# 6.055J/2.038J (Spring 2010)

# Solution set 1

*Do the following warmups and problems. Submit your answers, including the short explanation, online by* ***10pm on Wednesday, 17 Feb 2010****.*

***Open universe:*** *Collaboration, notes, and other sources of information are* ***encouraged.*** *However, avoid looking up answers to the problem, or to subproblems, until you solve the problem or have tried hard. This policy helps you learn the most from the problems.*

*Homework is graded with a light touch: P (made a decent effort), D (made an indecent effort), or F (did not make an effort).*

In the following questions, you are often asked to give your answer as a plausible range. For most of the questions, it is the exponent $x$ in $10^x$ that you are asked for. You can specify $10^x$ as $10^{a\pm b}$ or as $10^{c...d}$ (where $c = a - b$ and $d = a + b$). Think of $b$ as the sigma ($\sigma$) measuring your uncertainty, or $c \ldots d$ as the one-$\sigma$ range. Use the format that easier for you to think about in that question.

When you choose your plausible range, remember that the goal is not to be 'right' by choosing a giant, guaranteed-safe range or, at the other extreme, to pretend to have extra confidence by choosing an overly narrow range. Rather, the goal is to choose your range such that you would be somewhat surprised if the true value falls outside your range. Numerically, choose the range so that it has a 2/3 probability of containing the true value.

That criterion explains why the range narrows after you estimate using divide and conquer. At first, you have little idea about the true value, so you would not be surprised were it to fall outside a fairly large range; after the estimate, you know more, your confidence in the estimate increases, and your plausible range shrinks.

## Warmups

**1. One or few**

Use the 1 or few method of multiplication (and division) to estimate

$$161 \times 294 \times 280 \times 438$$

(a random multiplication problem generated by a short Python program).

$10^{\square \pm \square}$ ***or*** $10^{\square \ldots \square}$

Then compare your range with the actual answer.

The first step is to convert each factor in the product to the nearest power of ten, perhaps also including a factor of a few. For example, 161 contains two factors of 10 and a factor of 1.61; and 1.61 is closer, on a log scale, to 1 than it is to few ($\sqrt{10}$). So 161 becomes simply 100 or $10^2$. Here are the conversions for all four factors:

$161 \to 10^2$

$294 \to 10^2 \times \text{few};$

$280 \to 10^2 \times \text{few};$

$438 \to 10^2 \times \text{few}.$

Now the product is easy to do mentally. There are eight factors of 10 and three factors of a few. Since $(\text{few})^2 = 10$, three factors of a few becomes $10 \times \text{few}$. So

$$161 \times 294 \times 280 \times 438 \approx 10^8 \times 10 \times \text{few} \approx 3 \cdot 10^9.$$

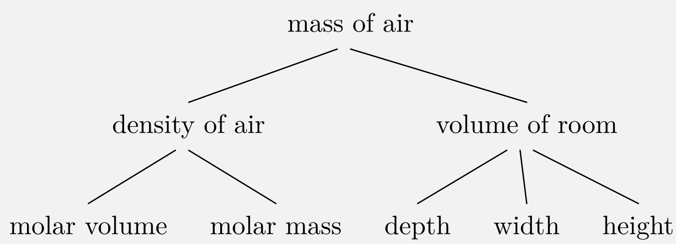
In the form $10^x$, the estimate is $10^{9.5}$ because 3 (or few) is one-half of a power of 10. The estimate is only a factor of 2 smaller than the actual value of 5805041760 or roughly $6 \cdot 10^9$.

**2. Air mass**

Use divide-and-conquer to estimate the mass of air in the 6.055J/2.038J classroom and explain your estimate with a tree. If you have not yet seen the classroom, try harder to attend lecture!

$10^{\square} \pm \square$ kg ***or*** $10^{\square} \ldots \square$ kg

One way to estimate the mass is to subdivide into the volume of the room and the density of air. The volume of the room subdivides into its length, height, and width. I remember that the density of air is roughly $1\,\text{g}\,\ell^{-1}$ (or $1\,\text{kg}\,\text{m}^{-3}$) because I have used the value often in estimation problems. Alternatively, you can use a useful fact from chemistry, that one mole of an ideal gas at standard temperature and pressure occupies 22 liters, and combine that fact with the molar mass of air. Using that method, the tree is

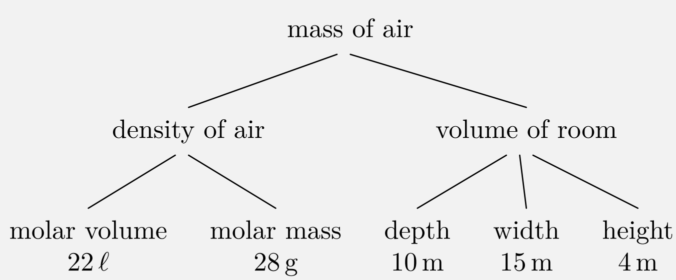



Now put values at the leaves.

For the room dimensions, the MIT schedules office webpage gives the room area, but let's estimate the dimensions by eye. Most rooms are 8 or 9 feet high but our classroom (4-163) has high ceilings, so let's say 12 feet high or 4 m. The room has about 10 rows, spaced around 1 m apart. So the depth is about 10 m. The room is perhaps 1.5 times as wide as it is deep, so the width is roughly 15 m. As a check, these estimates mean the area is $150\,\text{m}^2$ or about $1600\,\text{ft}^2$; the MIT classroom-inventory page (linked from the course website) says that the area is 1303 square feet, so our estimate of the area is accurate to 25%.

The molar volume for air (like any ideal gas) is 22 liters. The molar mass is, roughly, the molar mass of nitrogen, which is 14 g. But nitrogen is diatomic, so the actual molar mass is 28 g.

The tree with values is:



Now propagate values upward. The volume of the room is $600\,\mathrm{m}^3$. The density of air is roughly $28/22\,\mathrm{g}\,\ell^{-1}$, or roughly $1\,\mathrm{kg}\,\mathrm{m}^{-3}$. Therefore, the mass of air in the room is roughly 600 kg. In the form $10^x$ kg, it is halfway between (on a log scale) few $\times\, 10^2$ kg and $10^3$ kg. Because few is one-half of a power of 10, the mass is in the middle of the range $10^{2.5}..10^3$ kg. So let's call it $10^{2.75}$ kg. Either $10^{2.5}$ kg or $10^3$ kg would also be a reasonable estimate if you are rounding the exponent to the nearest 0.5.

## Problems

**3. Mass of the earth**

Estimate the mass of the earth.

$10^{\square \pm \square}$ kg ***or*** $10^{\square \ldots \square}$ kg

After choosing your range (in either form), check it against the measured value.

The mass breaks into density times volume:

$$m \sim \frac{4}{3}\pi r^3 \rho, \tag{1}$$

where $r$ is the radius of the earth, and $\rho$ is the density of the earth. Note that even in this first step we have already approximated by assuming that the earth is spherical and that it has a uniform density.

To estimate $r$, I remember that California is about 3000 mi away from Boston (a typical flight at 500 mph takes about 6 hr) and it is also 3 time zones away. So each time zone is about 1000 mi, meaning that the circumference of the earth (24 time zones) is $C \sim 2.4 \cdot 10^4$ mi giving a radius of $r = C/2\pi \sim 4000$ mi. In metric units, that is $6.4 \cdot 10^6$ m.

To estimate $\rho$, I start the density of water: $10^3\,\mathrm{kg}\,\mathrm{m}^{-3}$. The earth is made up mostly of iron and dense rock, both much denser than water – maybe by a factor of 5. Why a factor of 5? A factor of 3 would be too low, since that is the density of typical surface rocks, and they are the material that floated to the top when the earth was cooling, so they are less dense than the rest of the earth. A factor of 10, on the other hand, sounds way too dense. So I'll choose a factor of 5, making $\rho \sim 5 \cdot 10^3\,\mathrm{kg}\,\mathrm{m}^{-3}$.

Then the mass is, using $\pi \sim 3$,

$$m \sim 4 \times (6.4 \cdot 10^6\,\mathrm{m})^3 \times 5 \cdot 10^3\,\mathrm{kg}\,\mathrm{m}^{-3}. \tag{2}$$

Do the arithmetic by divide and conquer. The powers of 10 total to 21: 18 from the cubed radius and 3 from the density. Then there's the factor of 4, a factor of $6.4^3$, and a factor of 5. If the $6.4^3$ were $6^3$, it would be 216, so let's pretend that $6.4^3$ is 250. Then the factors are $4 \times 250 \times 5 = 5 \cdot 10^3$. The result is a mass of $5 \cdot 10^{24}$ kg or $10^{24.7}$ kg. (The true value is $6 \cdot 10^{24}$ kg.)

**4. Explain a UNIX pipeline**

What does this UNIX pipeline do?

```
ls -t | head | tac | head -1
```

If you are not familiar with the individual UNIX commands, use the `man` command on Athena or on any other handy UNIX or GNU/Linux system.

The `ls -t` lists all the filenames in the directory ordered by recency with the most recent first. The next step, `head`, takes the first 10 lines. Therefore so far we have a list of the 10 newest files. The `tac` reverses this list so that we still have a list of the 10 newest files but it is ordered from 10th newest at the top to newest at the bottom. The `head -1` takes the first line from this list, giving us the 10th-newest file.

**5. Atmospheric carbon dioxide**

What is the mass of $CO_2$ generated by the world annual oil consumption?

$10^{\boxed{\phantom{00}} \pm \boxed{\phantom{00}}}$ kg/year ***or*** $10^{\boxed{\phantom{00}} \ldots \boxed{\phantom{00}}}$ kg/year

Here is the (unbalanced!) combustion of a generic hydrocarbon (including oil, gasoline, and kerosene):

$$CH_2 + O_2 \longrightarrow CO_2 + H_2O.$$

I'll start with the US oil consumption, roughly $3 \cdot 10^9$ barrels/yr. Then increase it by a factor of 4 to get the world oil consumption: I often remember reading that although the United States has 5% of the world's population, it uses 25% of the energy. I don't remember whether the 25% was talking about energy overall or just oil, but maybe it doesn't matter. I'll then convert barrels to liters using $160\,\ell$ per barrel and then to mass using $1\,\mathrm{kg}\,\ell^{-1}$ (assuming oil and water have comparable density).

Finally, I'll convert this mass of oil into a mass of carbon dioxide. According to the (unbalanced) chemical reaction, one mole of hydrocarbon ($CH_2$) becomes one mole of carbon dioxide ($CO_2$). I might just either ignore the effect of balancing the equation; on the other hand, it is not hard to determine: No other products or reactants involve carbon, so the coefficients in front of $CH_2$ and $CO_2$ must be identical. In other words, balancing may give strange coefficients for the other products and reactants, but it leaves the $1:1$ mole ratio between $CH_2$ and $CO_2$. A mole of $CH_2$ weighs 14 g whereas a mole of $CO_2$ weights 44 g, almost 3 times as much as the mole of $CH_2$. So, to convert mass of oil into mass of carbon dioxide, I'll multiply by 3 (or few).

The overall calculation is then:

$$3 \cdot 10^9\,\mathrm{barrels/yr} \times 4 \times \frac{1.6 \cdot 10^2\,\ell}{1\,\mathrm{barrel}} \times \frac{1\,\mathrm{kg\ oil}}{1\,\ell} \times \frac{3\,\mathrm{kg\ CO_2}}{3\,\mathrm{kg\ oil}}. \quad (3)$$

Now do the numbers. There are 11 powers of 10 and then the following factors:

$$3 \times 4 \times 1.6 \times 3 \sim 60. \quad (4)$$

So the estimate is $6 \cdot 10^{12}$ kg per year.

Out of curiosity, I wanted to compare this number to the actual world production of carbon dioxide. It's hard to find the carbon-dioxide production due just to oil. But oil might be one-third of the world energy consumption (there's also natural gas, hydroelectric, coal, etc.). Multiplying the above estimate by 3 gives an estimate for the world production of carbon dioxide: $18 \cdot 10^{12}$ kg per year or roughly $2 \cdot 10^{13}$ kg per year. The actual total in 2006 was $3 \cdot 10^{13}$ kg.

**6. Piano tuners**

Here is the classic Fermi question: Roughly how many piano tuners are there in New York City? (These questions are called Fermi questions because the physicist Enrico Fermi was an acknowledged master of inventing and solving them.)

$10^{\square} \pm \square$ ***or*** $10^{\square} \ldots \square$

I'll break this one into several pieces:

1. The number of families in NYC. It is a big city, so maybe there are $10^7$ people and therefore $10^7/4$ families.

2. The fraction of families that have a piano. Having a piano is not common – often people say, "Oh, you have a piano!" when they come to our apartment – but it's not so uncommon that I am amazed when I see a house with a piano. So I'll estimate this fraction as 1/10 (i.e. 1 family in 10 has a piano).

3. How often a piano needs to be tuned. Judging by our own piano, it needs to be tuned every year, but we somehow don't arrange it that often; maybe once every 2 years is more realistic.

4. How long it takes to tune a piano. Piano tuning looks like an intricate task investigating all the strings, etc.; maybe it takes half a day. I'll estimate 3 hours for it.

5. How many hours of work a piano tuner needs to stay afloat. A regular work week of 40 hr times 50 weeks gives 2000 hr in the year. Perhaps piano tuning involves lots of traveling; plus it's hard work. So maybe a fulltime piano tuner spends 1500 hours per year tuning pianos.

Now I use convenient forms of unity to find the number of piano tuners:

$$10^7\,\text{people} \times \frac{1\,\text{family}}{4\,\text{people}} \times \frac{1\,\text{piano}}{10\,\text{families}} \times \frac{1\,\text{tuning/piano}}{2\,\text{yr}} \times \frac{3\,\text{hr}}{1\,\text{tuning}} \times \frac{1\,\text{yr of work}}{1500\,\text{hr tuning}}. \tag{5}$$

There are a total of 3 powers of 10: 7 from the $10^7$ and 4 in the denominators (10 families and 1500 hours of work). What's left is

$$\frac{1}{4} \times \frac{1}{2} \times 3 \times \frac{1}{1.5}. \tag{6}$$

The 3 and the $2 \times 1.5$ cancel leaving 1/4. The number of tuners is therefore $10^3/4$ or 300. In the form $10^x$, it is roughly $10^{2.5}$.

**7. Your turn to create**

Invent an estimation question that divide and conquer might help solve. You do not need to solve the question!

Particularly interesting or instructive questions might appear on the course website or as examples in lecture or the notes (let me know should you not want your name attributed in case your question gets selected).

# 6.055J/2.038J (Spring 2010)

# Solution set 2

*Do the following warmups and problems. Submit your answers, including the short explanation, online by* ***10pm on Wednesday, 24 Feb 2010****.*

***Open universe:*** *Collaboration, notes, and other sources of information are* **encouraged.** *However, avoid looking up answers to the problem, or to subproblems, until you solve the problem or have tried hard. This policy helps you learn the most from the problems.*

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On `linux.mit.edu` (the Athena GNU/Linux machine), an (American) English dictionary lives in the `/usr/share/dict/words` file.

For the (optional) question that references `decline.txt`: It is the plain-text file on the course website that contains volume 1 of Gibbon's *Decline and Fall*. It is also available – as is any other file on the course website – on any Athena machine as `/mit/6.055/data/decline.txt`

## Warmups

**1. Direct practice with one or few**

Here is another 'one or few' problem generated by my Python script:

$$985 \times 385 \times 721 \times 319 = ? \tag{1}$$

$10^{\square} \pm \square$ ***or*** $10^{\square} \ldots \square$

Here are the approximations for each number:

$$\begin{aligned} 985 &\to 10^3, \\ 385 &\to \text{few} \cdot 10^2, \\ 721 &\to 10^3, \\ 319 &\to \text{few} \cdot 10^2. \end{aligned} \tag{2}$$

The approximate product has 10 powers of 10 and two factors of a few, giving $10^{11}$. The exact value is $87,221,370,775$ or roughly $0.9 \cdot 10^{11}$.

**2. Land area per capita**

Here is another problem on which to practice the 'one or few' method of multiplication and division: Estimate how much land area each person would have if people were evenly distributed on the (land) surface of the earth.

$10^{\square} \pm \square$ $\text{m}^2$ per person ***or*** $10^{\square} \ldots \square$ $\text{m}^2$ per person

The surface area of the earth is $4\pi r^2$, where $r$ is the radius of the earth. The land area is some fraction $f$ of the total surface area. I half remember that $f$ is about 0.25, which seems plausible: There is a lot of land, but the oceans are giant. The radius of the earth (from Homework 1) is $r \sim 6.4 \cdot 10^6$ m. The world population is about $5 \cdot 10^9$.

So, the per-capita area $A$ is

$$A \sim \frac{4\pi \times (6.4 \cdot 10^6\,\mathrm{m})^2 \times 0.25}{5 \cdot 10^9}. \tag{3}$$

There are three powers of 10. In the remaining factors, the 4 and the 0.25 cancel each other. The $\pi \times 6.4^2/5$ is (using $6.4^2 \sim \text{few} \times 10$) few × few × 10/few or few × 10. The final per-capita area is few $\cdot 10^4\,\mathrm{m}^2$.

**3. Nested square roots**

Evaluate

$$\sqrt{2 \times \sqrt{2 \times \sqrt{2 \times \sqrt{\cdots}}}} \tag{4}$$

The computation is recursive in that it contains a copy of itself. To see that, define

$$P \equiv \sqrt{2 \times \sqrt{2 \times \sqrt{2 \times \sqrt{\cdots}}}}. \tag{5}$$

Notice that $P$ is repeated inside the square root:

$$P = \sqrt{2 \times P}. \tag{6}$$

The solution to this equation is $P = 2$.

**4. Searching for . . . gry words**

What English words, other than `angry`, end in `gry`?

Humans are much worse than computers at this question, because we store words not by their endings but more by their beginnings and meanings. For a computer, it's all bit strings, and computers don't care whether the bit string happens at the beginning or end of the word (and there's no meaning).

The regular expression that matches words ending in `gry` is `gry$`. In the following pipeline, the first `grep` finds all those words, and the second `grep` excludes `angry` from the list:

```
grep 'gry$' /usr/share/dict/words | grep -v '^angry$'
```

The result is just one line: 'hungry'.

# Problems

**5. Geometric series**

Use abstraction to find the sum of the infinite series

$$1 + r + r^2 + r^3 + \cdots. \tag{7}$$

Similar to **Problem 3**, look for a repeated motif or abstraction. Here, define $S$ as the sum

$$S = 1 + r + r^2 + r^3 + \cdots \tag{8}$$

and then notice that $S$ contains itself:

$$S = 1 + r\underbrace{(1 + r + r^2 + \cdots)}_{S}. \tag{9}$$

So, $S = 1 + rS$, whose solution is

$$S = \frac{1}{1 - r}. \tag{10}$$

**6. Pool temperature**

A large outdoor swimming pool in the Arizona desert has a time constant of 4 days for exchanging heat with the air. Roughly how large are the peak-to-peak fluctuations of the water temperature caused by 30 °F (peak-to-peak) night–day fluctuations in the air temperature?

$10^{\square} \pm \square$ °F ***or*** $10^{\square} \ldots \square$ °F

The daily temperature oscillation has an angular frequency

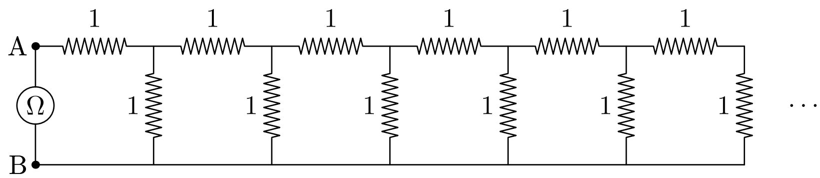
$$\omega = 2\pi f = \frac{2\pi}{1\,\text{day}}. \tag{11}$$

With a thermal time constant of $\tau = 4$ days, the dimensionless paramater $\omega\tau$ is $2\pi \times 4$ or about 25. Since $\omega\tau \gg 1$, we are in the limit of fast oscillations. In this limit, the low-pass filter – the combined system of thermal resistance and reservoir (the swimming pool) – attenuates the inputs oscillations by a factor of $|\omega\tau|$. So the 30 °F fluctuations in air temperature become a 1 °F fluctuation in pool temperature.

The practical consequence for swimming is as follows. These fluctuations happen around the average daily temperature (the DC or zero-frequency input signal). In the Arizona winter, the daytime temperature is often 70 °F, but the nighttime temperature can be only 40 °F (the 30 °F variation). Therefore, the pool will sit mostly at 55 °F. It is far too cold for swimming. The small fluctuation of 1 °F around 55 °F does not make the pool comfortable even at its peak temperature.

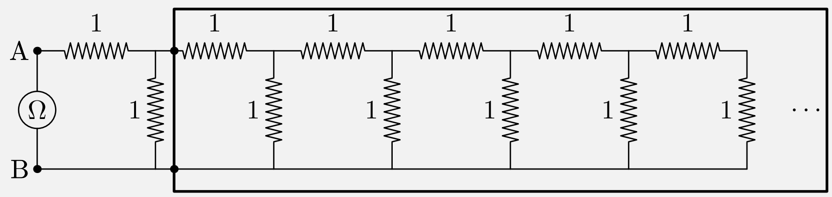
**7. Resistive network**

In the following infinite network of 1 Ω resistors, what is the resistance between points A and B? This measurement is indicated by the ohmmeter connected between these points. (If you want to read about series and parallel resistances, a useful reference is the Wikipedia article 'Series and parallel circuits'.)

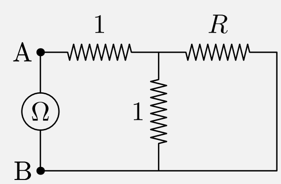



[ ] ± [ ] Ω ***or*** [ ] . . . [ ] Ω

This resistive network contains a copy of itself (enclosed in the box):




Call $R$ the resistance of the network inside the box, measured between the two dots as the terminals. Then the original network, which also has resistance $R$, is




It is a $1\,\Omega$ resistance in series with the parallel combination of $1\,\Omega$ and $R$. So

$$R = 1 + \underbrace{\frac{R}{1+R}}_{1\,\Omega \| R}, \tag{12}$$

or $R^2 - R - 1 = 0$. The positive solution is

$$R = \frac{1+\sqrt{5}}{2} \approx 1.618, \tag{13}$$

which is the Golden Ratio.

An alternative, direct method is the following continued fraction that accounts for the infinite cascase of series and parallel resistors:

$$R = 1 + \frac{1}{1 + \frac{1}{1+\cdots}}. \tag{14}$$

This famous continued fraction converges (slowly) to the Golden Ratio. (One special feature of the Golden Ratio is that it has the the slowest-converging continued fraction of any real number.)

# Optional

*These problems are optional in case you want more practice or want to try a (possibly large) project.*

**8. Email indexer**

Design a set of shell scripts for doing quick keyword searches of a large database of emails. Assume that each email is stored in its own plain-text file. Perhaps one shell script generates an index, and a second script searches the index.

**9. Running time**

Ordinary long multiplication requires $O(n^2)$ digit-by-digit multiplications. Show that the Karatsuba multiplication method explained in lecture requires $O(n^{\log_2 3}) \approx O(n^{1.58})$ digit-by-digit multiplications.

**10. Counting empires**

How often does the word `Empire` (uppercase E, then all lowercase) occur in `decline.txt`? [*Hint:* Look up the `tr` command.]

Divide and conquer! First turn all non-letters into newlines (squeezing out repeated newlines); second, look for lines that exactly match 'Empire'; and third, count the lines. Those three stages are the three stages of the following pipeline:

```
tr -cs 'a-zA-Z' '\012' < ./data/decline.txt | grep '^Empire$' | wc -l
```

It produces '37'.

# 6.055J/2.038J (Spring 2010)

# Solution set 3

*Do the following warmups and problems. Submit your answers, including the short explanation, online by **10pm on Wednesday, 3 Mar 2010**.*

***Open universe:** Collaboration, notes, and other sources of information are **encouraged.** However, avoid looking up answers to the problem, or to subproblems, until you solve the problem or have tried hard. This policy helps you learn the most from the problems.*

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## Warmups

**1. Fuel efficiency of a 747**

Use the cost of a plane ticket to estimate the fuel efficiency of a 747, in passenger–miles per gallon (passenger–mpg).

$10^{\square} \pm \square$ passenger–mpg ***or*** $10^{\square} \ldots \square$ passenger–mpg

A roundtrip economy ticket from New York to San Francisco costs roughly \$400. The journey is about 2500 miles each way, so a 5000-mile journey costs about \$500 (rounding up the \$400 to make the math easier). That's about 10 cents/mile. Perhaps one-half of that cost is fuel. [Although the service – in the air, on the phone, and at the counter – is so lousy due to understaffing that perhaps two-thirds of the cost being fuel would be a better estimate!] At 5 cents/mile for fuel, and at \$3/gallon for fuel, the fuel efficiency is 60 passenger–miles per gallon.

**2. High winds**

At roughly what wind speed is the force on your body from the wind approximately equal to your weight?

$\square \pm \square$ $\mathrm{m\,s^{-1}}$ ***or*** $\square \ldots \square$ $\mathrm{m\,s^{-1}}$

A typical person is maybe $m \sim 65\,\mathrm{kg}$, so a weight of $mg \sim 700\,\mathrm{N}$. The drag force is $F \sim \rho v^2 A$. Therefore,

$$v \sim \left(\frac{mg}{\rho A}\right)^{1/2}. \tag{1}$$

The density $\rho$ is roughly $1\,\mathrm{kg\,m^{-3}}$. For a person, the cross-sectional area is roughly $2\,\mathrm{m} \times 0.5\,\mathrm{m}$ (height times width) or $1\,\mathrm{m}^2$. So

$$v \sim \sqrt{\frac{700\,\mathrm{N}}{1\,\mathrm{kg\,m^{-3}} \times 1\,\mathrm{m}^2}} \sim 25\,\mathrm{m\,s^{-1}}. \tag{2}$$

That's roughly 55 mph.

This problem was inspired by the high winds (and rain) a couple weeks ago. As I was walking home in that miserable weather, I leaned sharply into the wind in order not to get toppled over – indicating that the drag force was comparable to my weight.

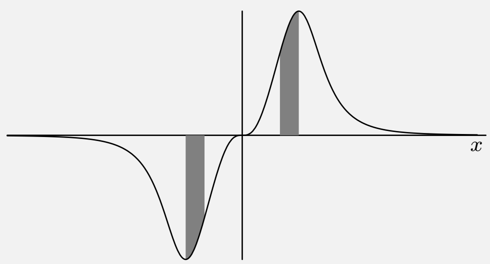
**3. Daunting integral**

Evaluate

$$\int_{-\infty}^{\infty} \frac{x^3}{1+7x^2+18x^8}\,dx. \tag{3}$$

☐ ± ☐ ***or*** ☐ … ☐

The integrand, $x^3/(1+7x^2+18x^8)$, is antisymmetric: When $x$ becomes $-x$, the integrand changes sign. So, for every sliver of rectangle in the negative-$x$ region, there's a corresponding sliver with the opposite sign in the positive-$x$ region. The net sum is therefore zero.




This problem was inspired by my days as a physics undergraduate. Physics problem sets often meant doing tons of complicated integrals. Our bible was Gradshteyn and Ryzhik's *Table of Integrals, Series, and Products*, now in its 7th edition. Often when we couldn't find an integral in Gradshteyn, we later realized, after much painful integration gymnastics, that the integral had to be zero by symmetry. So, don't miss those chances to use symmetry.

# Problems

**4. Solitaire**

You start with the numbers 3, 4, and 5. At each move, you choose any two of the three numbers – call the choices $a$ and $b$ – and replace them with $0.8a - 0.6b$ and $0.6a + 0.8b$. The goal is to reach 4, 4, 4. Can you do it? If yes, give a move sequence; if no, show that you cannot.

To see whether solitaire games are solvable, look for an invariant. Alas there is no algorithm for finding invariants; you have to use clues and make lucky guesses.

Speaking of clues, is it a happy coincidence that $0.8^2 + 0.6^2 = 1$? That convenient sum suggests looking at sums of squares, and how those are changed by making a move. Replacing $a$ and $b$ by $a' = 0.8a - 0.6b$ and $b' = 0.6a + 0.8b$ makes the sum of squares $a^2 + b^2$ into $a'^2 + b'^2$. Expand that expression:

$$\begin{aligned} a'^2 + b'^2 &= (0.8a - 0.6b)^2 + (0.6a + 0.8b)^2 \\ &= 0.64a^2 - 0.96ab + 0.36b^2 \ + \ 0.36a^2 + 0.96ab + 0.64b^2 \\ &= a^2 + b^2. \end{aligned}$$

Great! Each move leaves the sum of squares unchanged. That sum started out with the invariant at $3^2 + 4^2 + 5^2 = 50$, so it remains 50. The goal state, however, requires that the invariant become $4^2 + 4^2 + 4^2 = 48$. It's not possible to reach the goal.

The invariant has a nice geometric interpretation (a picture). To see it, let $P = (a, b, c)$ be the coordinates of a point in three-dimensional space. Then each move leaves unchanged the distance to the origin, which is $\sqrt{a^2 + b^2 + c^2}$. So each move shifts $P$ to another location equally distant from the origin, meaning that it moves $P$ on the surface of a sphere. But it cannot escape the surface.

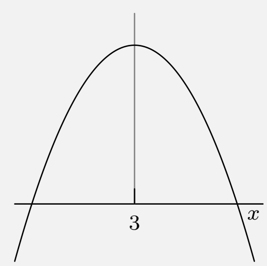
An interesting question to which I don't know the answer: Can you reach every point on the surface of the sphere? The distance invariant does not forbid it, but maybe other constraints do?

**5. Maximizing a polynomial**

Use symmetry to find the maximum value of $6x - x^2$.

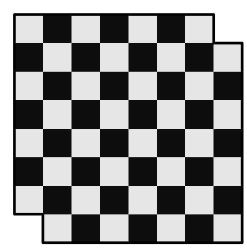
☐ ± ☐ *or* ☐ . . . ☐

The polynomial factors as $P = x(6 - x)$. As a symmetry operation, try replacing $x$ with $6 - x$. That operation is a reflection through the vertical line $x = 3$. It turns $P$ into $(6 - x)x$, which is again $P$ just with the factors swapped. Let's call $x_0$ the value of $x$ that maximizes $P$. Because changing $x$ to $6 - x$ doesn't change the curve, it doesn't change the location of the minimum, which is at $(x_0, P(x_0))$. Thus $x_0$ turns into $x_0$ under the symmetry operation $x \mapsto 6 - x$. The only value of $x$ that is unchanged by a reflection through the vertical line $x = 3$ is 3 itself, so $x_0 = 3$ and $P(x_0) = 9$.




**6. Tiling a mouse-eaten chessboard**

An $8 \times 8$ chessboard gets two diagonally opposite corners eaten away by a mouse. You have dominoes, each 2×1 in shape – i.e. each covers two adjacent squares. Can you tile the mouse-eaten chessboard with these dominoes? In other words, can you lay down the dominoes to cover every square exactly once (no empty squares and no overlaps)?



☐ yes

☐ no

Placing a domino on the board is one move in this solitaire game. For each move, you choose where to place the domino – which means you might have many choices for each move. Can you cover the whole board? The space of possible moves grows rapidly. Hence, look for an invariant: a quantity unchanged by any move of the game.

Because each domino covers one white square and one black square, the following quantity is invariant (unchanged):

$$I = \text{number of uncovered black squares} - \text{number of uncovered white squares}. \tag{4}$$

With a regular chess board, the initial position would have $I = 0$, from 32 white squares and 32 black squares. With this modified board, two black squares have vanished, so $I$ is $30 - 32 = -2$. However, in the winning position, all squares are covered; therefore $I = 0$. Because $I$ is invariant, no sequence of domino moves can turn the initial uncovered board into the winning board (with all squares covered).

# Optional!

**7. Symmetry for second-order systems**

This problem analyzes the frequency of maximum gain for an *LRC* circuit or, equivalently, for a damped spring–mass system. The gain of such a system is the ratio of the input amplitude to the output amplitude as a function of frequency.

If the output voltage is measured across the resistor, and you drive the circuit with a voltage oscillating at frequency $\omega$, the gain is (in a suitable system of units):

$$G(\omega) = \frac{j\omega}{1 + j\omega/Q - \omega^2},$$

where $j = \sqrt{-1}$ and $Q$ is quality factor, a dimensionless measure of the damping. Do not worry if you do not know where that gain formula comes from. The purpose of this problem is not its origin, but rather using symmetry to maximize its magnitude.

The magnitude of the gain is

$$|G(\omega)| = \frac{\omega}{\sqrt{\left(1 - \omega^2\right)^2 + \omega^2/Q^2}}.$$

Find a variable substitution (a symmetry operation) $\omega_{\text{new}} = f(\omega)$ that turns $|G(\omega)|$ into $|H(\omega_{\text{new}})|$ such that $G$ and $H$ are the same function (i.e. they have the same structure but with $\omega$ in $G$ replaced by $\omega_{\text{new}}$ in $H$). Use the form of that symmetry operation to maximize $|G(\omega)|$ without using calculus.

When maximizing a parabolic function such as $y = x(6 - x)$, the symmetry is reflection about the line $x = 3$. In symbols, the transformation is $x_{\text{new}} = 6 - x$.

Let's transfer a few lessons from the parabola example to the problem of maximizing the gain. In the parabola example, the symmetry is a reflection about an interesting point (there, the point halfway between the two roots $x = 0$ and $x = 6$). Analogously, an interesting frequency is $\omega = 1$ because it makes the real part of the denominator in $G(\omega)$ go to zero, and making the real part go to zero helps minimize the denominator.

Therefore reflecting about $\omega = 1$ is worth trying, perhaps $\omega_{\text{new}} = 1 - \omega$. For frequencies, however, differences are not as important as ratios. For example, a musical octave is a factor of 2 in frequency, rather than a difference. So reflect in a multiplicative way: $\omega_{\text{new}} = w^{-1}$.

This transformation works either in $G(\omega)$ or in the magnitude $|G(\omega)|$. It's slightly easier in $G(\omega)$:

$$G(\omega) = \frac{j\omega}{1 + j\omega/Q - \omega^2} \mapsto H(\omega_{\text{new}}) = \frac{j/\omega_{\text{new}}}{1 + j/Q\omega_{\text{new}} - 1/\omega_{\text{new}}^2}.$$

Multiply numerator and denominator by $\omega_{\text{new}}^2$:

$$H(\omega_{\text{new}}) = \frac{j\omega_{\text{new}}}{\omega_{\text{new}}^2 + j\omega_{\text{new}}/Q - 1},$$

which is the same function as $G(\omega)$, except for negating the real part in the denominator. Negating the real part in the denominator doesn't affect the magnitude of the denominator, so $|H(\omega_{\text{new}})|$ has the same form as $|G(\omega)|$.

Since $\omega_{new} = 1/\omega$, the maximum value of $\omega_{new}$ will be $\omega_{max}^{-1}$. That's one equation.

Since the two magnitudes $|G(\omega)|$ and $|H(\omega_{new})|$ are the same function, the maximum value of $\omega_{new}$ is also the maximum value of $\omega$. That's the second equation.

Together they produce $\omega = \omega_{new} = 1$ (ignoring the negative-frequency solution $\omega = -1$). At that frequency, $|G(\omega)|$ is $Q$. For the electrical and mechanical engineers: The quality factor $Q$ is also the gain at resonance.

8. **Inertia tensor**

*[For those who know about inertia tensors.]* Here is the inertia tensor (the generalization of moment of inertia) of a particular object, calculated in a lousy coordinate system:

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix}$$

Change coordinate systems to a set of principal axes. In other words, write the inertia tensor as

$$\begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

and give the values of $I_{xx}$, $I_{yy}$, and $I_{zz}$. *Hint:* What properties of a matrix are invariant when changing coordinate systems?

Whatever coordinate change I make, I will leave the $x$ axis alone because the $I_{xx}$ component is already separated from the $y$- and $z$ submatrix. That submatrix is

$$\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

I have to figure out how changing the coordinate system changes this submatrix. Rather than find the coordinate change explicitly, I use invariants to avoid that computation.

One invariant of any matrix, not just of this $2 \times 2$ matrix, is its determinant. Another invariant is its trace (the sum of the diagonal elements). In the nasty coordinate system, the trace of the $y$- and $z$ submatrix is $5 + 5 = 10$. So the trace is 10 in the nice coordinate system. The determinant is $5 \times 5 - 4 \times 4 = 9$, so it the determinant is 9 in the nice coordinate system.

Those facts are sufficient to deduce the submatrix in the nice coordinate system (without needing to figure out what the nice coordinate system is). In the nice coordinate system, the 2×2 submatrix looks like

$$\begin{pmatrix} I_{yy} & 0 \\ 0 & I_{zz} \end{pmatrix}$$

So I need to find $I_{yy}$ and $I_{zz}$ such that

$$I_{yy} + I_{zz} = 10 \quad \text{(from the trace invariant)}$$

and

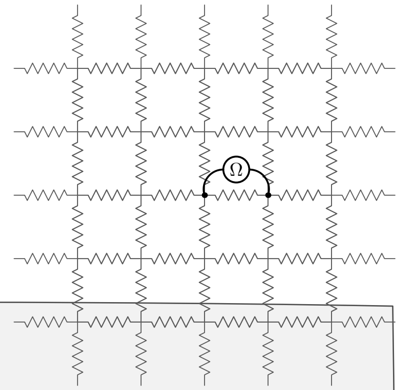
$$I_{yy} I_{zz} = 9 \quad \text{(from the determinant invariant)}$$

The solution is $I_{yy} = 1$ and $I_{zz} = 9$ (or vice versa). So the inertia tensor becomes

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

**9. Resistive grid**

In an infinite grid of 1-ohm resistors, what is the resistance measured across one resistor?




To measure resistance, an ohmmeter injects a current $I$ at one terminal (for simplicity, say $I = 1\,\text{Å}$), removes the same current from the other terminal, and measures the resulting voltage difference $V$ between the terminals. The resistance is $R = V/I$.

*Hint:* Use symmetry. But it's still a hard problem!

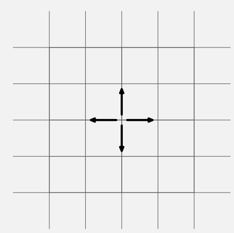
I'd like to find the current flowing through the resistor when 1 Å is sent into one terminal of the ohmmeter and removed from its other terminal. The solution has two steps, each subtle:

**1.** Break the resistance-measuring experiment into two parts, each having a lot of symmetry.

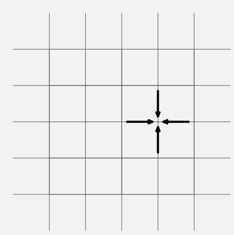
**2.** Analyze those parts using symmetry.

The current distribution that results from the full resistance-measuring experiment is not sufficiently symmetric because it has a preferred direction along the selected resistor. However, if I break the experiment into two parts – inserting current and removing current – then each part produces a symmetric current distribution.

By symmetry – because all four coordinate directions are equivalent – inserting 1 Å produces 1/4 Å flowing in each coordinate direction away from the terminal. Let's call this terminal the positive terminal. So inserting the 1 Å at the positive terminal produces 1/4 Å through the selected resistor, and this current flows away from the positive terminal.



By symmetry, removing 1 Å produces 1/4 Å in each coordinate direction, flowing toward the terminal. Let's call this terminal the negative terminal. So removing 1 Å produces 1/4 Å through the selected resistor, flowing toward the negative terminal. Equivalently, it produces 1/4 Å flowing away from the positive terminal.



Now superimpose the two pictures to reproduce the experiment of measuring the resistance. The experiment produces 1/2 Å through the resistor, flowing from the positive to the negative terminal. The voltage across the resistor is the current times its resistance, so the voltage is 1/2 V. Since a 1 Å test current produces a 1/2 V drop, the effective resistance is $1/2\,\Omega$.

If you want an even more difficult problem: Find the resistance measured across a diagonal!

# 6.055J/2.038J (Spring 2010)

# Solution set 4

*Submit your answers and explanations online by **10pm on Wednesday, 10 Mar 2010**.*

***Open universe:*** *Collaboration, notes, and other sources of information are **encouraged.** However, avoid looking up answers to the problem, or to subproblems, until you solve the problem or have tried hard. This policy helps you learn the most from the problems.*

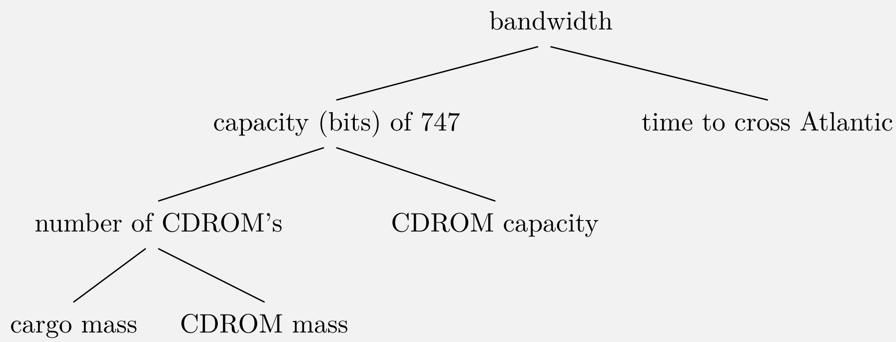
*Homework is graded with a light touch: P (made a decent effort), D (made an indecent effort), or F (did not make an effort).*

**Problem 1 Bandwidth**

To keep your divide-and-conquer muscles strong, here is an exercise from lecture: Estimate the bandwidth of a 747 crossing the Atlantic filled with CDROMs.

$10^{\square} \pm \square$ bits/s ***or*** $10^{\square} \ldots \square$ bits/s

Divide and conquer! Here's a tree on which to fill values:



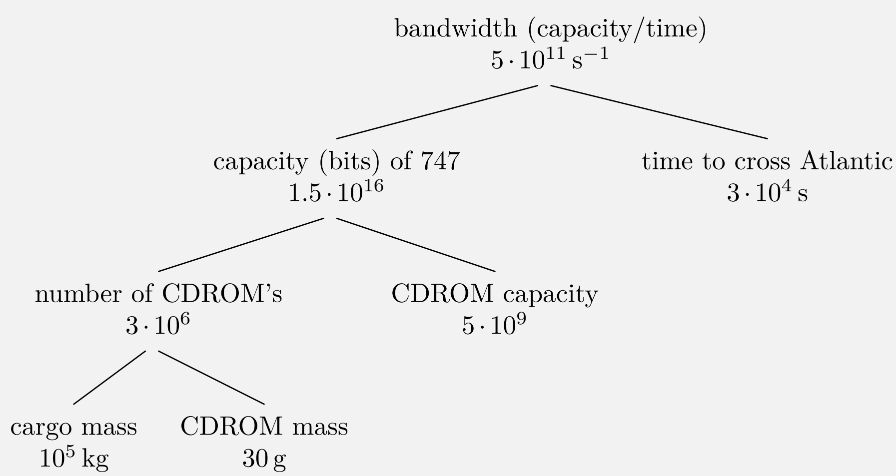

First I estimate the cargo mass. A 747 can easily carry about 400 people, each person having a mass (with luggage) of, say 140 kg. The total mass is

$$m \sim 400 \times 140\,\text{kg} \sim 6 \cdot 10^4\,\text{kg}.$$

A special cargo plane, with no seats or other frills for passengers, probably can carry $10^5$ kg.

Here are the other estimates. A CDROM's mass is perhaps one ounce or 30 g. So the number of CDROM's is $3 \cdot 10^6$. The capacity of a CDROM is 600 MB or about $5 \cdot 10^9$ bits. The time to cross the Atlantic is about 8 hours or $3 \cdot 10^4$ s.

Now propagate the values toward the root of the tree:




The bandwidth is 0.5 terabits per second or $10^{11.5}$ bits/second.

Despite the large bandwidth offered by a 747 carrying CDROM's (not to mention DVDROM's), trans-Atlantic Internet connections go instead via undersea fiber-optic cables. Low latency is important!

**Problem 2 Gravity versus radius**

Assume that planets are uniform spheres. How does $g$, the gravitational acceleration at the surface, depend on the planet's radius $R$? In other words, what is the exponent $n$ in

$$g \propto R^n? \tag{1}$$

☐ ± ☐ *or* ☐ . . . ☐

The gravitational force (the weight) on an object of mass $m$ is $GMm/R^2$, where $G$ is Newton's constant, and $M$ is the moon's mass. Thus the gravitational acceleration $g$ is $GM/R^2$. But the mass $M$ is proportional to $R^3$, so $g \propto R^1$. In other words, $n = 1$.

**Problem 3 Gravity on the moon**

The radius of the moon is one-fourth the radius of the earth. Use the result of **Problem 2** to predict the ratio $g_{\text{moon}}/g_{\text{earth}}$. In reality, $g_{\text{moon}}/g_{\text{earth}}$ is roughly one-sixth. How might you explain any discrepancy between the predicted and actual ratio?

The ratio $g_{\text{moon}}/g_{\text{earth}}$ should be proportional to the ratio of radii $R_{\text{moon}}/R_{\text{earth}}$, namely 1/4. The actual ratio is lower because of an effect neglected in the analysis of **Problem 2**: the differing density. When that effect is included, then the mass $M$ is $\rho R^3$ (except for a constant), so

$$g \sim \frac{G\rho R^3}{R^2} \propto \rho R. \tag{2}$$

If $\rho_{\text{moon}}/\rho_{\text{earth}}$ is 2/3, that reduction in concert with the radii ratio would explain the factor of 6 difference in $g$.

Moon rock, which is less dense than the average earth rock, is comparable in density to rock in the earth's crust. This equivalence suggests that the moon was once a piece of the earth's crust that got scooped out probably by a large meteor impact.

**Problem 4 Minimum power**

In the readings we estimated the flight speed that minimizes energy consumption. Call that speed $v_E$. We could also have estimated $v_P$, the speed that minimizes power consumption. What is the ratio $v_P/v_E$?

| | ± | | *or* | | ... | |
|---|---|---|---|---|---|---|

The zillions of constants (such as $\rho$) clutter the analysis without changing the result. So I'll simplify the problem by using a system of units where all the constants are 1. Then the energy is

$$E \sim v^2 + \frac{1}{v^2},$$

where the first term is from drag and the second term is from lift. The power is energy per time, and time is inversely proportional to $v$, so $P \propto Ev$ and

$$P \sim v^3 + \frac{1}{v}.$$

The first term is the steep $v^3$ dependence of drag power on velocity (which we used to estimate the world-record cycling and swimming speeds). The energy expression is unchanged when $v \to 1/v$, so it has a minimum at $v_E = 1$.

To minimize the power, use calculus (ask me if you are curious about calculus-free ways to minimize it):

$$\frac{dP}{dv} \sim 3v^2 - \frac{1}{v^2} = 0,$$

therefore $v_P = 3^{-1/4}$ (roughly 3/4), which is also the ratio $v_P/v_E$.

So the minimum-power speed is about 25% less than the minimum-energy speed. That result makes sense. Drag power grows very fast as $v$ increases – much faster than lift power decreases – so it's worth reducing the speed a little to reduce the drag a lot.

If you don't believe the simplification that I used of setting all constants to 1 – and it is not immediately obvious that it should work – then try using this general form:

$$E \sim Av^2 + \frac{B}{v^2},$$

where $A$ and $B$ are constants. You'll find that $v_E$ and $v_P$ each contain the same function of $A$ and $B$ and that this function disappears from the ratio $v_P/v_E$.

**Problem 5 Highway vs city driving**

Here is a measure of the importance of drag for a car moving at speed $v$ for a distance $d$:

$$\frac{E_{\text{drag}}}{E_{\text{kinetic}}} \sim \frac{\rho v^2 A d}{m_{\text{car}} v^2}.$$

This ratio is equivalent to the ratio

$$\frac{\text{mass of the air displaced}}{\text{mass of the car}}$$

and to the ratio

$$\frac{\rho_{\text{air}}}{\rho_{\text{car}}} \times \frac{d}{l_{\text{car}}},$$

where $\rho_{\text{car}}$ is the density of the car (its mass divided by its volume) and $l_{\text{car}}$ is the length of the car. Make estimates for a typical car and find the distance $d$ at which the ratio becomes significant (say, roughly 1).

$10^{\boxed{\phantom{00}} \pm \boxed{\phantom{00}}}$ m ***or*** $10^{\boxed{\phantom{00}} \ldots \boxed{\phantom{00}}}$ m

*To include in the explanation box:* How does the distance compare with the distance between exits on the highway and between stop signs or stoplights on city streets? What therefore are the main mechanisms of energy loss in city and in highway driving?

A typical car has mass $m_{\text{car}} \sim 10^3$ kg, cross-sectional area $A \sim 2\,\text{m} \times 1.5\,\text{m} = 3\,\text{m}^2$, and length $l_{\text{car}} \sim 4\,\text{m}$. So

$$\rho_{\text{car}} \sim \frac{m_{\text{car}}}{A l_{\text{car}}} \sim \frac{10^3\,\text{kg}}{3\,\text{m}^2 \times 4\,\text{m}} \sim 10^2\,\text{kg}\,\text{m}^{-3}.$$

Since $\rho_{\text{car}}/\rho_{\text{air}} \sim 100$, the ratio

$$\frac{\rho_{\text{air}}}{\rho_{\text{car}}} \frac{d}{l_{\text{car}}}$$

becomes 1 when $d/l_{\text{car}} \sim 100$, so $d \sim 400\,\text{m}$.

This distance $d$ is significantly farther than the distance between stop signs or stoplights on city streets. In Manhattan, for example, 20 east–west blocks are one mile, giving a spacing of approximately 80 m. So air resistance is not a significant loss in city driving. Instead the loss comes from engine friction, rolling resistance, and (mostly) braking.

However, the distance $d$ is comparable to the exit spacing on urban highways. So when you drive on the highway for even a few exit distances, air resistance is a significant loss.

Interestingly, highway fuel efficiencies are higher than city fuel efficiencies, even though drag gets worse at the higher, highway speeds, and presumably engine friction and rolling resistance also get worse at higher speeds. Only one loss mechanism, braking, is less prevalent in highway than in city driving. Therefore, braking must be a significant loss in city driving. Regenerative braking, used in some hybrid or electric cars, would therefore significantly improve fuel efficiency in city driving.

**Problem 6 Mountains**

Here are the heights of the tallest mountains on Mars and Earth.

| | | |
|---|---|---|
| Mars | 27 km | (Mount Olympus) |
| Earth | 9 km | (Mount Everest) |

Predict the height of the tallest mountain on Venus.

$10^{\square} \pm \square$ km *or* $10^{\square} \ldots \square$ km

*To include in the explanation box:* Then check your prediction in a table of astronomical data (or online).

One pattern is that the larger planet (earth) has the smaller mountain. Large planets presumably have stronger gravitational fields at their surface, which keeps the mountains closer to the ground. The derivation in lecture on mountain heights dropped the dependence on $g$ because we looked only at mountains on earth where all mountains share the value of $g$.

The same derivation can be repeated but retaining $g$. The weight of a mountain of size $l$ is $W \propto gl^3$, so the pressure at the base is $p \propto gl^3/l^2 \sim gl$. When the pressure $p$ exceeds the maximum pressure that rock can support, the mountain can no longer grow upward. This criterion is equivalent to holding $gl$ constant. Therefore,

$$l \propto g^{-1}.$$

Here are the gravitational field strengths on the three planets:

a. Mars: $3.7\,\mathrm{m\,s^{-2}}$

b. earth: $10\,\mathrm{m\,s^{-2}}$

c. Venus: $8.9\,\mathrm{m\,s^{-2}}$

The product $gl$ for each planet should be the same. That hypothesis works for Mars and earth:

a. Mars: $10^5\,\mathrm{m^2\,s^{-2}}$

b. earth: $0.9 \cdot 10^5\,\mathrm{m^2\,s^{-2}}$

If Venus follows the predicted scaling, then $gl$ should be roughly $10^5\,\mathrm{m^2\,s^{-2}}$ with $g \sim 8.9\,\mathrm{m\,s^{-2}}$. Therefore $l$ should be roughly 11 km. Indeed, the tallest mountain on Venus, which is Maxwell Montes, has just that height. Scaling triumphs!

Here is a fun question: Why aren't mountains on the moon 60 km tall (the Moon's surface gravity is about one-sixth of earth's surface gravity, as analyzed in **Problem 3**)?

**Problem 7 Raindrop speed**

Use the drag-force results from the readings to estimate the terminal speed of a typical raindrop (diameter of about 0.5 cm).

$10^{\square} \pm \square$ m s$^{-1}$ ***or*** $10^{\square} \ldots \square$ m s$^{-1}$

*To include in the explanation box:* How could you check this result?

The weight of the raindrop is the density times the volume times $g$:

$$W \sim \rho r^3 g,$$

where I neglect dimensionless factors such as $4\pi/3$.

At terminal velocity, the weight equals the drag. The drag is

$$F \sim \rho_{\text{air}} v^2 A \sim \rho_{\text{air}} v^2 r^2.$$

Equating the weight to the drag gives an equation for $v$ and $r$:

$$\rho_{\text{air}} v^2 r^2 \sim \rho r^3 g,$$

so $v \propto r^{1/2}$.

Bigger raindrops fall faster but – because of the square root – not much faster.

With the $g$ and the densities, the terminal velocity is

$$v \sim \sqrt{\frac{\rho}{\rho_{\text{air}}} gr}.$$

A typical raindrop has a diameter of maybe 5 or 6 mm, so $r \sim 3$ mm. Since the density ratio between water and air is roughly 1000,

$$v \sim \sqrt{1000 \times 10\,\text{m}\,\text{s}^{-2} \times 3 \cdot 10^{-3}\,\text{m}} \sim 5\,\text{m}\,\text{s}^{-1}.$$

First convert the speed into a more familiar value: 11 mph (miles per hour). If one drives at a speed $v_{\text{car}}$, then raindrops appear to move at an angle $\arctan(v_{\text{car}}/v)$. When $v_{\text{car}} = v$, the drops come at a 45° angle. So one way to measure the terminal speed is to drive in a rainstorm, slowly accelerating while the passenger (not the driver!) says when the drops hit at a 45° angle.

You could also run in a rainstorm and note the speed at which a small umbrella has to held at 45° to keep you perfectly dry.

**Problem 8 Cruising speed versus air density**

For geometrically similar animals (same shape and composition but different size), how does the minimum-energy speed $v$ depend on air density $\rho$? In other words, what is the exponent $\beta$ in $v \propto \rho^{\beta}$?

☐ ± ☐ ***or*** ☐ . . . ☐

From the lecture notes,

$$Mg \sim C^{1/2}\rho v^2 L^2,$$

where $C$ is the modified drag coefficient. So

$$v \sim \left(\frac{Mg}{C^{1/2}\rho L^2}\right)^{1/2}.$$

The only dependence on $\rho$ is the $\rho$ itself in the denominator, leaving

$$v \propto \rho^{-1/2}$$

and $\beta = 1/2$.

The inverse relationship between the speed and density explains why planes fly at a high altitude. The energy consumption at the minimum-energy speed is proportional to the drag force, which is proportional to $\rho v^2$. Because $v \propto \rho^{-1/2}$, the powers of $\rho$ cancel in the energy consumption; in other words, the energy consumption (at the minimum-energy speed for that $\rho$) is independent of $\rho$. By flying high, where $\rho$ is low, planes can fly faster without increasing their energy consumption.

**Problem 9 Cruising speed versus mass**

For geometrically similar animals (same shape and composition but different size), how does the minimum-energy speed $v$ depend on mass $M$? In other words, what is the exponent $\beta$ in $v \propto M^{\beta}$?

☐ ± ☐ ***or*** ☐ . . . ☐

Again from the lecture notes,

$$Mg \sim C^{1/2}\rho v^2 L^2,$$

where $C$ is the modified drag coefficient. So

$$v \sim \left(\frac{Mg}{C^{1/2}\rho L^2}\right)^{1/2}.$$

For geometrically similar animals, $g$ is independent of size (they all fight the same gravity) and $C$ is also independent of size (because the drag coefficient depends only on shape). But $M$ depends on $L$ according to $M \propto L^3$ or $L \propto M^{1/3}$. Because $L^2$ is proportional to $M^{2/3}$, the denominator contains $M^{2/3}$. The numerator contains $M^1$, so the ratio of numerator to denominator is $M^{1/3}$. After taking the square root, we find the scaling

$$v \propto M^{1/6}.$$

In other words, $\beta = 1/6$.

Large birds (and planes) fly slightly faster than small birds and planes. The design of the 737 was affected by this fact. The 737 is for medium-range flights and carries fewer passengers than a 747. However, if the 737 were merely a geometrically scaled 747 – retaining the shape but reducing $M$ by, say, a factor of 3 – then it would have a cruising speed roughly 20% lower than a 747 (because $3^{1/6} \approx 1.2$). That reduction would be fine if the 737 were the only plane traveling the skies. But planes are directed along fixed flight paths where it is dangerous to have planes overtaking one another. Therefore, the 737 was designed not to be geometrically similar to the 747 but instead to have the same cruising speed as the 747. Scaling matters!

**Problem 10 Speed of a bar-tailed godwit**

Use the results of **Problem 8** and **Problem 9** to write the ratio $v_{747}/v_{\text{godwit}}$ as a product of dimensionless factors, where $v_{747}$ is the minimum-energy speed of a 747, and $v_{\text{godwit}}$ is the minimum-energy speed of a bar-tailed godwit (i.e. its cruising speed). By estimating the dimensionless factors and their product, estimate the cruising speed of a bar-tailed godwit. [Useful information: $m_{\text{godwit}} \sim 0.4\,\text{kg}$; $v_{747} \sim 600\,\text{mph}$.]

$10^{\square} \pm \square\ \text{m s}^{-1}$ ***or*** $10^{\square} \ldots \square\ \text{m s}^{-1}$

*To include in the explanation box:* Compare your result with the speed of the record-setting bar-tailed godwit, which made its 11,570 km journey in 8.5 days.

Assuming that the animals and planes fly at the minimum-energy speed,

$$\frac{v_{747}}{v_{\text{godwit}}} = \left(\frac{\rho_{\text{high}}}{\rho_{\text{sea level}}}\right)^{-1/2} \times \left(\frac{m_{747}}{m_{\text{godwit}}}\right)^{1/6}.$$

A plane flies at around 10 km where the density is roughly one-third of the sea-level density. The mass of a 747 is roughly $4 \cdot 10^5$ kg, so the mass ratio between a 747 and a godwit is $10^6$. Therefore, the speed ratio is roughly

$$\frac{v_{747}}{v_{\text{godwit}}} \sim (1/3)^{-1/2} \times (10^6)^{1/6} = \sqrt{3} \times 10 \sim 17.$$

A 747 flies at around 550 mph so the godwit should fly around 550/17 mph $\sim$ 32 mph. The actual speed of record-setting godwit is almost identical:

$$v_{\text{actual}} \sim \frac{11,570\,\text{km}}{8.5\,\text{days}} \times \frac{0.6\,\text{mi}}{1\,\text{km}} \times \frac{1\,\text{day}}{24\,\text{hours}} \sim 35\,\text{mph}.$$

# 6.055J/2.038J (Spring 2010)

# Solution set 5

*Submit your answers and explanations online by* ***10pm on Wednesday, 07 Apr 2010****.*

***Open universe:*** *Collaboration, notes, and other sources of information are* ***encouraged.*** *However, avoid looking up answers to the problem, or to subproblems, until you solve the problem or have tried hard. This policy helps you learn the most from the problems.*

*Homework is graded with a light touch: P (made a decent effort), D (made an indecent effort), or F (did not make an effort).*

**Problem 1  9V battery**

Roughly how much energy is stored in a typical (disposable) 9V battery?

$10^{\square} \pm \square$ J  ***or***  $10^{\square} \ldots \square$ J

I'll estimate it by working out the energy in my laptop battery, and then adjusting the estimate to compensate for the smaller size of a 9V battery. The energy in my laptop battery is the ideal candidate for divide and conquer: the power drawn by the laptop times the battery life. I know the battery life well: about 4 or 5 hours (I have the slightly larger 9-cell Thinkpad battery).

The power draw is harder to estimate. The screen, the CPU, and the disk drive probably use comparable amounts of power, since reducing the power consumption of each item seems to be comparably important in extending the battery's life. The methods include using a lower screen brightness, putting the CPU into idle (technically, C2 and C3 states), or spinning down the disk. The screen is an LCD screen, which is much more efficient than an incandescent (standard) light bulb. So, although it is bright like a (weak) light bulb, say a 30-watt bulb, it may draw only 5 or 10 W.

Three such items – which includes the disk drive and the CPU – add up to perhaps 20 W. (As a check, the Powertop utility that comes with my Debian GNU/Linux installation says that the laptop is using 16.6 W.)

The product of power and time is energy stored in the battery:

$$E_{\text{laptop}} \sim 20\,\text{W} \times 4.5\,\text{hours} \times \frac{3600\,\text{s}}{1\,\text{hour}} \sim 3 \cdot 10^{5}\,\text{J}. \tag{1}$$

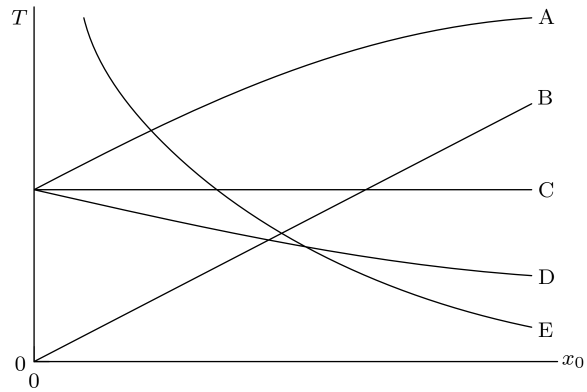
Now let's shrink that energy to account for the smaller size of a 9V battery. As a simple method, I'll assume that all batteries have a comparable energy density (energy stored per mass). In mass, my laptop battery feels like about 15 or maybe 20 9V batteries. So I'll divide $3 \cdot 10^{5}$ J by 15 or 20:

$$E_{9\text{V}} \sim \frac{E_{\text{laptop}}}{15 \text{ or } 20} \sim 15\,\text{kJ}. \tag{2}$$

For a rough comparison with actual values, Wikipedia quotes 20 kJ as the energy stored in a 9V battery.

**Problem 2 Non-Hooke's law spring**

Imagine a mass connected to a spring with force law $F = Cx^3$ (instead of the usual Hooke's law behavior $F = kx$) and therefore potential energy $V \sim Cx^4$ (where $C$ is a constant). Which curve shows how the system's oscillation period $T$ depends on the amplitude $x_0$?




- ☐ Curve A
- ☐ Curve B
- ☐ Curve C
- ☐ Curve D
- ☐ Curve E

Dimensional analysis:

| | | |
|---|---|---|
| $T$ | period | T |
| $x_0$ | amplitude | L |
| $C$ | spring constant | $\mathrm{ML^{-2}T^{-2}}$ |
| $m$ | mass | M |

The trickiest entry in the table is the dimensions of $C$. Since $Cx^3$ is a force, $C$ has dimensions of force over length cubed, namely $\mathrm{ML^{-2}T^{-2}}$. These four quantities made out of three dimensions produce one independent dimensionless group. Its simplest form is $Cx_0^2T^2/m$. Because there is only one dimensionless group, it must be a constant. In other words, $T \propto 1/x_0$. The only matching curve is curve E.

**Problem 3 Power radiated by an accelerating charge**

If the velocity and acceleration of a (nonrelativistic) electric charge are doubled, how does the power radiated by the charge change?

- [ ] The power increases by a factor of 16.
- [ ] The power increases by a factor of 8.
- [ ] The power increases by a factor of 4.
- [ ] The power increases by a factor of 2.
- [ ] The power increases by a factor of $\sqrt{2}$.

The first question is: On what does the radiated power depend? First, $c$ (the speed of light), because that is the speed at which the power travels; second, the charge's acceleration or velocity (or both); and third, the charge itself $q$. We probably also need $\epsilon_0$, the horrible constant when using SI units for electromagnetism. But $\epsilon_0$ and $q$ will show up only together as $q^2/4\pi\epsilon_0$, so let's combine those two quantities accordingly.

The remaining question is what to include from among acceleration or velocity. If the power radiated depended on the velocity, then we could use relativity to make a perpetual motion machine: Generate more energy simply by using a different reference frame, one moving with just the right velocity. No way! So, the power depends on the acceleration but not the velocity.

The list of variables, including the radiated power, is:

| | | |
|---|---|---|
| $P$ | radiated power | $\mathrm{ML^2T^{-3}}$ |
| $q^2/4\pi\epsilon_0$ | | $\mathrm{ML^3T^{-2}}$ |
| $c$ | speed of light | $\mathrm{LT^{-1}}$ |
| $a$ | acceleration | $\mathrm{MLT^{-2}}$ |

These four variables, again with three dimensions, result in one independent dimensionless group–for example,

$$\Pi_1 \equiv \frac{P}{q^2/4\pi\epsilon_0}\frac{c^3}{a^2}. \tag{3}$$

With only one group, it must be a constant, so

$$P \sim q^2/4\pi\epsilon_0\,\frac{a^2}{c^3}. \tag{4}$$

Except for needing a factor of 2/3, this result is correct (the full result is called the Larmor formula).

To answer the particular problem, doubling the velocity and acceleration quadruples the power radiated.

**Problem 4 Local black hole**

What is roughly the largest radius the earth could have, with its current mass, and be a black hole (i.e. light cannot escape from its surface)?

$10^{\square} \pm \square$ m *or* $10^{\square} \ldots \square$ m

In gravity problems, the quantity $GM/Rc^2$ is dimensionless. For most objects, it is very tiny. A likely candidate criterion for a black hole is when this quantity is around 1. Therefore, the radius of the black-hole earth would be

$$R \sim \frac{GM}{c^2}. \quad (5)$$

Putting in numbers,

$$R \sim \frac{7 \cdot 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}} \times 6 \cdot 10^{24}\,\mathrm{kg}}{10^{17}\,\mathrm{m^2\,s^{-2}}} \sim 4\,\mathrm{mm}. \quad (6)$$

(The true black-hole radius, based on general-relativity calculations, is twice this value based on dimensional analysis and a bit of guessing.)

**Problem 5 Wire**

Roughly what is the number density of free (conduction) electrons in a copper wire?

$10^{\square} \pm \square\ \mathrm{m^{-3}}$ ***or*** $10^{\square} \ldots \square\ \mathrm{m^{-3}}$

Copper has, as a guess, one conduction electron per atom. Each atom occupies roughly a 3-Angstrom cube – i.e., a volume of roughly $3 \cdot 10^{-29}\,\mathrm{m^3}$. The number density is the reciprocal of the atomic volume (since there is only one conduction electron per atom). Thus $n \sim 3 \cdot 10^{28}\,\mathrm{m^{-3}}$. (The true value is roughly 2.5 times greater, due to the atomic volume being slightly smaller than I estimated here using the 3-Angstrom rule of thumb.)

**Problem 6 Yield from an atomic bomb**

Geoffrey Taylor, a famous Cambridge fluid mechanic, annoyed the US government by doing the following analysis. The question he answered: 'What was the energy yield of the first atomic blast (in the New Mexico desert in 1945)?' Pictures declassified by the US government – the pictures even had a scale bar! – provided the tabulated data on the radius of the explosion at various times.

| $t$ (ms) | $R$ (m) |
|---|---|
| 3.26 | 59.0 |
| 4.61 | 67.3 |
| 15.0 | 106.5 |
| 62.0 | 185.0 |

Use dimensional analysis to work out the relation between radius $R$, time $t$, blast energy $E$, and air density $\rho$. Then use the data in the table to estimate the blast energy $E$:

$10^{\square} \pm \square\ \mathrm{J}$ ***or*** $10^{\square} \ldots \square\ \mathrm{J}$

Four quantities composed of three independent dimensions make one independent dimensionless group. The only quantities whose dimensions contain mass are $E$ and $\rho$. So $E$ and $\rho$ appear in the group as $E/\rho$, whose dimensions are $\mathrm{L^5T^{-2}}$. Therefore the following choice is dimensionless:

$$\Pi_1 \equiv \frac{Et^2}{\rho R^5}.$$

With only one dimensionless group, the most general statement connecting those four quantities is

$$\frac{Et^2}{\rho R^5} \sim 1.$$

or

$$E \sim \frac{\rho R^5}{t^2}.$$

For each row of data in the table, I'll estimate $\rho R^5/t^2$, using $\rho \sim 1\,\mathrm{kg\,m^{-3}}$:

| $t$ (ms) | $R$ (m) | $E$ ($10^{13}$ J) |
|---|---|---|
| 3.26 | 59.0 | 6.7 |
| 4.61 | 67.3 | 6.5 |
| 15.0 | 106.5 | 6.1 |
| 62.0 | 185.0 | 5.6 |

The data are not perfectly consistent about the predicted blast energy $E$, but they hover pretty closely around $6 \cdot 10^{13}$ J.

This blast energy, expressed in more common units for such devices, is roughly 15 kilotons of TNT – in close agreement with the then-classified value of 20 kilotons. Dimensional analysis triumphs again!

# 6.055J/2.038J (Spring 2010)

# Solution set 6

*Submit your answers and explanations online by* ***10pm on Wednesday, 14 Apr 2010****.*

***Open universe:*** *Collaboration, notes, and other sources of information are* ***encouraged.*** *However, avoid looking up answers to the problem, or to subproblems, until you solve the problem or have tried hard. This policy helps you learn the most from the problems.*

*Homework is graded with a light touch: P (made a decent effort), D (made an indecent effort), or F (did not make an effort).*

**Problem 1 Guessing an integral using easy cases**

Use easy cases to choose the correct value of the integral

$$\int_{-\infty}^{\infty} e^{-ax^2}\, dx. \tag{1}$$

- [ ] $\sqrt{\pi a}$
- [ ] $\sqrt{\pi / a}$

The most useful special cases here are $a \to 0$ and $a \to \infty$. When $a$ is zero, the Gaussian becomes the flat line $y = 1$, which has infinite area. The first choice, $\sqrt{\pi a}$, goes to zero in this limit, so it cannot be right. The second choice, $\sqrt{\pi / a}$, has the correct behavior.

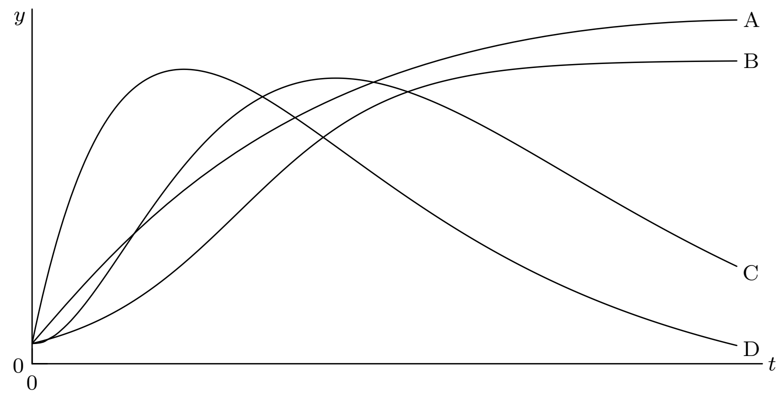
The limit $a \to \infty$ gives the same conclusion: The first choice cannot be right, and the second one might be right.

**Problem 2 Differential-equation solution**

Which sketch shows a solution of the differential equation

$$\frac{dy}{dt} = Ay(M - y),$$

where $A$ and $M$ are positive constants?




☐ Curve A

☐ Curve B

☐ Curve C

☐ Curve D

Use easy cases by choosing the solution that behaves correctly in all the easy cases. Here, one easy case is small $t$ ($t \approx 0$), when $y$ is small – in particular, small compared to $M$. Then the $M - y$ term is approximately $M$, making the differential equation

$$\frac{dy}{dt} = AMy \propto y.$$

It is the equation for exponential growth (since $AM$ is positive). Therefore, for small $t$, the curve should follow an exponential, which is concave upwards ('holds water'). Only curves $B$ and $C$ satisfy this test.

In the large-$t$ extreme case, $y$ approaches $M$. Then $dy/dt = 0$, which makes $y$ constant (consistent with the assumption $y \to M$). Among curves $B$ and $C$, the only curve that becomes flat is curve $B$.

As a further piece of evidence in favor of curve $B$, the derivative $dy/dt$ must always be positive. Why? For it to be negative, $y$ would have to exceed $M$. But when $y$ reaches $M$, then $dy/dt$ becomes 0 and $y$ stops changing. Therefore, $y$ can never exceed $M$. Contradiction! Therefore, the derivative cannot be negative. Curve C, however, has a region of negative slope.

**Problem 3 Fog**

Fog is a low-lying cloud, perhaps 1 km tall and made up of tiny water droplets (radius $r \sim 10\,\mu$m). By estimating the terminal speed of fog droplets, estimate the time that the cloud takes to settle to the ground.

$10^{\square \pm \square}$ s *or* $10^{\square \ldots \square}$ s

*To include in the explanation box:* What is the everyday consequence of this settling time?

At low Reynolds number, the drag is

$$F = 6\pi\rho_{\text{fl}}\nu v r, \tag{2}$$

where $\rho_{\text{fl}}$ is the density of the fluid. The weight of the object is

$$W = \frac{4}{3}\pi r^3 \rho_{\text{obj}} g, \tag{3}$$

where $\rho_{\text{obj}}$ is the density of the object. At the terminal speed $v$, the drag and weight balance:

$$6\pi\rho_{\text{fl}}\nu v r \sim \frac{4}{3}\pi r^3 \rho_{\text{obj}} g. \tag{4}$$

Therefore, the terminal speed $v$ is

$$v \sim \frac{2}{9}\frac{gr^2}{\nu}\frac{\rho_{\text{obj}}}{\rho_{\text{fl}}}.$$

(This calculation neglects buoyancy, which is a small effect for water droplets falling in air.)

Calling $2/9 = 1/5$ and using $\nu \sim 10^{-5}\,\text{m}^2\,\text{s}^{-1}$ gives

$$v \sim \frac{1}{5} \times \frac{10\,\text{m}\,\text{s}^{-2} \times 10^{-10}\,\text{m}^2}{10^{-5}\,\text{m}^2\,\text{s}^{-1}} \times 1000 \sim 2\,\text{cm}\,\text{s}^{-1}.$$

As a check on the initial assumption, let's calculate the Reynolds number:

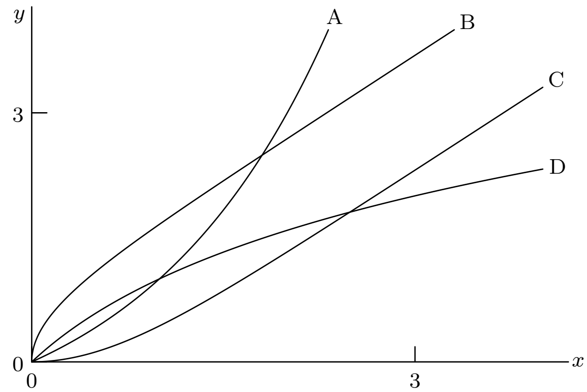
$$\text{Re} \sim \frac{10^{-5}\,\text{m} \times 2 \cdot 10^{-2}\,\text{m}\,\text{s}^{-1}}{10^{-5}\,\text{m}^2\,\text{s}^{-1}} \sim 0.02.$$

It is much less than 1, validating the assumption of low-Reynolds-number flow.

At $v \sim 2\,\text{cm}\,\text{s}^{-1}$, the droplet takes $5 \cdot 10^4\,\text{s}$ to fall 1 km. A day is roughly $10^5\,\text{s}$, so the fall time is about one-half of a day. The everyday consequence is that fog settles overnight. You go to sleep with a pea-soup fog, and by the time you wake up, it's mostly settled onto the ground – and maybe evaporated as the morning sun warms the ground.

**Problem 4 Hyperbolic-function sketch**

Which graph is $\ln\cosh x$ (where $\cosh x \equiv (e^x + e^{-x})/2$)?




☐ Curve A

☐ Curve B

☐ Curve C

☐ Curve D

Use easy cases: $|x| \to \infty$ and $x \to 0$. In the $x \to \infty$ case, $\cosh x \approx e^x/2$, so $\ln\cosh x \approx x - \ln 2$. In the $x \to -\infty$ case, $\cosh x \approx e^{-x}/2$, so $\ln\cosh x \approx -x - \ln 2$. In other words,

$$\ln\cosh x = |x| - \ln 2 \qquad (|x| \to \infty). \tag{5}$$

This is enough information to select curve C.

But let's check that curve C is correct also in the $x \to 0$ case. There, a Taylor series for $e^x$ gives

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left[(1 + x + \frac{x^2}{2} + \cdots) + (1 - x + \frac{x^2}{2} - \cdots)\right]. \tag{6}$$

The result is $\cosh x \approx 1 + x^2/2$. For the logarithm, the Taylor series is

$$\ln(1+z) \approx z. \tag{7}$$

So,

$$\ln\cosh x \approx \ln\left(1 + \frac{x^2}{2}\right) \approx \frac{x^2}{2}. \tag{8}$$

Thus, near the origin, $\ln\cosh x$ looks like an upward-facing parabola (concave up). Curve C passes this test.

**Problem 5 Guessing an integral**

Choose the correct value of the integral

$$\int_{-\infty}^{\infty} \frac{1}{a^2 + x^2}\, dx, \tag{9}$$

where $a$ is a positive constant.

☐ $\pi a$

☐ $\pi/a$

☐ $\sqrt{\pi}a$

☐ $\sqrt{\pi}/a$

The easiest special case is $a \to \infty$. In that limit, the integrand is zero everywhere, so the integral is zero. The first and third choices are therefore incorrect.

To decide between the second and fourth choices, use the special case $a = 1$. The integral becomes

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^2}\, dx$$

The integral is $\arctan x$. At $\infty$ it contributes $\pi/2$, and at $-\infty$ it subtracts $-\pi/2$, so the integral is $\pi$. Only the second choice, $\pi/a$, has the correct behavior when $a = 1$.

**Problem 6 Debugging**

Use special (i.e. easy) cases of $n$ to decide which of these two C functions correctly computes the sum of the first $n$ odd numbers:

☐ Program A:

```
int sum_of_odds (int n) {
  int i, total = 0;
  for (i=1; i<=2*n+1; i+=2)
    total += i;
  return total;
}
```

☐ Program B:

```
int sum_of_odds (int n) {
  int i, total = 0;
  for (i=1; i<=2*n-1; i+=2)
    total += i;
  return total;
}
```

Special cases are useful in debugging programs. The easiest cases are often $n = 0$ or $n = 1$. Let's try $n = 0$ first. In the first program, the $2n + 1$ in the loop condition means that $i = 1$ is the only case, so the total becomes 1. Whereas the sum of the first 0 odd numbers should be zero! So the first program looks suspicious.

Let's confirm that analysis using $n = 1$. The first program will have $i = 1$ and $i = 3$ in the loop, making the total $1 + 3 = 4$. The second program will have $i = 1$ in the loop, making the total 1. Since the correct answer is 1, Program A has a bug, and Program B looks good.

**Problem 7 Damped, driven spring**

A damped, driven spring–mass system (e.g., in 18.03, 2.003, 2.004, 6.003, and maybe also 8.01) is described by the differential equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 e^{i\omega t}, \tag{10}$$

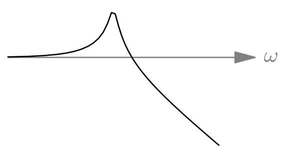
where $m$ is the mass of the object, $b$ is the damping constant, $k$ is the spring constant, $x$ is the displacement of the mass, $\omega$ is the (angular) frequency of the driving force, and $F_0$ is the amplitude of the driving force. The solution has the form

$$x = x_0 e^{i\omega t}, \tag{11}$$

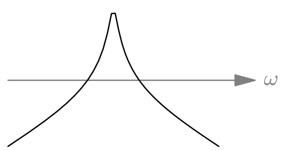
where $x_0$ is the (possibly complex) amplitude.

Which graph, on log–log axes, correctly shows the transfer function $F_0/x_0$? Don't solve the differential equation – use an approximation method to guess the answer!

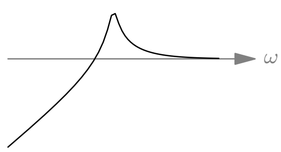
☐ Curve A




☐ Curve B




☐ Curve C




[In writing the solution, I realized that I made a mistake in the problem statement by asking for $F_0/x_0$ (input/output) instead of $x_0/F_0$ (output/input). Additionally, I should have used absolute value and asked about the magnitude of the transfer function $|x_0/F_0|$. I'll write the solution as if I had written the problem correctly. Apologies if you spent extra time because of those mistakes!]

Use easy cases. At low frequencies ($\omega \to 0$), the spring moves very slowly, meaning that derivatives with respect to time become tiny. Therefore, the time-derivative terms $m(d^2x/dt^2)$ and $b(dx/dt)$ become much smaller than the $kx$ term. The remaining equation is

$$kx \approx F_0 e^{i\omega t}. \tag{12}$$

With $x = x_0 e^{i\omega t}$, the transfer function $x_0/F_0$ is $1/k$. This function is independent of frequency, so the curve must be flat at low frequencies. The only curve that matches this criterion is curve A.

As a check, let's try really high frequencies ($\omega \to \infty$). Then the second-derivative term $m(d^2x/dt^2)$ is the dominant term, so the differential equation simplifies to

$$m\frac{d^2x}{dt^2} = F_0 e^{i\omega t}. \tag{13}$$

Using $x = x_0 e^{i\omega t}$ gives

$$-mx_0\omega^2 = F_0, \tag{14}$$

so the magnitude of the transfer function $|x_0/F_0|$ is $1/m\omega^2$. On a log–log graph, that is a $-2$ slope, which could be curves B or C but not curve A.

# 6.055J/2.038J (Spring 2010)

# Solution set 7

*Submit your answers and explanations online by* ***10pm on Friday, 23 Apr 2010****.*

***Open universe:*** *Collaboration, notes, and other sources of information are* **encouraged.** *However, avoid looking up answers to the problem, or to subproblems, until you solve the problem or have tried hard. This policy helps you learn the most from the problems.*

*Homework is graded with a light touch: P (made a decent effort), D (made an indecent effort), or F (did not make an effort).*

**Problem 1 No need to remember lots of constants**

Many atomic problems, such as the size or binding energy of hydrogen, end up in expressions with $\hbar$, the electron mass $m_e$, and $e^2/4\pi\epsilon_0$. You can avoid remembering those constants by instead remembering the following values:

$$\begin{aligned} \hbar c &\approx 200\,\text{eV nm} = 2000\,\text{eV Å} \\ m_e c^2 &\sim 0.5 \cdot 10^6\,\text{eV} \\ \frac{e^2/4\pi\epsilon_0}{\hbar c} &\equiv \alpha \approx \frac{1}{137} \quad \text{(fine-structure constant).} \end{aligned}$$

Use those values to evaluate the energy of a visible-light photon. Note: The photon energy is $E = hf$, where $h$ is Planck's constant and $f$ is the frequency of the radiation; equivalently, $E = \hbar\omega$, where $\hbar = h/2\pi$ and $\omega$ is the angular frequency of the radiation.

$10^{\square} \pm \square$ eV ***or*** $10^{\square} \ldots \square$ eV

$$E = hf = 2\pi\hbar f = 2\pi\hbar\frac{c}{\lambda},$$

where $f$ is its frequency and $\lambda$ is its wavelength. For green light, $\lambda \sim 600\,\text{nm}$, so

$$E \sim \frac{\overbrace{6}^{2\pi} \times \overbrace{200\,\text{eV nm}}^{\hbar c}}{\underbrace{600\,\text{nm}}_{\lambda}} \sim 2\,\text{eV}.$$

**Problem 2 Boundary-layer thickness**

How thick is the boundary layer on a golf ball traveling at, say, $v \sim 40\,\mathrm{m\,s^{-1}}$?

$10^{\square} \pm \square$ m *or* $10^{\square} \ldots \square$ m

The thickness $\delta$ is roughly $\sqrt{\nu t}$, where $\nu$ is the kinematic viscosity of air, and $t$ is the time for air to travel a distance comparable to $r$, the radius of the golf ball. So

$$\delta \sim \sqrt{\frac{\nu r}{v}}, \tag{1}$$

A golf ball has a diameter of about 5 cm so $r \sim 2\,\mathrm{cm}$. The kinematic viscosity of air is $\nu \sim 10^{-5}\,\mathrm{m^2\,s^{-1}}$. So

$$\delta \sim \sqrt{\frac{10^{-5}\,\mathrm{m^2\,s^{-1}} \times 2 \cdot 10^{-2}\,\mathrm{m}}{4 \cdot 10^{1}\,\mathrm{m\,s^{-1}}}} \sim 10^{-4}\,\mathrm{m} \tag{2}$$

(after neglecting a factor of 0.7). Compared to this thickness, the dimples are plenty thick enough to trip the boundary layer into turbulence and thereby reduce the drag coefficient.

**Problem 3 Viscous versus form drag**

The form drag (drag due to moving fluid aside) is

$$F_{\mathrm{d}} \sim \rho v^2 A. \tag{3}$$

The viscous (skin-friction) drag is

$$F_\nu \sim \rho\nu \times \text{surface area} \times \text{velocity gradient}, \tag{4}$$

where $\rho\nu$ is the dynamic viscosity $\eta$. The velocity gradient is $v/\delta$, where $v$ is the flow speed, and $\delta$ is the boundary-layer thickness.

The ratio $F_{\mathrm{d}}/F_\nu$ is dimensionless, and must therefore be a function of the only dimensionless measure of the flow, namely the Reynolds number Re. In fact, the function is a power law:

$$\frac{F_{\mathrm{d}}}{F_\nu} \sim \mathrm{Re}^n, \tag{5}$$

where $n$ is the scaling exponent. What is $n$?

☐ ± ☐ ***or*** ☐ … ☐

Using $v/\delta$ as the velocity gradient and $A$ as the surface area, the skin-friction drag becomes

$$F_\nu \sim \rho\nu A \frac{v}{\delta}. \tag{6}$$

Therefore, the ratio of drag forces is

$$\frac{F_{\mathrm{d}}}{F_\nu} \sim \frac{\rho v^2 A_{\mathrm{cs}}}{\rho\nu A v/\delta}, \tag{7}$$

where $A_{\mathrm{cs}}$ is the cross-sectional area. For objects that are not too elongated (e.g. not a long train), the surface and cross-sectional areas are comparable. Then the areas in the numerator and denominator cancel out. Additionally, the factors of $\rho$ and one factor of $v$ also cancel. What's left is

$$\frac{F_{\mathrm{d}}}{F_\nu} \sim \frac{v\delta}{\nu}. \tag{8}$$

From the reading (`r27-lumping-boundary-layers`), $\delta \sim r/\sqrt{\mathrm{Re}}$, so

$$\frac{F_{\mathrm{d}}}{F_\nu} \sim \frac{vr}{\nu} \times \mathrm{Re}^{-1/2}. \tag{9}$$

The fraction $vr/\nu$ is the Reynolds number, so

$$\frac{F_{\mathrm{d}}}{F_\nu} \sim \mathrm{Re}^{1/2}. \tag{10}$$

Thus, $n = 1/2$.

For most everyday flows, $\mathrm{Re} \gg 1$. Thus, most of the drag is form drag rather than skin-friction drag. The exception to this rule is very long objects (freight trains), where the surface area is much greater than the cross-sectional area.

**Problem 4 Viscous versus form drag while walking**

Use the result of **Problem 3** to estimate the ratio

$$\frac{\text{form drag}}{\text{viscous drag}} = \frac{F_{\text{d}}}{F_{v}} \tag{11}$$

for a person walking.

$10^{\square} \pm \square$ ***or*** $10^{\square} \ldots \square$

The ratio is roughly the square root of the Reynolds number, where

$$\text{Re} \sim \frac{\text{size} \times \text{speed}}{\text{kinematic viscosity}}. \tag{12}$$

For a person, the size is roughly $r \sim 1\,\text{m}$ (using the geometric mean of $2\,\text{m}$ for the height and $0.5\,\text{m}$ for the width). For walking, $v \sim 1.5\,\text{m}\,\text{s}^{-1}$. The viscosity of air is, conveniently, $1.5 \cdot 10^{-5}\,\text{m}^2\,\text{s}^{-1}$, so the Reynolds number is roughly

$$\text{Re} \sim \frac{1\,\text{m} \times 1.5\,\text{m}\,\text{s}^{-1}}{1.5 \cdot 10^{-5}\,\text{m}^2\,\text{s}^{-1}} \sim 10^5. \tag{13}$$

The square root is $10^{2.5}$ or 300. Form drag, which is mostly independent of viscosity, is the big source of drag.

**Problem 5 Rolling down the plane**

Four objects, made of identical steel, roll down an 30-degree inclined plane without slipping. The large objects have three times the radius of the small objects. Which object has the greatest acceleration?

☐ a large spherical shell

☐ a large disc

☐ a small solid sphere

☐ a small ring

The goal is to find the acceleration $a$ along the plane. It depends on $g$, $\theta$ (which is 30° here), the object's moment of inertia $I$, its mass $m$, and the rolling radius $r$.

$$a = f(g, \theta, I, m, r).$$

Use dimensional analysis to simplify this function of five variables. The six variables in total make three independent dimensionless groups:

$$\frac{a}{g} \qquad \theta \qquad \frac{I}{mr^2}.$$

Therefore,

$$\frac{a}{g} = f\left(\theta, \frac{I}{mr^2}\right). \tag{14}$$

Probably

$$\frac{a}{g} = f\left(\frac{I}{mr^2}\right)\sin\theta.$$

$$a = f\left(\frac{I}{mr^2}\right) g\sin\theta.$$

The ratio $I/mr^2$ is a dimensionless measure of where the mass of an object lies. The farther toward the edge and away from the rolling axis, the greater the ratio. Most importantly, $I/mr^2$ is independent of an object's radius; for example, a big and a small ring have the same ratio.

The bigger the ratio, the bigger the fraction of energy consumed by rolling motion compared to translational motion. Therefore, the object with the smallest $I/mr^2$ will have the greatest acceleration. The solid sphere (choice C) has the most mass near the rolling axis, so it will be the fastest.

**Problem 6 Hydrogen binding energy**

In lecture and readings we analyzed hydrogen (`r26-lumping-hydrogen.pdf` on NB), which is one electron bound to one proton. Using those results, one can show that the binding energy is

$$E \sim \frac{1}{2} m_\mathrm{e} (\alpha c)^2, \tag{15}$$

where $\alpha$ is the fine-structure constant, $c$ is the speed of light, and $m_\mathrm{e}$ is the mass of the electron.

Use the methods of **Problem 1** to calculate the binding energy in electron–volts.

$10^{\boxed{\phantom{00}}} \pm \boxed{\phantom{00}}$ eV ***or*** $10^{\boxed{\phantom{00}}} \ldots \boxed{\phantom{00}}$ eV

Rearranging the powers of $c$,

$$E \sim \frac{1}{2} \times m_\mathrm{e} c^2 \times \alpha^2. \tag{16}$$

Because $\alpha \approx 1/137$, which is roughly $1/141$,

$$\alpha^2 \sim \frac{1}{1.41 \cdot 10^2} \sim \frac{1}{2} \cdot 10^{-4}. \tag{17}$$

Since $m_\mathrm{e} c^2 \sim 0.5 \cdot 10^6$ eV, the binding energy is

$$E \sim \frac{1}{2} \times \frac{1}{2} \cdot 10^6 \,\mathrm{eV} \times \frac{1}{2} \cdot 10^{-4} \sim \frac{1}{8} \cdot 10^2 \,\mathrm{eV}. \tag{18}$$

The result is 13 eV.

**Problem 7 Heavy nuclei**

In this problem you study the innermost electron in an atom with many protons (i.e. with a heavy nucleus). So, imagine a nucleus with $Z$ protons around which orbits just one electron. Let $E(Z)$ be the binding energy. The case $Z = 1$ (**Problem 6**) is hydrogen.

Find how $E(Z)$ depends on $Z$. Namely, what is the scaling exponent $n$ in

$$E(Z) \propto Z^n \tag{19}$$

or, equivalently, in

$$\frac{E(Z)}{E(1)} = Z^n\,? \tag{20}$$

☐ ± ☐ ***or*** ☐ . . . ☐

With $Z$ protons pulling on one electron, the electrostatic energy contains the factor $Ze^2/4\pi\epsilon_0$. So instead of using $e^2/4\pi\epsilon_0$ as one quantity in the dimensional analysis, we should use $Ze^2/4\pi\epsilon_0$. The other quantities – $a_Z$, $m_e$, and $\hbar$ – are unchanged except for $a_Z$ replacing $a_0$. The $Z$ propagates along with the $e^2$ through the calculation of the radius $a_Z$ and the energy $E(Z)$.

Since the radius $a_0$ has one factor of $e^2$ in the denominator, the $a_Z$ picks up a factor of $Z$ in the denominator relative to $a_0$. Therefore,

$$a_Z = \frac{a_0}{Z}.$$

The electrostatic binding energy is inversely proportional to the radius $a_Z$:

$$E(Z) \sim \frac{Ze^2}{4\pi\epsilon_0}\frac{1}{a_Z}. \tag{21}$$

One factor of $Z$ is directly visible, and the second factor is part of $1/a_Z$. The energy $E(Z)$ thus has a factor of $Z^2$:

$$E(Z) = E(1) \times Z^2.$$

Therefore, $n = 2$.

**Problem 8 Heaviest nuclei**

Consider again the system of **Problem 7**: a nucleus with $Z$ protons surrounded by one electron.

When the binding energy $E(Z)$ is comparable to $m_e c^2$ – the rest energy of the electron – then the electron has enough kinetic energy to produce, out of nowhere, a positron (an anti-electron). As a result of this process, which is known as pair creation, the positron leaves the nucleus, turning one proton into a neutron. That makes the atomic number $Z$ drop by one. The consequence is that, for large-enough $Z$, the nucleus is unstable! Relativity sets an upper limit for $Z$.

Use the results of **Problem 7** to estimate this maximum $Z$ set by relativity (feel free to ignore factors of 1/2 in $E(1)$).

$\square \pm \square$ ***or*** $\square \ldots \square$

*To include in the explanation box:* Compare your estimate with the $Z$ for the heaviest stable nucleus (uranium).

Since the binding energy $E(Z)$ is $E_0 \times Z^2$ and $E_0 \sim m(\alpha c)^2$, the binding energy is

$$E(Z) \sim mc^2 (Z\alpha)^2. \tag{22}$$

Whe $Z\alpha \sim 1$, this energy is comparable to the electron's rest energy: That is when the electron becomes significantly relativistic, which permits pair creation to destabilize the nucleus. So the maximum $Z$ is roughly $\alpha^{-1}$ or about 140. The heaviest stable nucleus is uranium with $Z = 92$, so the explanation for the stability of the elements looks pretty good.

# 6.055J/2.038J (Spring 2010)

# Solution set 8

*Submit your answers and explanations online by **10pm on Wednesday, 05 May 2010**.*

***Open universe:*** *Collaboration, notes, and other sources of information are **encouraged.** However, avoid looking up answers to the problem, or to subproblems, until you solve the problem or have tried hard. This policy helps you learn the most from the problems.*

*Homework is graded with a light touch: P (made a decent effort), D (made an indecent effort), or F (did not make an effort).*

**Problem 1 Should you be worried?**

Assume that 1 in $10^4$ bridges in the United States are in danger of collapse unless repaired soon, and that a new test for bridge integrity has been devised. This test is 90% accurate: It always detects an unsafe bridge (no false negatives); and 10% of the time it says that a safe bridge is unsafe (10% false positives).

You learn that a nearby bridge, on which you often walk, failed the test (the test said it was unsafe). What are the odds that the bridge is safe?

$10^{\square \pm \square}$ ***or*** $10^{\square \ldots \square}$

*Note:* If $p_{\text{safe}}$ is the probability that the bridge is safe, then the corresponding odds are defined by

$$\text{odds} \equiv \frac{p_{\text{safe}}}{1 - p_{\text{safe}}}.$$

(Odds, unlike probabilities, range from 0 to $\infty$ and are thus more suitable for describing in the form $10^{a \pm b}$.)

Let's do it by the natural-frequencies approach. Imagine a population of $10^4$ US bridges. Given the base rate of 1 in $10^4$, assume that one bridge among them is actually in danger of collapse.

Now imagine applying the bridge-integrity test to all $10^4$ bridges. It will spot the one unsafe bridge. But from among the nearly $10^4$ safe bridges, it will also mark 10% or $10^3$ bridges as unsafe. The bridge you use is among the roughly $10^3$ bridges with a positive test. But only one of those bridges is actually unsafe, so $p_{\text{unsafe}} \approx 10^{-3}$. Therefore, the odds are $10^3$ to 1 that the bridge is safe (or simply $10^3$).

Now let's use Bayes theorem to get the same result. The odds form of Bayes theorem is

$$O(H|E) = O(H)\,\frac{P(E|H)}{P(E|\overline{H})}, \tag{1}$$

where $H$ is the hypothesis that the bridge is unsafe, $O(H)$ is the odds in favor of that hypothesis being true, $E$ is the evidence that the bridge failed the integrity test, $P(E|H)$ is the probability of a failed integrity test given that the bridge is unsafe (the false-negative rate), and $P(E|\overline{H})$ is the probability of a failed test given that the bridge is safe (the false-positive rate).

The initial odds $O(H)$ are $10^{-4}$ (the bridge is very probably safe). The likelihood ratio is

$$\frac{P(E|H)}{P(E|\overline{H})} = \frac{1}{0.1} = 10. \tag{2}$$

Therefore, the new odds are $10^{-3}$ in favor of the bridge being unsafe (or $10^3$ in favor of it being safe, as computed above).

For an excellent article in the newspaper (of all places) on how to do Bayes theorem using natural frequencies, see Steven Strogatz's recent column in the *New York Times* (thanks to Sean Clarke for pointing me to it), available at **http://opinionator.blogs.nytimes.com/2010/04/25/chances-are/**

**Problem 2 Reusing plausible-range combinations**

In lecture, we saw that if the width of an object has a plausible range $w = 1 \ldots 10\,\mathrm{m}$ and the length has a plausible range $l = 1 \ldots 10\,\mathrm{m}$, then the area $A = lw$ has the range $2 \ldots 50\,\mathrm{m}^2$.

If instead the plausible ranges are $w = 2 \ldots 20\,\mathrm{m}$ and $l = 5 \ldots 50\,\mathrm{m}$, what is the plausible range for the area?

[ ] ± [ ] $\mathrm{m}^2$ ***or*** [ ] . . . [ ] $\mathrm{m}^2$

Multiplying everything (the top and bottom) by a constant changes the midpoint of the range and the lower and upper endpoints, but does not change the width itself (the ratio of the upper to lower endpoints). For example, a factor of 25 uncertainty is still a factor of 25, just around a new midpoint. In fancy words, the width of a range is invariant to changes of scale.

From lecture, we are given that $1 \ldots 10 \times 1 \ldots 10 \approx 2 \ldots 50$. Now multiply the lower and upper endpoints of the first range by 2; and multiply the lower and upper endpoints of the second range by 5. Those changes multiply the lower and upper endpoints of the product by $2 \times 5$. So,

$$2 \ldots 20 \times 5 \ldots 50 \approx 20 \ldots 500. \tag{3}$$

**Problem 3 Singing a logarithm**

Estimate $1.5^{40}$ using the singing-logarithms method from lecture (a copy of the handout is on the course website).

$10^{[\ ]}$ ± [ ] ***or*** $10^{[\ ]}$ . . . [ ]

1.5 is 3/2, which is 7 semitones (a perfect fifth). Each semitone is a factor of $2^{1/12}$ which is also $10^{1/40}$ (40 semitones make a factor of 10). Therefore, $1.5 \approx 10^{7/40}$ and

$$1.5^{40} \approx \left(10^{7/40}\right)^{40} = 10^7. \tag{4}$$

The true value is just above $1.1 \cdot 10^7$.

**Problem 4 Estimating a mass**

You are trying to estimate the mass of an object. Suppose that your plausible range for its density is $1 \ldots 5\,\mathrm{g\,cm^{-3}}$ and for its volume is $1 \ldots 5\,\mathrm{cm^3}$. What is (roughly) your plausible range for its mass?

☐ ± ☐ g *or* ☐ … ☐ g

Each range is a factor of 5 in width. In semitones,

$$5 = \underbrace{\frac{5}{4}}_{4\ \text{semitones}} \times \underbrace{2}_{12\ \text{semitones}} \times \underbrace{2}_{12\ \text{semitones}} = 28\ \text{semitones}. \tag{5}$$

Because 40 semitones make a factor of 10, the 28 semitones means a base-10 logarithm of 0.7. Add the squares of (logarithmic) widths to get the new (logarithmic) width squared:

$$0.7^2 + 0.7^2 \approx 1. \tag{6}$$

So, the new plausible range has a width of 1 decade (a factor of 10). The range is centered at 5 g:

$$m = \rho V \sim \underbrace{\sqrt{5}}_{1...5}\,\mathrm{g\,cm^{-3}} \times \underbrace{\sqrt{5}}_{1...5}\,\mathrm{cm^3} = 5\,\mathrm{g}. \tag{7}$$

So the plausible range is $1.7 \ldots 15\,\mathrm{g}$. (A full calculation, without using the semitones approximation, gives $1.6 \ldots 15.6\,\mathrm{g}$.)

**Problem 5 Which is the wider range?**

Suppose that your knowledge of the quantities $a$, $b$, and $c$ is given by these plausible ranges:

$$\begin{aligned} a &= 1 \ldots 10 \\ b &= 1 \ldots 10 \\ c &= 1 \ldots 10. \end{aligned} \tag{8}$$

Which quantity – $abc$ or $a^2b$ – has the wider plausible range? (The 'width' is the ratio of the upper to lower endpoints; so $a$, $b$, and $c$ are each a factor of 10 wide.)

☐ $abc$

☐ $a^2b$

☐ Both quantities have the same width.

Both choices have $b$ in them, so ignore it and instead compare $ac$ versus $a^2$. When computing $ac$ there is a chance that an overestimate in $a$ will compensate an underestimate in $c$ (and vice versa). However, when computing $a^2$, any error in estimating $a$ is magnified – a factor of 2 error in $a$ becomes a factor of 4 error in $a^2$. So, $a^2$ has a wider plausible range than $ac$. Numerically,

$$\begin{aligned} ac &= 2 \ldots 50. \\ a^2 &= 1 \ldots 100. \end{aligned} \tag{9}$$

**Problem 6 Golf-ball dimples**

Why do golf balls have dimples?

- ☐ The dimples make the main airflow around the ball become turbulent.
- ☐ The dimples stabilize the flight.
- ☐ The dimples are there by tradition but have no physical justification.
- ☐ The dimples make the airflow turbulent in the thin boundary layer adjacent to the ball.

Let's first calculate the Reynolds number (a good first instinct in understanding a fluid flow). Let's say that the golf ball is hit at $30\,\mathrm{m\,s^{-1}}$ (70 mph!). It's diameter is a few centimeters, say 3 cm. Using $\nu \sim 10^{-5}\,\mathrm{m^2\,s^{-1}}$ for the viscosity of air, the Reynolds number is

$$\mathrm{Re} \sim \frac{30\,\mathrm{m\,s^{-1}} \times 3 \cdot 10^{-2}\,\mathrm{m}}{10^{-5}\,\mathrm{m^2\,s^{-1}}} \sim 10^5. \tag{10}$$

That means the Reynolds number in the boundary layer is roughly $\sqrt{\mathrm{Re}} \sim 10^{2.5} \sim 300$. This is not high enough for turbulence, so the boundary layer is laminar.

A laminar boundary layer separates easily on the back of the golf ball, creating a large turbulent, low-pressure region behind the ball – that means lots of drag. If only the boundary layer could be made turbulent! Then the boundary layer would stick to the golf ball farther along the back side, and the drag would be lower. That's just what the dimples do: They trip the boundary layer into turbulence at a lower Reynolds number than is otherwise required (Choice D).

**Problem 7 Singing logarithms to combine plausible ranges**

You are trying to esimate the plausible range for the volume of an object. You have assigned the length, width, and height the plausible ranges

$$\begin{aligned} l &= 1\ldots 10\,\mathrm{m} \\ w &= 1\ldots 10\,\mathrm{m} \\ h &= 1\ldots 10\,\mathrm{m}. \end{aligned} \tag{11}$$

In other words, each range is a factor of 10 wide (the 'width' is the ratio of the upper to lower endpoints). Convince yourself that the plausible range for the volume $V = lwh$ is a factor of $10^{\sqrt{3}}$ wide and is centered on $10^{1.5}\,\mathrm{m}^3$.

Each factor in $lwh$ is centered at $10^{0.5}\,\mathrm{m}$ (the geometric mean of the lower and upper endpoints). Therefore $lwh$ is centered on

$$\left(10^{0.5}\,\mathrm{m}\right)^3 = 10^{1.5}\,\mathrm{m}^3. \tag{12}$$

To compute the width of the range for $lwh$, note that each factor in $lwh$ is 1 factor of 10 in width. For plausible ranges, add the squares of the (logarithmic) widths to get the square of the final (logarithmic) width:

$$1^2 + 1^2 + 1^2 = 3. \tag{13}$$

So the plausible range for $V$ is $\sqrt{3}$ wide (in its base-10 logarithm); in other words, the range has width $10^{\sqrt{3}}$.

For the answer box, use $\sqrt{3} \approx 1.7$ or $\sqrt{3} \approx 1.73$ and the singing-logarithm method from lecture (a copy of the handout is on the course website) to estimate $10^{\sqrt{3}}$.

☐ ± ☐ *or* ☐ . . . ☐

I'll first use $\sqrt{3} = 1.7$. Then

$$10^{\sqrt{3}} = 10^{1.7} = 10 \times 10^{0.7}. \tag{14}$$

But $10^{0.7}$ is 28 semitones (40 semitones is a factor of 10).

$$28\text{ semitones} = \underbrace{\frac{5}{4}}_{4\text{ semitones}} \times \underbrace{2}_{12\text{ semitones}} \times \underbrace{2}_{12\text{ semitones}} = 5. \tag{15}$$

So, $10^{1.7} \approx 50$.

For fun, let's correct that estimation slightly by using $\sqrt{3} = 1.73$. The extra factor is $10^{0.03}$. Since 0.03 is roughly 1/40, $10^{0.03}$ is roughly 1 semitone. Based on the observation that 1.25 is 4 semitones, 1 semitone is given by

$$1.25^{1/4} = (1 + 0.25)^{1/4} \approx 1 + \frac{0.25}{4} \approx 1.06. \tag{16}$$

So, we should raise the earlier estimate of $10^{1.7}$ by 6%, which gives 53. [An exact calculation gives $10^{\sqrt{3}} \approx 53.96$.]

**Problem 8 Perfume**

If the diffusion constant (in air) for small perfume molecules is $10^{-6}\,\mathrm{m^2\,s^{-1}}$, estimate the time for perfume molecules to diffuse across the lecture room.

$10^{\square \pm \square}$ s *or* $10^{\square \ldots \square}$ s

*To include in the explanation box:* Now try the experiment, at least mentally. How long does it actually take to smell the perfume from across the room? Explain the large discrepancy between the theoretical estimate and the experimental value.

The dimensions of a diffusion constant $D$ are $\mathrm{L^2T^{-1}}$, so the diffusion time is given by $\tau \sim x^2/D$, where $x$ is a length. The lecture room is perhaps 10 m deep (and maybe 15 m wide). It doesn't matter exactly which length I use, so I'll use the one that is simpler to square: $x \sim 10\,\mathrm{m}$. Then

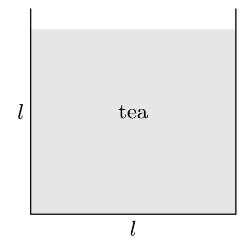
$$\tau \sim \frac{10\,\mathrm{m}^2}{10^{-6}\,\mathrm{m}^2\,\mathrm{s}^{-1}} \sim 10^8\,\mathrm{s} \sim 3\,\text{years}. \tag{17}$$

That time does not agree with experiment! In reality, it takes perhaps a minute to notice that someone has opened a bottle of aromatic stuff. The discrepancy is that the molecules must travel not just by diffusion; in fact, the unavoidable air currents in the room transport the molecules much farther and faster than diffusion can.

**Problem 9 Teacup spindown**

You stir your afternoon tea to mix the milk (and sugar if you have a sweet tooth). Once you remove the stirring spoon, the rotation starts to slow. In this problem you'll estimate the spindown time $\tau$: the time for the angular velocity of the tea to drop by a significant fraction.

To estimate $\tau$, consider first a physicist's idea of a teacup: a cylinder with height $l$ and diameter $l$, filled with a water-like liquid. Tea near the edge of the teacup – and near the base, but for simplicity we'll neglect the effect of the base – is slowed by the presence of the edge.




Because of the no-slip boundary condition, the edge creates a velocity gradient. Because of the tea's viscosity, the velocity gradient produces a force along the edge. This force tries to accelerate each piece of the edge along the direction of the tea's motion. The piece in return exerts an equal and opposite force on the tea. That is how the edge slows the rotation. Now analyze this model quantitatively using the following steps. Keep the results in symbolic form until the final step (**Step e**) when you get a numerical value for $\tau$.

a. Convince yourself that the spindown time $\tau$ is given by

$$\tau \sim \frac{\rho l^5 \omega}{\sigma l^3} = \frac{\rho l^2 \omega}{\sigma}, \tag{18}$$

where $\rho$ is the density of tea, $\sigma$ is the viscous stress (the viscous tangential force per unit area), and $\omega$ is the initial angular velocity. *Hint:* Consider the torque on and the angular momentum of the rotating blob of tea. In addition, drop all dimensionless constants like $\pi$ and 2 by invoking the Estimation Theorem 1 = 2.

If the tea is spinning at angular velocity $\omega$, then it has angular momentum $L = I\omega$, where $I$ is the moment of inertia. The moment of inertia is given by mass times a squared distance from the origin:

$$I \sim \underbrace{\rho l^3}_{m} \times l^2 = \rho l^5. \tag{19}$$

Not all of the mass is at a distance $l$ from the center, but the twiddle accounts for the omitted dimensionless constant. With that $I$, the angular momentum is

$$L \sim \rho l^5 \omega. \tag{20}$$

The viscous stress produces a torque that reduces this angular momentum. The viscous torque is

$$\text{viscous torque} \sim \underbrace{\text{viscous stress}}_{\sigma} \times \underbrace{\text{area}}_{l^2} \times \underbrace{\text{lever arm}}_{l} \sim \sigma l^3. \tag{21}$$

Because torque is $dL/dt$, it has dimensions of $L/t$. So a time is given by $L/\text{torque}$:

$$\tau \sim \frac{\rho l^5 \omega}{\sigma l^3}. \tag{22}$$

b. Now estimate the viscous stress $\sigma$ by using the idea that

$$\text{viscous stress} \sim \rho \nu \times \text{velocity gradient}. \tag{23}$$

The velocity gradient is determined by the thickness of the region over which the the edge significantly affects the flow; this region is the boundary layer. Let $\delta$ be its thickness (you'll find $\delta$

in **Step d**). In terms of $\delta$, estimate the velocity gradient near the edge. Then estimate the viscous stress $\sigma$.

The velocity gradient is

$$\frac{\Delta v}{\Delta x} \sim \frac{\omega l}{\delta}. \tag{24}$$

Therefore the viscous stress is

$$\sigma \sim \rho\nu\frac{\omega l}{\delta}. \tag{25}$$

c. Insert your expression for the viscous stress $\sigma$ into the earlier estimate for the spindown time $\tau$. Your new expression for $\tau$ should contain only the boundary-layer thickness $\delta$, the cup's size $l$, and the viscosity $\nu$.

After substituting,

$$\tau \sim \frac{\rho l^5 \omega}{l^3 \times \rho\nu\omega l/\delta} \sim \frac{l\delta}{\nu}. \tag{26}$$

[The information in the problem statement is sufficient to arrive at this result, because $l\delta/\nu$ is the only way to make a time from $l$, $\delta$, and $\nu$.]

d. Now estimate the boundary-layer thickness $\delta$ using your knowledge of random walks. The boundary layer is a result of momentum diffusion – and $\nu$ is the momentum-diffusion coefficient. In a given time $t$, how far can momentum diffuse? This distance is $\delta$. Estimate a reasonable $t$ for the rotating blob of tea. [*Hint:* After rotating by 1 radian, the fluid is moving in a significantly different direction than before, so the momentum fluxes no longer add.] Use that time to estimate $\delta$.

A reasonable time is the time to rotate 1 radian, namely $t \sim 1/\omega$. In that time, the diffusion distance $\delta$ is

$$\delta \sim \sqrt{\nu t} \sim \sqrt{\nu/\omega}. \tag{27}$$

e. Now put it all together. For a typical teacup stirred with a typical stirring motion, what is the predicted spindown time $\tau$? [Tea is roughly water, and $\nu_{\text{water}} \sim 10^{-6}\,\text{m}^2\,\text{s}^{-1}$.]

$10^{\square \pm \square}$ s ***or*** $10^{\square \ldots \square}$ s

Substituting for $\delta$ in the expression for the spindown time $\tau$ gives

$$\tau \sim \frac{l\delta}{\nu} \sim \frac{l}{\sqrt{\nu\omega}}. \tag{28}$$

Now put in numbers. My nearby teacup is a few inches across, so $l \sim 10\,\text{cm}$. When I stir the tea, it rotates at a frequency $f \sim$ few Hz, so $\omega = 2\pi f \sim 20\,\text{s}^{-1}$. The result is

$$\tau \sim \frac{0.1\,\text{m}}{\sqrt{10^{-6}\,\text{m}^2\,\text{s}^{-1} \times 20\,\text{s}^{-1}}} \sim 20\,\text{s}. \tag{29}$$

*To include in the explanation box:* Estimate $\tau_{exp}$ experimentally by stirring tea. Compare the experimental time with the predicted time.

I just tried it, and $\tau_{experimental}$ (the time for the rotation to slow significantly) was around 10 s. Not bad!

# 6.055J/2.038J (Spring 2010)

# Solution set 9

*This homework is* ***not for turning in*** *(by MIT rules, no assignment may fall due after May 7). But I hope that you enjoy thinking about the problems.*

**Problem 1 Xylophones**

If a 10-cm long slat of a xylophone rings with middle C ($f \sim 260\,\text{Hz}$), how long is the slat that rings with low C ($f \sim 130\,\text{Hz}$)? (The slats of a xylophone all have the same thickness.)

- ☐ 5 cm
- ☐ 7 cm
- ☐ 14 cm
- ☐ 20 cm

I'll follow the approach used in lecture: dimensional analysis along with the result of the wood-blocks demonstration. But for fun I'll do it slightly differently, by using dimensional analysis to compute the slat's bending stiffness.

The stiffness $k$ depends on $Y$ (the Young's modulus) and the three dimensions of length, width, and height ($l$, $w$, and $h$). The dimensions of $k$ are force per length (it's a spring constant) or energy per area; the dimensions of $Y$ are force per area or energy per volume. The other quantites are all lengths. So, these five variables can be produced by two independent dimensions (energy and length, for example). Therefore, there are three independent dimensionless groups.

One group can be $k/Yl$. Another is $w/l$, and a third is $h/l$. The most general statement using these three groups is

$$\frac{k}{Yl} = f\left(\frac{w}{l}, \frac{h}{l}\right). \tag{1}$$

The goal in this problem is to find out how $k$ depends on the length $l$ (and from there to find how the frequency depends on the length). To that end, the first observation is that the stiffness is proportional to the width $w$: Two identical slats side by side act like one wide slat, and have twice the stored energy, therefore twice the stiffness $k$. The only way to make $k$ proportional to $w$ is

$$\frac{k}{Yl} = \frac{w}{l} f\left(\frac{h}{l}\right). \tag{2}$$

The second observation is the argument from the reading (`r33-springs-wood-blocks.pdf`) and lecture that the stiffness is proportional to $h^3$. The only way to make that true is to make $f$ a cubic:

$$\frac{k}{Yl} = \frac{w}{l} \times \frac{h}{l^3}. \tag{3}$$

Thus, the stiffness $k$ is proportional to $l^{-3}$. Because the mass is proportional to $l$, the oscillation frequency is proportional to $l^{-2}$:

$$\omega = \sqrt{\frac{k}{m}} \propto \sqrt{\frac{l^{-3}}{l}} = l^{-2}. \tag{4}$$

Thus, to decrease the frequency by a factor of 2 (from 260 Hz to 130 Hz) requires increasing the length by a factor of $\sqrt{2}$, i.e. from 10 cm to roughly 14 cm.

$$\text{new length} \approx 14\,\text{cm (choice C)}. \tag{5}$$

**Problem 2 Speed of sound**

The speed of sound in oxygen (molecular mass 32) at standard temperature and pressure is $316\,\mathrm{m\,s^{-1}}$. What is the speed of sound for hydrogen (molecular mass 2)?

- ☐ $78\,\mathrm{m\,s^{-1}}$
- ☐ $157\,\mathrm{m\,s^{-1}}$
- ☐ $316\,\mathrm{m\,s^{-1}}$
- ☐ $637\,\mathrm{m\,s^{-1}}$
- ☐ $1284\,\mathrm{m\,s^{-1}}$

The speed of sound in a gas depends on the density and pressure:

$$c_s \sim \sqrt{\frac{p}{\rho}}. \tag{6}$$

The ideal gas law is

$$p = nkT, \tag{7}$$

where $n$ is the molecule's number density, $k$ is Boltzmann's, constant and $T$ is the temperature. The density is $\rho = mn$, where $m$ is the molecular mass. So,

$$c_s \sim \sqrt{\frac{p}{\rho}} = \sqrt{\frac{nkT}{mn}} \propto m^{-1/2}. \tag{8}$$

(Everything else, including $T$, is held constant.)

Decreasing the molecular mass from 32 atomic mass units to 2 atomic mass units – a factor of 16 decrease – will increase the speed of sound by a factor of $\sqrt{16} = 4$. Therefore,

$$c_s^{\text{hydrogen}} \approx 1264\,\mathrm{m\,s^{-1}} \text{ (choice E)}. \tag{9}$$

**Problem 3 Blue skies**

Which reason is part of the explanation for why the sky looks blue?

☐ The power radiated by an oscillating charge decreases strongly with frequency.

☐ The power radiated by an oscillating charge increases strongly with frequency.

☐ Our eyes are more sensitive to blue than to red light.

☐ The sun radiates more energy in the short-wavelength (blue) end of the visible spectrum than in the long-wavelength (red) end of the spectrum.

☐ The sun radiates more energy in the long-wavelength (red) end of the visible spectrum than in the short-wavelength (blue) end of the spectrum.

In an earlier problem set (Problem 3 on Homework 5), you found that an accelerating charge's radiation power is proportional to the square of its acceleration. The accelerating itself is proportional to the square of the frequency (as is true of any spring). So, the power radiated is proportional to the fourth power of frequency – a strongly increasing function.

Thus, choice A is incorrect. The remaining question is which of choices B-E are true and relevant. First, truth. Choice B is true. Choice C is false, but even if it were true, it would be a small effect. Choices D and E, even if true, would again be only small effects.

Whereas choice B is true and is a large effect. The frequency ratio between red and blue light is roughly 2 (the visible spectrum spans an octave). So any atoms, e.g. in the atmopshere, receiving red and blue solar radiation will oscillate in a combination of red and blue frequencies, and that combination will radiate predominantly (by a factor of $2^4 = 16$) blue light. That's why the sky looks blue.

Choice B (10)

**Problem 4 Boiling away mercury**

The surface tension of mercury (a liquid at room temperature) is roughly $0.5\,\mathrm{N\,m^{-1}}$. Roughly how much energy is required to boil away $1\,\mathrm{m^3}$ of mercury?

$10^{\square} \pm \square$ J ***or*** $10^{\square} \ldots \square$ J

The simplest argument is from dimensions. Surface tension is energy per area. The required quantity $L_{\mathrm{vap}}$ is energy per volume. Their ratio $\gamma/L_{\mathrm{vap}}$ is a length. Because both quantities are microscopic in origin (they are based on atomic spacings and bond energies), the only plausible length is the interatomic spacing $a$. Thus,

$$\frac{\gamma}{L_{\mathrm{vap}}} \sim a. \tag{11}$$

A slightly more accurate result, as discussed in lecture, is to include a factor of 1/6:

$$\frac{\gamma}{L_{\mathrm{vap}}} \sim \frac{1}{6}a, \tag{12}$$

to account for the fact that making surface breaks only 1 out of, say, 6 bonds. Then

$$L_{\mathrm{vap}} \sim \frac{6\gamma}{a}. \tag{13}$$

For mercury,

$$L_{\mathrm{vap}} \sim \frac{6 \times 0.5\,\mathrm{N\,m^{-1}}}{3 \cdot 10^{-10}\,\mathrm{m}} \sim 10^{10}\,\mathrm{J\,m^{3}}. \tag{14}$$

The true value is about $0.4 \cdot 10^{10}\,\mathrm{J\,m^{-3}}$, so this simple dimensions argument is reasonably accurate.

**Problem 5 Waves on a swimming pool**

Roughly what is the minimum speed of waves on the surface of a swimming pool filled with mercury instead of water? [Surface tension $\gamma_{\mathrm{Hg}} \sim 0.5\,\mathrm{N\,m^{-1}}$; density $\rho_{\mathrm{Hg}} \sim 1.4 \cdot 10^4\,\mathrm{kg\,m^{-3}}$.]

$10^{\square} \pm \square\ \mathrm{m\,s^{-1}}$ ***or*** $10^{\square} \ldots \square\ \mathrm{m\,s^{-1}}$

Do it by dimensions. The minimum speed $v$ depends on $\gamma$ and $\rho$ – and also on gravity $g$. The four variables $v$, $\gamma$, $\rho$, and $g$ are made up of three dimensions, so there is one independent dimensionless group. The choice that gives the result most directly is

$$v \Big/ \left(\frac{\gamma g}{\rho}\right)^{1/4}. \tag{15}$$

Therefore,

$$v \sim \left(\frac{\gamma g}{\rho}\right)^{1/4}. \tag{16}$$

Numerically,

$$v \sim \left(\frac{0.5\,\mathrm{N\,m^{-1}} \times 10\,\mathrm{m\,s^{-2}}}{1.4 \cdot 10^4\,\mathrm{kg\,m^{-3}}}\right)^{1/4} \sim 0.14\,\mathrm{m\,s^{-1}}. \tag{17}$$

For more on waves and the physical reasoning behind this result, see Section 9.3.11 of the full book (now on the course website).

**Problem 6 Cold day**

You stand outside on a calm (i.e. not very windy) but cold winter day wearing only a thin T-shirt and equally thin pants. Roughly at what rate does your body lose heat?

$10^{\square} \pm \square\ \mathrm{W}$ ***or*** $10^{\square} \ldots \square\ \mathrm{W}$

This situation is analyzed in the readings (`r31-probabilistic-random-walks.pdf` or Section 8.3 of the full book). The result is

$$P \sim 600\,\mathrm{W}. \tag{18}$$

**Problem 7 Blackbody temperature of the earth**

The earth's surface temperature is mostly due to solar radiation.

The solar flux $S \approx 1350\,\mathrm{W\,m^{-2}}$ is the amount of solar energy reaching the top of the earth's atmosphere. But that energy is spread over the surface of a sphere, so $S/4$ is the relevant flux for calculating the surface temperature. Some of that energy is reflected back to space by clouds or ocean before it can heat the ground, so the heating flux is slightly lower than $S/4$. A useful estimate is $S' \approx S/5 \sim 250\,\mathrm{W\,m^{-2}}$.

Look up the Stefan–Boltzmann law and use it to find the blackbody temperature of the earth.

☐ ± ☐ K *or* ☐ . . . ☐ K

Your value should be close to room temperature but enough colder to make you wonder about the discrepancy. Why is the actual average surface temperature warmer than the value calculated in this problem?

According to the Stefan–Boltzmann law, the power per area radiated from a blackbody is $F = \sigma T^4$, where $\sigma$ is the Stefan–Boltzmann constant and $T$ is the temperature of the body (the object). So

$$T = (F/\sigma)^{1/4}.$$

The Stefan–Boltzmann constant $\sigma$ is constructed from other fundamental constants (you can derive most of $\sigma$ using the method of ??). It's value is

$$\sigma \approx 5.7 \cdot 10^{-8}\,\mathrm{J\,s^{-1}\,m^{-2}\,K^{-4}}.$$

So,

$$T \approx \left( \frac{250\,\mathrm{W\,m^{-2}}}{5.7 \cdot 10^{-8}\,\mathrm{J\,s^{-1}\,m^{-2}\,K^{-4}}} \right)^{1/4} \sim 257\,\mathrm{K}.$$

In normal units, that's $-16\,^\circ\mathrm{C}$ or $3\,^\circ\mathrm{F}$. That's very cold, colder than the average Boston winter day.

It is close to room temperature, but the discrepancy is a bit large. What's wrong with the calculation? The greenhouse effect! The earth absorbs the $250\,\mathrm{W\,m^{-2}}$ from the sun, and it radiates it to space. Those parts of the calculation are correct. But the outgoing radiation is mostly infrared, which is well absorbed by carbon dioxide and water molecules in the atmosphere. The absorbed radiation is radiated in all directions, including back to the earth – warming the surface, and making life bearable.

So, we need the greenhouse effect, just not too much of it.

**Problem 8 Painful integral**

Estimate

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{200}}\,dx \tag{19}$$

with only paper and pencil (i.e. no calculators or computers).

☐ ± ☐ *or* ☐ . . . ☐

Following the recipe for the Landau-Institute entrance problem, notice that

$$(1+x^2)^{200} \approx e^{200x^2}, \tag{20}$$

as long as $x^2$ is small. (And when $x^2$ is not small, the error is irrelevant because the denominator is so huge anyway.) So, the integral is approximately

$$\int_{-\infty}^{\infty} e^{-200x^2}\,dx = \sqrt{\frac{\pi}{200}}. \tag{21}$$

Since $\pi/50$ was about $1/16$, here we'll get

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{200}}\,dx \approx \int_{-\infty}^{\infty} e^{-200x^2}\,dx = \sqrt{\frac{\pi}{200}} \approx \sqrt{\frac{1}{64}} = 0.125. \tag{22}$$

The exact value, as integrated by `maxima` (an MIT invention), is

$$\frac{80633223790268196250764317293547921958968456595455551562381782249313386883342699403459113031779834705172343826368087 5\pi}{20173827172553973356686868531273530268200826506478308693989526222973809547006571833044104322501076808092993531037089792} \tag{23}$$

which is approximately 0.12556702371249.