential equation.

b/c the small was made easy y comparing y to M, it holds to do the same for a large case of t. Make y–> M in order to simplify your max case into an easy case...i think...

Couldn't we also assume the case as a y parabola?

exactly what I was thinking!

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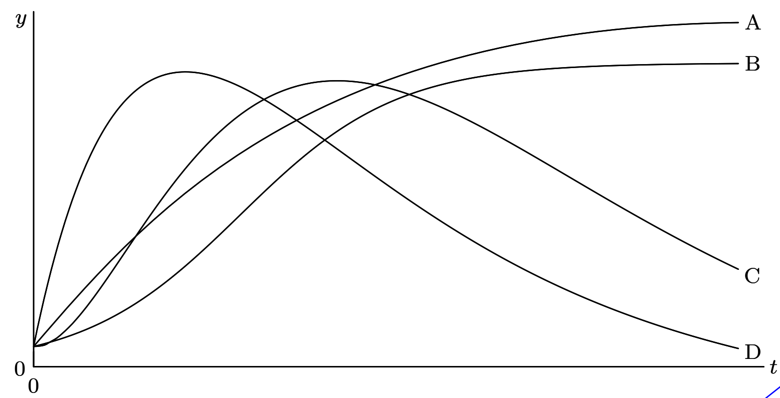

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but if the slope is 0 (y=m) this could be a maximum right? In which case, C could be a solution.

This was my assumption and I also chose C, however The graph follows from the equation with y as the dependent variable. As time increases, and the derivative gets to zero, the y will no longer increase any more, which is what it would have to do in order to have a negative slope.

i hadn't even thought of this!

This is a great explanation for further proof but I would have never thought to reason this myself!

Why does it have to stop changing when it becomes 0? If there was a local maximum it would also be zero but then the slope would continue to change afterwards.

That's an interesting point. The reason is that the equation is first order. The only way for y to change is for dy/dt to be nonzero. Once dy/dt becomes zero, y won't change, and dy/dt will stay nonzero.

With a second- (or higher-) order equation, that argument doesn't work. For example, in spring motion, dy/dt=0 at the extreme of the motion, but d^2y/dt^2 is not zero, and it makes dy/dt eventually nonzero, which means y can change (so the spring oscillates).

I think I'm still confused about why y can't grow larger than M.... is it because y=dy/dt is a function of y? As I was thinking about this, I realized I was thinking of the situation where dy/dt is a function of t, and I'm not sure I have any intuition for when it's a function of y.

I'm glad you answered that. It might be good to mention in the solutions that this is all due to it being a first-order system.

Also, don't you mean that once dy/dt becomes 0, y won't change, and dy/dt will remain 0? you wrote "stay nonzero"

Ah, I see. I was debating between D and C, but this makes sense.

I completely got this one wrong. I figured that the derivative was a linear term (AM)y minus a quadratic term (AM)y^2, and that there was some point where the graph would go pos–>neg. I realize I was really off after reading this explanation.

Me too. I thought it was C or D, but I was pretty confident it was C.

Makes sense. I just misinterpreted the graph, by think of y as the dependent variable of the equation, instead of just the slope.

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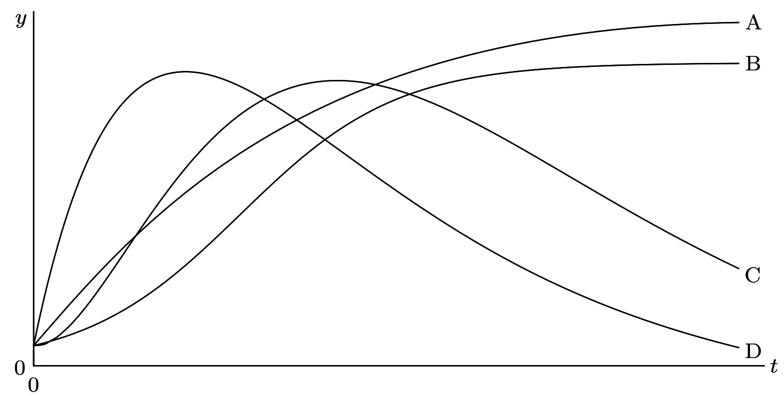

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could you explain this in class? I reasoned that if y goes to infinity then the slope should be negative after multiplying out the equation to get dy/dt=ayM-ay^2. The negative term will dominate.

if y rises above M, then yes the slope will be negative. But since all of these curves start between 0 and M, the correct curve is confined between 0 and M and is always increasing. y=M is a "sink", because for all y>0, y will tend towards M with time (from below if y<M, from above if y>M). Similarly, for y<M, y tends away from 0.

But how do we know that all of the curves start between 0 and M and are always increasing. I've forgotten a lot of my 18.03 so please bear with me.

@11:31 Thank you! I get it now....I was definitely one of the people that started thinking as t gets big.

More interestingly, we can show that every solution curve is monotonously increasing or decreasing (not strictly, since there are solutions y=0, y=M).

First, note the fact that two solutions can never cross. A rough proof of this is that if two curves were to cross, then they would have different derivatives dy/dt at the same point (t,y). But dy/dt is a function of y and t (well, just y in this case, but that's ok), so every correct solution needs to have the same derivative at (t,y) as every other solution. Moreover, if two solutions did go through the same point, they would necessarily be the same solution.

Now, here's where the fact that dy/dt=f(y) comes in useful. Since the slope doesn't depend on t, any solution curve can be translated left or right, and will still be a solution. If a solution ever increased and then decreased, it would be possible to translate it horizontally such that the new curve crossed the old curve, violating the theorem "proved" in the previous paragraph. So every solution is monotonously increasing or decreasing, or constant!

At y=0, the slope=0. The only solution that is horizontal at y=0 is B.

You won't know this is true unless you see the rest of the graph (x less than 0)...maybe all the graphs move steeply back up in the y direction.

from my interpretation of the graph, B & C look like the might have a slope of 0 at y=0

Problem 3 Fog

Fog is a low-lying cloud, perhaps 1 km tall and made up of tiny water droplets (radius $r \sim 10\,\mu m$). By estimating the terminal speed of fog droplets, estimate the time that the cloud takes to settle to the ground.

$10^{\square} \pm \square$ s *or* $10^{\square} \ldots \square$ s

To include in the explanation box: What is the everyday consequence of this settling time?

At low Reynolds number, the drag is

$$F = 6\pi\rho_{fl}\nu v r, \tag{2}$$

where ρ_{fl} is the density of the fluid. The weight of the object is

$$W = \frac{4}{3}\pi r^3 \rho_{obj} g, \tag{3}$$

where ρ_{obj} is the density of the object. At the terminal speed v, the drag and weight balance:

$$6\pi\rho_{fl}\nu v r \sim \frac{4}{3}\pi r^3 \rho_{obj} g. \tag{4}$$

Therefore, the terminal speed v is

$$v \sim \frac{2}{9}\frac{gr^2}{\nu}\frac{\rho_{obj}}{\rho_{fl}}.$$

(This calculation neglects buoyancy, which is a small effect for water droplets falling in air.) Calling $2/9 = 1/5$ and using $\nu \sim 10^{-5}\,m^2\,s^{-1}$ gives

$$v \sim \frac{1}{5} \times \frac{10\,m\,s^{-2} \times 10^{-10}\,m^2}{10^{-5}\,m^2\,s^{-1}} \times 1000 \sim 2\,cm\,s^{-1}.$$

As a check on the initial assumption, let's calculate the Reynolds number:

$$Re \sim \frac{10^{-5}\,m \times 2 \cdot 10^{-2}\,m\,s^{-1}}{10^{-5}\,m^2\,s^{-1}} \sim 0.02.$$

It is much less than 1, validating the assumption of low-Reynolds-number flow.

At $v \sim 2\,cm\,s^{-1}$, the droplet takes $5 \cdot 10^4$ s to fall 1 km. A day is roughly 10^5 s, so the fall time is about one-half of a day. The everyday consequence is that fog settles overnight. You go to sleep with a pea-soup fog, and by the time you wake up, it's mostly settled onto the ground – and maybe evaporated as the morning sun warms the ground.

Comments on page 3

this one was hard

I find I am very limited by my unfamiliarity with physics equations when solving these types of problems, but once I know which ones to use I can figure out how to estimate the solution

I had NO idea how to do this problem using the terminal speed/ reynolds number.. I took my gut feeling that when fog appears at night it is usually gone in the morning and so I figured it takes about 12 hours to settle... my intuitive answer was close but lacked any mathematical proof!

it was neat knowing the consequence before trying to approximate!

We were to use this equation for low reynolds number, which occurs when the density is low, the velocity is low, the diameter is small, and the viscosity is large. Viscosity of air has magnitude 10^-3, density has magnitude 1, and the diameter is 20*10^-6 micrometers. This gives a reynold's number of 10^-3*v, so it's fair to use this drag equation. When I first thought about it though, I didn't do it this analytically - I just thought: "oh, air isn't viscous, I'll use the other drag equation".

oh! I totally make a silly mistake, I don't know why I didn't think of this! I instead just used the terminal speed equation - but I got an answer much smaller (like 24 minutes). Drag really slows it down!

> haha, I did the EXACT same thing apparently–I also got 24 minutes. I forgot to use the Reynolds number completely, and I used the drag equation we had from before. this answer here makes much more sense.

i used drag force as F rhoAv^2 and the drop as a cube. still go the same order of magnitude though so approximating still works out

can we get an equation sheet for the important things to remember like this- I always forget them and have to go and dig them back up

can you maybe list in class when reynolds numbers are high vs low?

I should have used this equation. Instead I used F=rho*A*c_d*V^2

> Yes this formula would have been very useful!

> Yeah, I made the same mistake.

where is this equation from again?

what does the two different v's mean?

> viscosity and velocity. this came from at least two different readings

> The v with a curve on the left stands for viscosity and the v with the curve on the right stands for velocity. I agree that it is difficult to distinguish sometimes.

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(This calculation neglects buoyancy, which is a small effect for water droplets falling in air.) Calling $2/9 = 1/5$ and using $\nu \sim 10^{-5}\,m^2\,s^{-1}$ gives

$$v \sim \frac{1}{5} \times \frac{10\,m\,s^{-2} \times 10^{-10}\,m^2}{10^{-5}\,m^2\,s^{-1}} \times 1000 \sim 2\,cm\,s^{-1}.$$

As a check on the initial assumption, let's calculate the Reynolds number:

$$Re \sim \frac{10^{-5}\,m \times 2 \cdot 10^{-2}\,m\,s^{-1}}{10^{-5}\,m^2\,s^{-1}} \sim 0.02.$$

It is much less than 1, validating the assumption of low-Reynolds-number flow.

At $v \sim 2\,cm\,s^{-1}$, the droplet takes $5 \cdot 10^4$ s to fall 1 km. A day is roughly 10^5 s, so the fall time is about one-half of a day. The everyday consequence is that fog settles overnight. You go to sleep with a pea-soup fog, and by the time you wake up, it's mostly settled onto the ground – and maybe evaporated as the morning sun warms the ground.

are these just equations we should know, because we've taken this class?

I'm not sure where we got this either. The formula I came up with was based on the viscous drag force equation and was different.

Yes! That equation (Stokes drag) was the subject of reading 23 (r23... on NB) and used in the reading on the conductivity of seawater (r24... on NB), which I admit was not the most easy-to-follow reading. But hopefully the application in this problem helps make that equation more meaningful.

This equation took forever to find within the readings. Maybe we could create a class list of equations from the readings? We could use NB to comment on them and better understand their complexities and relationships.

We should have a review day and go over everything that we learned, including a recap of the equations!

I didn't think to use this equation and ended with the wrong answer.

wow... forgot gravity. There goes a factor of ten.

I forgot that there were two densities in this problem, that of the air and that of water, since both of them are fluids. Oops!

The object is air? I made this same mistake.

I think the object is water and the fluid is air.

I made the same mistake. I didn't think about air as a contributing factor. That's why my answer made no sense :(

This is something that makes a lot of sense, but I didn't realize before; at terminal velocity these balance. It makes it so much clearer and I finally understand this. Thanks.

In pset 4 you neglected this 4pi/3 term...why include it here this time?

I wanted to get a reasonably accurate value of the settling time (how long the droplet takes to fall) because otherwise the everyday consequence doesn't work out so well.

where did the (2/9) derive from?

In the previous step, we discovered that at the terminal speed v, the drag and weight balance. We then solve for the terminal speed v, by dividing the right-hand side by everything on the left except the terminal speed v. Since the pis cancel out, 4/3 divided by 6 is 4/18, or 2/9!

Problem 3 Fog

Fog is a low-lying cloud, perhaps 1 km tall and made up of tiny water droplets (radius $r \sim 10\,\mu m$). By estimating the terminal speed of fog droplets, estimate the time that the cloud takes to settle to the ground.

$10^{\square} \pm \square$ s *or* $10^{\square} \ldots \square$ s

To include in the explanation box: What is the everyday consequence of this settling time?

At low Reynolds number, the drag is

$$F = 6\pi\rho_{fl}\nu v r, \tag{2}$$

where ρ_{fl} is the density of the fluid. The weight of the object is

$$W = \frac{4}{3}\pi r^3 \rho_{obj} g, \tag{3}$$

where ρ_{obj} is the density of the object. At the terminal speed v, the drag and weight balance:

$$6\pi\rho_{fl}\nu v r \sim \frac{4}{3}\pi r^3 \rho_{obj} g. \tag{4}$$

Therefore, the terminal speed v is

$$v \sim \frac{2}{9}\frac{gr^2}{\nu}\frac{\rho_{obj}}{\rho_{fl}}.$$

(This calculation neglects buoyancy, which is a small effect for water droplets falling in air.) Calling $2/9 = 1/5$ and using $\nu \sim 10^{-5}\,m^2\,s^{-1}$ gives

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I don't think this is the same velocity that was calculated for a rain droplet, is it? Why does this have more terms?

When i used the terminal velocity for the rain droplet instead with fog-droplet radius, it only took 2.5 hours to settle. What's wrong with this?

The raindrop speed was calculated assuming high Reynolds number, where the drag coefficient is approximately 1 (equivalently, where the drag force is proportional to vˆ2). But the Reynolds number is too small for that assumption to be valid.

I was slightly over on this, which made me underestimate the time.

Is this a necessary check?

Checks are always a good thing!

yes. yes. yes. It should have been done in the reading, too. Always check assumptions!

What is the check intended for? Fog movement doesn't seem like something that is getting turbulent.

I gave mine .5 for a sphere. It didn't work that nice

My final answer didn't work out nicely either; it was off by a HUGE factor. Instead, i resorted to guessing the time fog takes to settle, and backsolved a speed of 10cm/s.

For this problem I used the formula for terminal velocity of a raindrop from 2 psets ago, and then changed the radius....however my answer was about an hour, guess I did something wrong

So I attacked this problem first using the everyday example. But I was steered pretty wrongly. I forgot to analyze fog at 1km and used my example form home. During which I only saw fog settling in about an hour, as it wasn't 1 km above the ground.

ah, when you said everyday consequences i for some reason thought you meant a much more dramatic conclusion

This is a pretty interesting consequence, definitely something we don't think about often!

This is not the way I thought of it at all. Fog is one of those things that is always around you but you never really think about.

Do all lay lowing clouds act as fog does from this demonstration? Is it something in the property of the cloud or is it just its relative distance to the ground that makes a cloud lower itself onto the ground?

I really like how the problem had us check to see if our approximation was realistic. It's definitely just as important as the approximation itself.

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doesn't the fog settle because the temperature drops also and allows the water to condense? not just because it is falling?

Yeah, that's how my parents explained it to me when I was little. Some insight, anyone?

Does the fog also get 'burned' off by the morning sun? Or is this just something that people say.

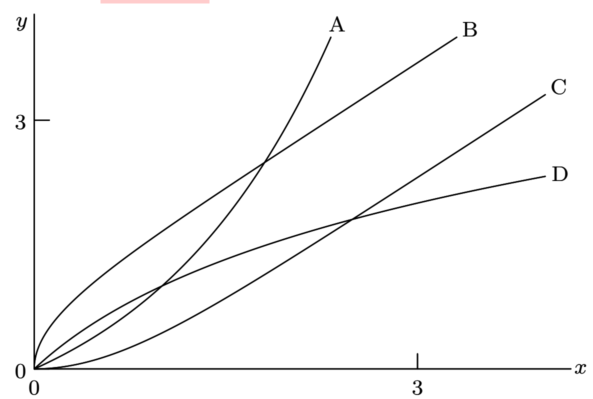
I've never heard the word "burned" used..they probably mean evaporated.

The temperature drops, which makes the water condense out of the air (formerly it was water vapor, i.e. a gas). That condensation is the fog droplets! But that is a separate process from the settling, which is gravity pulling the droplet downward.

I just took the question to mean what happens in the morning, I too have heard that fog "burns" off...when it gets hot enough the dew point rises and the vapor evaporates

Problem 4 Hyperbolic-function sketch

Which graph is $\ln \cosh x$ (where $\cosh x \equiv (e^x + e^{-x})/2$)?



☐ Curve A

☐ Curve B

☐ Curve C

☐ Curve D

Use easy cases: $|x| \to \infty$ and $x \to 0$. In the $x \to \infty$ case, $\cosh x \approx e^x/2$, so $\ln \cosh x \approx x - \ln 2$. In the $x \to -\infty$ case, $\cosh x \approx e^{-x}/2$, so $\ln \cosh x \approx -x - \ln 2$. In other words,

$$\ln \cosh x = |x| - \ln 2 \qquad (|x| \to \infty). \tag{5}$$

This is enough information to select curve C.

But let's check that curve C is correct also in the $x \to 0$ case. There, a Taylor series for e^x gives

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2} + \cdots\right) + \left(1 - x + \frac{x^2}{2} - \cdots\right)\right]. \tag{6}$$

The result is $\cosh x \approx 1 + x^2/2$. For the logarithm, the Taylor series is

$$\ln(1 + z) \approx z. \tag{7}$$

So,

$$\ln \cosh x \approx \ln\left(1 + \frac{x^2}{2}\right) \approx \frac{x^2}{2}. \tag{8}$$

Thus, near the origin, $\ln \cosh x$ looks like an upward-facing parabola (concave up). Curve C passes this test.

Comments on page 4

What exactly is cosh?

* Hyperbolic cosine:

coshx= 1/2 *(e^x +e^-x)

In a circle with radius 1 x^2+y^2=1, you can plot x=cos t and y = sin t.

In the hyperbola x^2-y^2=1, you can plot x = cosh t and y = sinh t.

Hence, hyperbolic cosine and hyperbolic sine.

I did use the easy cases for this one. But when I analyzed them, I did not have enough understanding for graphing to narrow down much. In the end it came down to a guess.

Same. seeing the cosh function just threw me totally off.

When doing my approximating, I left out this - ln 2 term which prevented me from figuring out whether the answer was curve B or C. I wish I hadn't dropped what I thought was extra luggage.

> To decide between curves B and C, you can use the x -> 0 easy case, which says that the curve looks like an upward-facing parabola at the origin. That test knocks out curve B.

I looked at easy cases for the derivative. As x-> infinity, f' goes to the indeterminate zero times infinity. Does this tell us anything?

I dont get the coshx notation...

do you mean x -> 0?

No this is correct, but not shown on the graph, it would be in the third quadrant. I didn't look at this case either because the graph does not show this area.

No this is correct, but not shown on the graph, it would be in the third quadrant. I didn't look at this case either because the graph does not show this area.

x -> negative infinity. I missed the - sign too.

I don't think I used easy cases to do this problem.

I used a similar approach, but I only addressed the +infinity situation, and not the -infinity situation, probably because the graph doesn't show that side. Why is it important that we look at the -infinity situation here, but not in other problems?

I just used the fact that y x at high x. I then took the derivative of the function (not nearly as hard as it looks) and found that at x=0, the slope was also zero

This is an interesting idea. I like the notion of looking at derivatives as easy cases, because sometimes they can be.

This is how I did it too. I first eliminated A and B by plugging in x=3 (y should be less than 3) and then by realizing that the slope should be close to 1.

I did the exact same steps...but I confused the infinity and negative infinity cases...oops...

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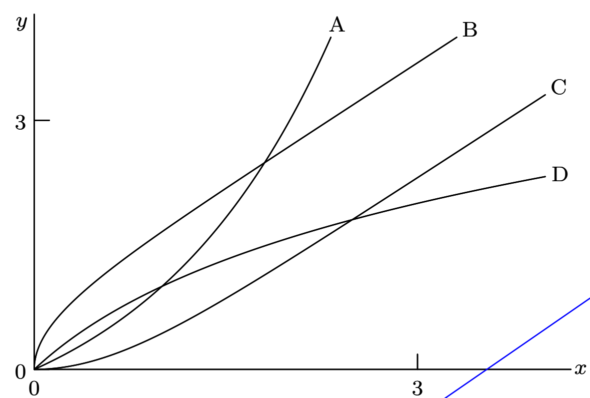

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This one was a little to math intensive for me, I had to get help on this problem. I'd never heard of cosh before really.

Interesting. I chose x=3 as my easy case, where the cosh function goes to 20 and ln coshx goes to a little over 2. C was closest so I chose C.

I wasn't completely convinced that this meant it was C. I did this same analysis and came up with the right answer, but I took this piece of information to mean that the end behaviour would have to be linear.

> I think this is indistinguishable from curve B. I used this to narrow it down to B and C, but they both look like the slope goes to 1 as x goes to infinity. I guess you could use the -ln(2) to guess where the y-intercept would be, but i didn't think that was accurate enough to guess off the graph.
>
> > I sort of agree. When I first read the solution, I wasn't convinced. But come to think about it, for large x, the difference between B and C is the offset. Since teh solution comes out to x - ln(2), the curve that has a negative offset is correct.
> >
> > > when I first read the solution, I didn't see the '-ln2' ... I feel like ln[cosh(x)] = |x| narrows it down to either B or C and ln[cosh(x)] = -ln2 narrows it down to C or D.
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>
> This statement isn't helpful. I see how the negative y-int. as x->inf would point to C, but not how the x->-inf tells us anything, since we don't know what that part of Curve C looks like.

I should have used this method. The method I used was more complicated and had more room for failure.

Not really good w/ log manipulation. I think it threw me on this problem. Reasoning is solid though.

ooh i forgot about those

I didn't think about incorporating Taylor series into this problem. That is actually really cool.

Does this count as an easy case? I wouldn't have known how to deal with this equation even in this case.

Woah I did not even consider a Taylor series expansion, but it's nice that the math works out so nicely.

> I tried to use Taylor series at first, but my math gave me unusual terms. Good to know that I was at least heading in the right direction.
>
> > yeah I didn't think of this at all!

totally got this competely wrong

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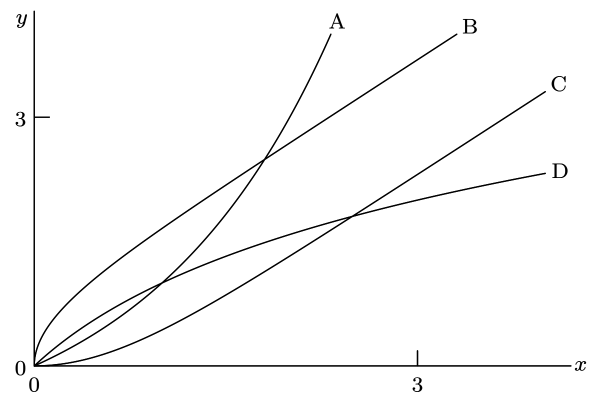

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Here's an alternate solution using easy cases:

At x = 0, coshx = 2/2 = 1. Thus, y = ln(coshx) = ln(1) = 0. All 4 curves fit that description so no elimination yet. If we take the derivative of y and see how dy/dx reacts for large x, we can eliminate curve A since curve A increases very quickly and we can see from simple calculus that this curve has a constant slope at large values of x. I chose to eliminate curve B because according to the graph, curve B has a value of y > 3 for x = 3, and we can see that ln(coshx) has a value of y < 3 for x = 3. So know we decide between curves C and D. If we take the second derivative, we see that it is = 1 > 0, which implies positive concavity. Curve D has negative concavity, so I eliminate that. Curve C has positive concavity, so I choose curve C.

Hmmm. Cool process of elimination.

I used easy cases in a different way, by letting x=1 and getting a quick approximation. Since we have x and y =3 marked on the axis it was easy to check the curves and I arrived at the same answer.

That's really interesting, I didn't think of it that way.

I did the same thing, by calculating whether or not the formula would be greater than 3 when x=3. I suppose if there were no numbers labelled on the graph that would be a little harder to do

Totally forgot about this. Should have done it.

I think using a bit of calculus is easier (and less error-prone) than using a Taylor series expansion. At least that is the case for me.

Could we talk about the smart use of Taylor series in class?

How do we know this is C rather than A? Isn't A also an upward-facing parabola?

I think because we divide by 2 we expect it to be flatter than A.

I also went with the "lets enter a few more numbers" approach and if you take the x-ln2 approach you can enter a few numbers and see that it must be C. Especially since values for all curves are given for x=3!

I got this one right, but for a possibly shady reason. I know ln(e) =1, so I figured any function of e transforms into something roughly linear when you take the ln of it. Using the property of logs, I guessed that C was the only curve that does this.

Curve B also has a linear portion though... You would have to decide whether the curve should be concave up or down at x=0. Also, curve B is above the y=x line, whereas curve C is below, in the limit of x -> infinity.

Problem 5 Guessing an integral

Choose the correct value of the integral

$$\int_{-\infty}^{\infty} \frac{1}{a^2 + x^2}\,dx, \tag{9}$$

where a is a positive constant.

- ☐ πa
- ☐ π/a
- ☐ $\sqrt{\pi a}$
- ☐ $\sqrt{\pi/a}$

The easiest special case is $a \to \infty$. In that limit, the integrand is zero everywhere, so the integral is zero. The first and third choices are therefore incorrect.

To decide between the second and fourth choices, use the special case $a = 1$. The integral becomes

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^2}\,dx$$

The integral is $\arctan x$. At ∞ it contributes $\pi/2$, and at $-\infty$ it subtracts $-\pi/2$, so the integral is π. Only the second choice, π/a, has the correct behavior when $a = 1$.

Comments on page 5

can we go over the different ways to think about this

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This was actually one of the integrals I had to memorize in school, so I knew the answer without even guessing. The explanation below is more satisfying though.

sort of, but it's only applicable when you're given four well-constructed multiple choice answers. Otherwise, dimensional analysis to give the 1/a and the a=1 trick could do it for you.

It was also the case for me that this is one integral drilled into my head in school, so I knew the definite answer without needing to do approximations. I like that there is a way to go through it if I ever forget the integral.

This one, at least, I did right. ><

I used dimensional analysis for this one too...

I decided moving a to zero and got a similar answer for the integral of 1/x^2.

I reached the same conclusion using dimensional analysis but then I got stuck figuring out if it was answer choice 2 or 4.

yeah, that's the part dimensional analysis can't do for you.

me too. I resorted to approximating an area using rectangles and triangles, but that gave me answer 4 instead of 2.

I guess I never thought to vary "a" instead of "x".

i think that's one of the points the pset was trying to get across. i varied x for the first problem, but thought about a for this one instead.

instead of using logic such as a-> to infinity, i prefer to think in terms of. as a increases/decreases... is this an okay application of "easy cases"?

I like this approach

It was really helpful for me to go over this problem in class... it helped me better understand how to use easy cases in approaching this problem.

Whoops. That seems really obvious in hindsight now.

Wow..this problem is a lot easier than I made it out to be. This was the problem that took me the most time on the problem set and I was no where close...

I thought it looked familiar to something...

I still don't quite understand a lot of these integral problems; I feel like that's a hard thing for me to grasp in this estimation course.

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This seems to illustrate that there is a limit to easy cases. integral of 1/(1+xˆ2) dx is certainly much easier than 1/(aˆ2+xˆ2), but it still doesn't help much if you're trying to do back of the envelope calculations and don't happen to have an integral table ready. Also, why doesn't a=0 work?

Unless you happen to have memorized this particular integral. I had used it enough times to recognize that there was something that looked like arctan of x going on here right from the beginning

yeah, I used this exact same method–the case of a approaching infinity is easy, so you can easily eliminate choices 1 and 3. I happened to recognize this integral, so I got lucky...but if you didn't remember this integral then i guess you were out of luck and had to guess between choices 2 and 4.

I agree, I couldn't choose between 2 and 4 b/c I did not realize this was arctan.

Agreed, it's still not very easy.

I did use a=0, and graphed the function. It has infinite area under it, so I think you draw the same conclusions as the first part of the solution..

You can also keep 'a' as a constant by realizing that the integral comes out to (1/a)arctan(x/a) and then plugging in infinity. π/2a - (-π/2a) = π/a

This was tricky! Forgot my 18.01 integrals for a second there...

Bah! I didn't even think of that.

I relied on my intuition and figured that a sqrt(pi) on top just didn't seem right, so I guessed the right answer.

Had the option for a sqrt(pi/a) been present, I fear I likely would have chosen that over pi/a.

heh, i did the same...couldn't reason out why the sqrt(pi) was there.

but: sqrt(pi/a) would not have the right dimensions, so that wouldn't make sense either.

the choice is sqrt(pi)/a not sqrt(pi/a) ... so the dimensions would work even though it's wrong.

...i think.

Haha. Me too. I figured out a=1 was the best easy case, but for some reason I forgot the integral would be arctanx.

Yah, i did not recognize the arctan!

This is something that I probably should have remembered from my math but slipped my mind

I definitely didn't remember this either. I tried to use lumping to come up with a good area underneath the curve in terms of triangles, and it worked fairly well.

I was pretty stoked that I actually remembered this technique

I got suck here because I couldn't remember the integral of arctan() . Was there any way of solving this without knowing that?

I recognized this, but just reasoned that getting a sqrt(pi) out would be difficult, so I concluded 2 as well.

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$$\int_{-\infty}^{\infty} \frac{1}{1 + x^2}\,dx$$

The integral is $\arctan x$. At ∞ it contributes $\pi/2$, and at $-\infty$ it subtracts $-\pi/2$, so the integral is π. Only the second choice, π/a, has the correct behavior when $a = 1$.

I recognized the integral as arctan(x), but i couldn't for the life of me remember or reason the behavior of arctan(x) at either infinity, so i had to look it up.

Is there an easy way to visualize this?

I also think that the pi/2 that arctan(x) contributes is very hard to intuit. I would also love to see a way to explain/visualize this.

I couldn't figure out how I was supposed to learn something useful from the normal easy case of setting a to zero. Using 1 in this case to generate the integral of arctan really was a clever trick for this problem.

What was the correct behavior when a=1? Please state it so I can learn.

Yeah and the a isn't in the square root in any of them so it's unlikely the pi would be either

That makes sense, my reasoning for not choosing the square root was that it just looked funny for a problem so nicely written.

I couldnt pick between the two answers, but I was tempted by the "prettiness" of pi*a!

Is there some way that we could figure this out (other than Wikipedia) if we didn't recognize or memorize the integral or how to solve the arctan?

Again, I had a hard time figuring out if this was the second or fourth answer.

Problem 6 Debugging

Use special (i.e. easy) cases of n to decide which of these two C functions correctly computes the sum of the first n odd numbers:

☐ Program A:

```
int sum_of_odds (int n) {
  int i, total = 0;
  for (i=1; i<=2*n+1; i+=2)
    total += i;
  return total;
}
```

☐ Program B:

```
int sum_of_odds (int n) {
  int i, total = 0;
  for (i=1; i<=2*n-1; i+=2)
    total += i;
  return total;
}
```

Special cases are useful in debugging programs. The easiest cases are often $n = 0$ or $n = 1$. Let's try $n = 0$ first. In the first program, the $2n + 1$ in the loop condition means that $i = 1$ is the only case, so the total becomes 1. Whereas the sum of the first 0 odd numbers should be zero! So the first program looks suspicious.

Let's confirm that analysis using $n = 1$. The first program will have $i = 1$ and $i = 3$ in the loop, making the total $1 + 3 = 4$. The second program will have $i = 1$ in the loop, making the total 1. Since the correct answer is 1, Program A has a bug, and Program B looks good.

Comments on page 6

This one was a fun problem, even for someone who doesn't code much.

I found this one really trivial, just one case of n=1 and you can get the answer

Maybe easy, but a nice confidence builder.

I agree; this was a very easy problem, but I think it's actually a good way to teach someone who doesn't know how to code about coding. I think you can figure out a bit about the actual coding process from this problem. Very nice.

I thought this was the simplest problem on the pset, but it paralleled nicely with what we did in class

I am totally unfamiliar with programming but when a friend gave me a hint for how to solve this problem, it was definitely much simpler than I thought!

I'm glad to see something familiar to me (code).

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wouldn't this return a -1 when with n = 0, and shouldn't the answer be 0? If I am right, how are we supposed to know that the program is right if it is wrong on one of our easy cases?

You are right - that condition will return a -1, which is NOT less than i (which is 1).

With n = 0, this condition that i <= 2n-1 will not be met, so the body of the for loop is not executed. The total that is returned is just 0, since that's what total starts at.

It may be a little unfamiliar if you're not used to seeing code.

using n=0 here seems suspect here since normally you count at least one odd. And negative odd numbers wouldn't work. Nevertheless a very intuitive problem solved by simple cases

This was the logic I used, I'm pretty happy I did a problem right (even though there were only 2 options to choose)

I got this problem incorrect because I'm not familiar with the code....

The important thing about this problem is knowing that computers start indexing at zero whereas our gut intuition is to begin at 1.

That doesn't really matter here. The for loop dictates that we start iterating i at 1 anyway.

you might note how the second program behaves for i=0. counting from i=1 to -1 by +2 equates to not running at all, I assume?

I had to rerun through the numbers several times to finally get the correct results when I was doing the problem.

It's good to see I got this one right on the head. I used the same analysis as well.

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I just tested n = 0, is that enough seeing as there are only two choices and one of them did not work with n = 0, or should we always check multiple values?

I used the same test case. A simpler reasoning would be if you are counting odd numbers, you can't subtract from the lowest one, which is 1.

yup

I did this with n=2, just because, as we established in class, 0 and 1 are sometimes 'special cases' and I didn't want to risk them outputting the same answer and wasting my time.

i used n=4 for the same reason...also because I was able to better understand it with 4

Problem 7 Damped, driven spring

A damped, driven spring–mass system (e.g., in 18.03, 2.003, 2.004, 6.003, and maybe also 8.01) is described by the differential equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 e^{i\omega t}, \tag{10}$$

where m is the mass of the object, b is the damping constant, k is the spring constant, x is the displacement of the mass, ω is the (angular) frequency of the driving force, and F_0 is the amplitude of the driving force. The solution has the form

$$x = x_0 e^{i\omega t}, \tag{11}$$

where x_0 is the (possibly complex) amplitude.

Which graph, on log–log axes, correctly shows the transfer function F_0/x_0? Don't solve the differential equation – use an approximation method to guess the answer!

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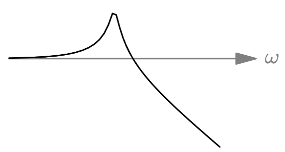

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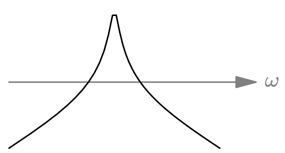

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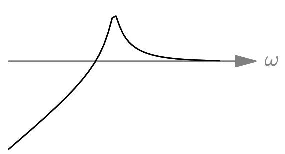

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Use easy cases. At low frequencies ($\omega \to 0$), the spring moves very slowly, meaning that derivatives with respect to time become tiny. Therefore, the time-derivative terms $m(d^2x/dt^2)$ and $b(dx/dt)$ become much smaller than the kx term. The remaining equation is

$$kx \approx F_0 e^{i\omega t}. \tag{12}$$

With $x = x_0 e^{i\omega t}$, the transfer function x_0/F_0 is $1/k$. This function is independent of frequency, so the curve must be flat at low frequencies. The only curve that matches this criterion is curve A.

As a check, let's try really high frequencies ($\omega \to \infty$). Then the second-derivative term $m(d^2x/dt^2)$ is the dominant term, so the differential equation simplifies to

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so the magnitude of the transfer function $|x_0/F_0|$ is $1/m\omega^2$. On a log–log graph, that is a -2 slope, which could be curves B or C but not curve A.

I couldn't really figure out a good way to do this problem, partly since you explicity said not to solve the differential equation- which was the only way I could think of doing it!

Yeah, I was also confused on how to approach this problem. I ended up going with my "gut" based on the limited information I remember about spring systems from 18.03.

I too was really confused on this problem (having only taken 8.01 on this list of classes). I made a guess based on low w - which sounds like it might be correct - but I don't really understand what the graphs mean in relation to the system.

8.01 would never get into things like this...

Brings back unpleasant memories of extreme cases...

I was a little confused here. What exactly does the transfer function tell you? I was able to make a reasonable guess on the homework, but I really wasn't sure what was going on in the problem.

I thought something was funny

That makes a lot more sense.

I actually followed your methods below while solving and was able to justify Curve A for low frequencies. But using the F/x, I couldn't justify the high-frequency behavior. So I went with Curve A, anyway.

me too. The F/x really threw me off in justifying every part of the graph for any answer.

I blissfully assumed it was output/input, ignored the equation, and instead just thought about a physical spring-dashpot system. When you drive it at low frequency, it behaves like just a spring, that is, with no frequency dependence. But if you try to drive it at high frequency, the dashpot absorbs all the energy and you don't get anything out.

Yikes, I solved for the original transfer function F_0/ x_0 and got the right answer (A)... I probably should have gotten (C) though since the real question is opposite, right?

I solved for the original transfer function as well, but wouldn't it only flip the graph vertically? The function should still be flat for small w (therefore A is still the best answer), correct?

I think this is why I picked C, the opposite curve of A.

I like this qualitative analysis better than all the math. This part is very intuitive.

I can't believe I never considered that option. It would have helped me greatly.

I will forever remember this demo that they did in 18.03 of the mass and spring. Makes soo much physical intuitive sense. very useful picture.

Whereas I will always remember Prof. Leeb swinging a toilet plunger back and forth in 6.131...

i agree it's usually easier to figure it out in your head this way if you know that the transfer function is getting at.

Finally 8.01 intuition paid off!

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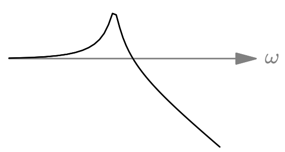

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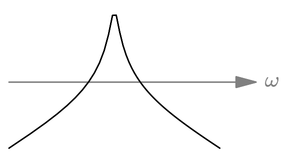

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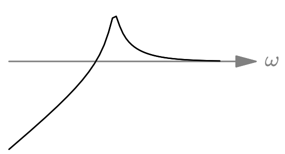

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I was sort of confused on how to start this problem, but this makes sense.

Me too. I didn't think to use frequencies right away, but this makes so much sense.

I looked at this problem for a long time, and still ended up taking a pretty random guess because I had no idea how to approach this problem. I didn't think of using frequency at all. but this makes sense

cool I think I got this one. This problem looked so intimidating.

Yeah I thought so too that it was quite intimidating. I didn't see how simple it was to disregard the time derivatives as w –> 0 which makes the problem really easy.

I think this is a good solution

I put curve A because it looked like all the second order systems we studied in 6.003, but I had no idea how to solve this problem using approximations, without doing the actual derivation.

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I like that we picked the more useful area first, but then as a double check knocked out the second two as a check

I like how this worked out but for some reason my logic ended up with the complete opposite answer.

I didn't do this so carefully and got A, because after getting my results from the first limit, I really just wanted to check that the slope was negative in the second limit, so I didn't catch the error

I was so confused about this. since the problem was flipped I had that it was a +2 slope. I still picked the right answer because of the approximation for small omega, but I couldn't figure out why the slope was +2 instead of -2 like in the picture, which made me think it had something to do with the -w^2, even though that made no sense on a log-log graph...

What exactly does log-log mean?

It means that log(gain) is plotted against log(frequency), the way we plotted log(drag coefficient) versus log(Reynolds number).

This problem maybe assumed too high a level of comfort with log-log plots and differential equations. But it was very well proposed and I thought the solutions were very helpful, aside from the last sentence about curve A being eliminated...isn't the answer curve A?

I got pretty lucky and guessed C on this problem just because of the definition of damping.

I did the exact same thing haha.

Am I crazy, or does he mean it could curves A and B but not curve C? That's what it looks like to me in the graphs..

Yeah, it must be. Curve A is definitely right.

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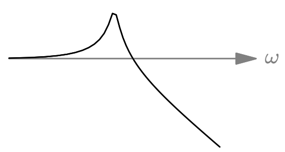
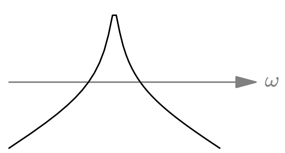
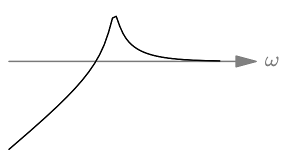
☐ Curve A 

☐ Curve B 

☐ Curve C 

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The analysis here makes a lot more sense than what I tried to work out. I don't have a very good intuition into these types of problems. What was the actual answer (to the question you intended)?

i thought it was C, from what i learned in 2.004. i don't see how B could be correct.

I had no idea – so I guessed C as well because it looked familiar.

I decided it was curve C based on the physical intuition but I really don't know what is supposed to be happening in B. Is the motion just disappearing...? acting like there isn't any force at all?

B is like a band-pass filter, where only certain frequencies resonate, and high and low are damped.

So none of the solutions are correct?

so what is the actual answer to this problem? Could you also go over this one in class?

I agree with you. After reading the explanation it seems he gives argument for multiple answers. Is that what he's actually saying? Or am I just getting too caught up on the right answer and ignoring the thought process?

you mean to say it could be either A or B, but not C

I would hope so. Otherwise this question has no answer anyways he writes it.

It's A, this is a tough typo though.

Yeah, I was quite confused by this last sentence. It definitely has to be curve A, though, just based on the way second-order systems like this respond.

Why can it be two of the options?

Because only option A fits with the requirement that the function be constant for small values of omega. The second piece of evidence just confirms that it can't be option C.

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