

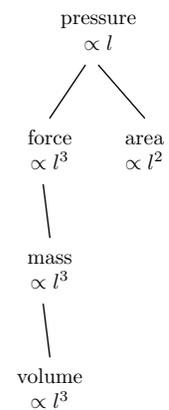
The force and area results show that the pressure is proportional to  $l$ :

$$p \sim \frac{F}{A} \propto \frac{l^3}{l^2} = l.$$

With a large-enough mountain, the pressure is larger than the maximum pressure that the rock can withstand. Then the rock flows like a liquid, and the mountain cannot grow taller.

This estimate shows only that there is a maximum height but it does not compute the maximum height. To do that next step requires estimating the strength of rock. Later in this book when we estimate the strength of materials, I revisit this example.

This estimate might look dubious also because of the assumption that mountains are cubical. Who has seen a cubical mountain? Try a reasonable alternative, that mountains are pyramidal with a square base of side  $l$  and a height  $l$ , having a  $45^\circ$  slope. Then the volume is  $l^3/3$  instead of  $l^3$  *but the factor of one-third does not affect the proportionality between force and length*. Because of the factor of one-third, the maximum height will be higher for a pyramidal mountain than for a cubical mountain. However, there is again a maximum size (and height) of a mountain. In general, the argument for a maximum height requires only that all mountains are similar – are scaled versions of each other – and does not depend on the shape of the mountain.



## 5.4 Animal jump heights

We next use proportional reasoning to understand how high animals jump, as a function of their size. Do kangaroos jump higher than fleas? We study a jump from standing (or from rest, for animals that do not stand); a running jump depends on different physics. This problem looks underspecified. The height depends on how much muscle an animal has, how efficient the muscles are, what the animal's shape is, and much else. The first subsection introduces a simple model of jumping, and the second refines the model to consider physical effects neglected in the crude approximations.

### 5.4.1 Simple model

We want to determine only how jump height varies with body mass. Even this problem looks difficult; the height still depends on muscle efficiency, and so on. Let's see how far we get by just plowing along, and using symbols for the unknown quantities. Maybe all the unknowns cancel.

We want an equation for the height  $h$  in the form  $h \sim m^\beta$ , where  $m$  is the animal's mass and  $\beta$  is the so-called scaling exponent.

Jumping requires energy, which must be provided by muscles. This first, simplest model equates the required energy to the energy supplied by the animal's muscles.

The required energy is the easier estimation: An animal of mass  $m$  jumping to a height  $h$  requires an energy  $E_{\text{jump}} \propto mh$ . Because all animals feel the same gravity, this relation does not contain the gravitational acceleration  $g$ . You could include it in the equation, but it would just carry through the equations like unused baggage on a trip.

The available energy is the harder estimation. To find it, divide and conquer. It is the product of the muscle mass and of the energy per mass (the energy density) stored in muscle.

To approximate the muscle mass, assume that a fixed fraction of an animal's mass is muscle, i.e. that this fraction is the same for all animals. If  $\alpha$  is the fraction, then

$$m_{\text{muscle}} \sim \alpha m$$

or, as a proportionality,

$$m_{\text{muscle}} \propto m,$$

where the last step uses the assumption that all animals have the same  $\alpha$ .

For the energy per mass, assume again that all muscle tissues are the same: that they store the same energy per mass. If this energy per mass is  $\mathcal{E}$ , then the available energy is

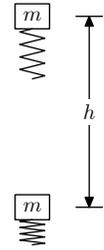
$$E_{\text{avail}} \sim \mathcal{E} m_{\text{muscle}}$$

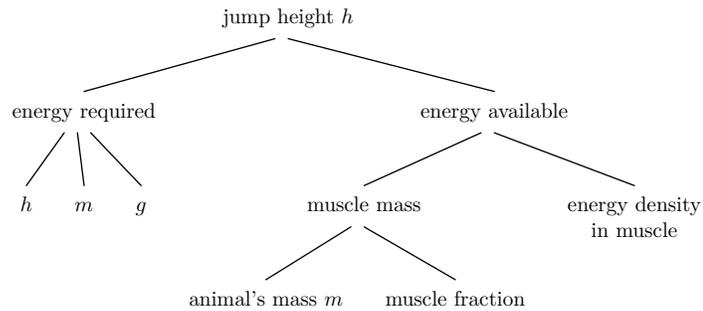
or, as a proportionality,

$$E_{\text{avail}} \propto m_{\text{muscle}},$$

where this last step uses the assumption that all muscle has the same energy density  $\mathcal{E}$ .

Here is a tree that summarizes this model:





Now finish propagating toward the root. The available energy is

$$E_{\text{avail}} \propto m.$$

So an animal with three times the mass of another animal can store roughly three times the energy in its muscles, according to this simple model.

Now compare the available and required energies to find how the jump height as a function of mass. The available energy is

$$E_{\text{avail}} \propto m$$

and the required energy is

$$E_{\text{required}} \propto mh.$$

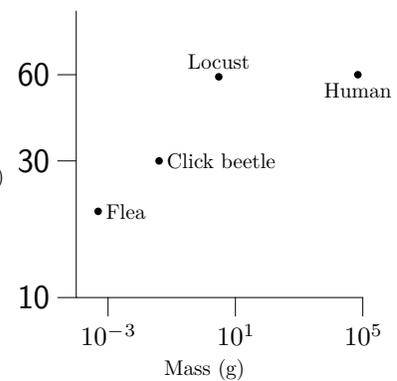
Equate these energies, which is an application of conservation of energy. Then  $mh \propto m$  or

$$h \propto m^0.$$

In other words, all animals jump to the same height.

The result, that all animals jump to the same height, seems surprising. Our intuition tells us that people should be able to jump higher than locusts. The graph shows jump heights for animals of various sizes and shapes [source: *Scaling: Why Animal Size is So Important* [31, p. 178]. Here is the data:

Animal	Mass (g)	Height (cm)
Flea	$5 \cdot 10^{-4}$	20
Click beetle	$4 \cdot 10^{-2}$	30
Locust	3	59
Human	$7 \cdot 10^4$	60



The height varies almost not at all when compared to variation in mass, so our result is roughly correct! The mass varies more than eight orders of magnitude (a factor of  $10^8$ ), yet the jump height varies only by a factor of 3. The predicted scaling of constant  $h$  ( $h \propto 1$ ) is surprisingly accurate.

### 5.4.2 Power limits

Power production might also limit the jump height. In the preceding analysis, energy is the limiting reagent: The jump height is determined by the energy that an animal can store in its muscles. However, even if the animal can store enough energy to reach that height, the muscles might not be able to deliver the energy rapidly enough. This section presents a simple model for the limit due to limited power generation.

Once again we'd like to find out how power  $P$  scales (varies) with the size  $l$ . Power is energy per time, so the power required to jump to a height  $h$  is

$$P \sim \frac{\text{energy required to jump to height } h}{\text{time over which the energy is delivered}}.$$

The energy required is  $E \sim mgh$ . The mass is  $m \propto l^3$ . The gravitational acceleration is independent of  $l$ . And, in the energy-limited model, the height  $h$  is independent of  $l$ . Therefore  $E \propto l^3$ .

The delivery time is how long the animal is in contact with the ground, because only during contact can the ground exert a force on the animal. So, the animal crouches, extends upward, and finally leaves the ground. The contact time is the time during which the animal extends upward. Time is length over speed, so

$$t_{\text{delivery}} \sim \frac{\text{extension distance}}{\text{extension speed}}.$$

The extension distance is roughly the animal's size  $l$ . The extension speed is roughly the takeoff velocity. In the energy-limited model, the takeoff velocity is the same for all animals:

$$v_{\text{takeoff}} \propto h^{1/2} \propto l^0.$$

So

$$t_{\text{delivery}} \propto l.$$

The power required is  $P \propto l^3/l = l^2$ .

That proportionality is for the power itself, but a more interesting scaling is for the specific power: the power per mass. It is

$$\frac{P}{m} \propto \frac{l^2}{l^3} = l^{-1}.$$

Ah, smaller animals need a higher specific power!

A model for power limits is that all muscle can generate the same maximum power density (has the same maximum specific power). So a small-enough animal cannot jump to its energy-limited height. The animal can store enough energy in its muscles, but cannot release it quickly enough.

More precisely, it cannot do so unless it finds an alternative method for releasing the energy. The click beetle, which is toward the small end in the preceding graph and data set, uses the following solution. It stores energy in its shell by bending the shell, and maintains the bending like a ratchet would (holding a structure motionless does require energy). This storage can happen slowly enough to avoid the specific-power limit, but when the beetle releases the shell and the shell snaps back to its resting position, the energy is released quickly enough for the beetle to rise to its energy-limited height.

But that height is less than the height for locusts and humans. Indeed, the largest deviations from the constant-height result happen at the low-mass end, for fleas and click beetles. To explain that discrepancy, the model needs to take into account another physical effect: drag.

## 5.5 Drag

### 5.5.1 Effect of drag on fleas jumping

The drag force

$$F \sim \rho A v^2$$

affects the jumps of small animals more than it affects the jumps of people. A comparison of the energy required for the jump with the energy consumed by drag explains why.

The energy that the animal requires to jump to a height  $h$  is  $mgh$ , if we use the gravitational potential energy at the top of the jump; or it is  $\sim mv^2$ , if we use the kinetic energy at takeoff. The energy consumed by drag is

$$E_{\text{drag}} \sim \underbrace{\rho v^2 A}_{F_{\text{drag}}} \times h.$$

The ratio of these energies measures the importance of drag. The ratio is

$$\frac{E_{\text{drag}}}{E_{\text{required}}} \sim \frac{\rho v^2 A h}{m v^2} = \frac{\rho A h}{m}.$$

Since  $A$  is the cross-sectional area of the animal,  $Ah$  is the volume of air that it sweeps out in the jump, and  $\rho Ah$  is the mass of air swept out in the jump. So the relative importance of drag has a physical interpretation as a ratio of the mass of air displaced to the mass of the animal.

To find how this ratio depends on animal size, rewrite it in terms of the animal's side length  $l$ . In terms of side length,  $A \sim l^2$  and  $m \propto l^3$ . What about the jump height  $h$ ? The simplest analysis predicts that all animals have the same jump height, so  $h \propto l^0$ . Therefore the numerator  $\rho Ah$  is  $\propto l^1$ , the denominator  $m$  is  $\propto l^3$ , and

$$\frac{E_{\text{drag}}}{E_{\text{required}}} \propto \frac{l^2}{l^3} = l^{-1}.$$

So, small animals have a large ratio, meaning that drag affects the jumps of small animals more than it affects the jumps of large animals. The missing constant of proportionality means that we cannot say at what size an animal becomes 'small' for the purposes of drag. So the calculation so far cannot tell us whether fleas are included among the small animals.

The jump data, however, substitutes for the missing constant of proportionality. The ratio is

$$\frac{E_{\text{drag}}}{E_{\text{required}}} \sim \frac{\rho A h}{m} \sim \frac{\rho l^2 h}{\rho_{\text{animal}} l^3}.$$

It simplifies to

$$\frac{E_{\text{drag}}}{E_{\text{required}}} \sim \frac{\rho}{\rho_{\text{animal}}} \frac{h}{l}.$$

As a quick check, verify that the dimensions match. The left side is a ratio of energies, so it is dimensionless. The right side is the product of two dimensionless ratios, so it is also dimensionless. The dimensions match.

Now put in numbers. A density of air is  $\rho \sim 1 \text{ kg m}^{-3}$ . The density of an animal is roughly the density of water, so  $\rho_{\text{animal}} \sim 10^3 \text{ kg m}^{-3}$ . The typical jump height – which is where the data substitutes for the constant

of proportionality – is 60 cm or roughly 1 m. A flea's length is about 1 mm or  $l \sim 10^{-3}$  m. So

$$\frac{E_{\text{drag}}}{E_{\text{required}}} \sim \frac{1 \text{ kg m}^{-3}}{10^3 \text{ kg m}^{-3}} \frac{1 \text{ m}}{10^{-3} \text{ m}} \sim 1.$$

The ratio being unity means that if a flea would jump to 60 cm, overcoming drag would require roughly as much as energy as would the jump itself in vacuum.

Drag provides a plausible explanation for why fleas do not jump as high as the typical height to which larger animals jump.

### 5.5.2 Swimming

The last section's analysis of cycling helps predict the world-record speed for swimming. The last section showed that

$$v_{\text{max}} \sim \left( \frac{P_{\text{athlete}}}{\rho A} \right)^{1/3}.$$

To evaluate the maximum speed for swimming, one could put in a new  $\rho$  and  $A$  directly into that formula. However, that method replicates the work of multiplying, dividing, and cube-rooting the various values.

Instead it is instructive to scale the numerical result for cycling by looking at how the maximum speed depends on the parameters of the situation. In other words, I'll use the formula for  $v_{\text{max}}$  to work out the ratio  $v_{\text{swimmer}}/v_{\text{cyclist}}$ , and then use that ratio along with  $v_{\text{cyclist}}$  to work out  $v_{\text{swimmer}}$ .

The speed  $v_{\text{max}}$  is

$$v_{\text{max}} \sim \left( \frac{P_{\text{athlete}}}{\rho A} \right)^{1/3}.$$

So the ratio of swimming and cycling speeds is

$$\frac{v_{\text{swimmer}}}{v_{\text{cyclist}}} \sim \left( \frac{P_{\text{swimmer}}}{P_{\text{cyclist}}} \right)^{1/3} \times \left( \frac{\rho_{\text{swimmer}}}{\rho_{\text{cyclist}}} \right)^{-1/3} \times \left( \frac{A_{\text{swimmer}}}{A_{\text{cyclist}}} \right)^{-1/3}.$$

Estimate each factor in turn. The first factor accounts for the relative athletic prowess of swimmers and cyclists. Let's assume that they generate equal amounts of power; then the first factor is unity. The second factor accounts for the differing density of the mediums in which each athlete moves. Roughly, water is 1000 times denser than air. So the second factor