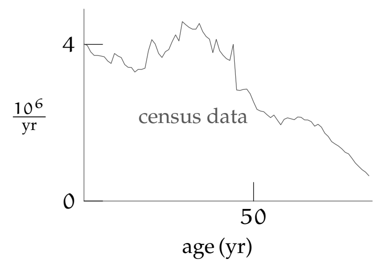
4 Discretization

Discretization, the next technique, is the opposite extreme to calculus. Calculus was invented to analyze changing processes such as planetary orbits or, as a one-dimensional illustration, the distance a ball free falls during a time t. The simple computation, distance is velocity times time, fails because the velocity is not constant. Therefore the fundamental idea of calculus: Over short time intervals, the velocity is almost constant, allowing the simple distance computation to be used.

The shorter the intervals, the smaller the error. Discretization turns this noble goal on its head: Lump all processes into at most a few fat rectangles. At the cost of larger errors, calculations simplify drastically.

4.1 How many babies?

The first example is to estimate the number of babies in the United States. To define the problem, let's call a child a baby if it is less than two years old. The most accurate estimate of their numbers would come from US census data. From the data, make a graph showing the number of people with a given age. Then integrate the curve over the range $t = 0 \ldots 2$ years.



Problem 4.1 Dimensions of the vertical axis
Why is the vertical axis labeled in units of people per year rather than in units simply of people? Equivalently, why does the vertical axis have dimensions of T^{-1}?

This method has two problems. First, it depends on the huge statistical resources of the US Census Bureau. A method that requires such a deus ex machina is not generalizable or usable on a desert island. Second, even with all that data, the method requires integrating a curve with no analytic form, so the integration must be done numerically. That requirement makes the method specific to this problem. Mathematics, however, is about generality and patterns. Surely a method exists with potential to transfer to other problems?

The mention of calculus suggests, to a sufficiently ornery mind, its opposite: discretization. Rather than integrating the population curve exactly – a difficult task because of its fluctuations – replace it by a single rectangle.

▶ *What are the dimensions of this rectangle?*

The rectangle's width is a time, and a natural population-related time is life expectancy. So take $\tau \sim 80$ years as its width. In this discretized model, the population curve is flat in the range $t = 0..80$ yr, so all people live happily, then die abruptly on their 80th birthday. The height does not have such an obvious natural value. However, we know the rectangle's area: It is the population of the United States, roughly $3 \cdot 10^8$ in 2008. Therefore, the rectangle's height is

$$\text{height} \sim \frac{\text{area}}{\text{width}} \sim \frac{3 \cdot 10^8}{75\,\text{yr}},$$

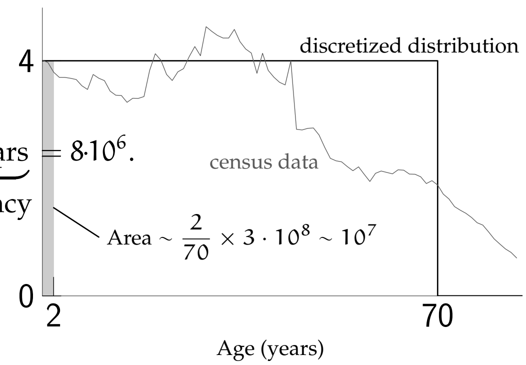
▶ *Why did the life expectancy change from 80 to 75 years?*

Fudging the life expectancy simplifies the mental calculations: The new number 75 divides into 3 and 300 more easily than 80 does. The resulting inaccuracy is no worse than in replacing a varying population curve with a rectangle. With luck, the numerical error may compensate for the rectangle-replacement error. With the numerical fudge, the height is

$$\text{height} \sim 4 \cdot 10^6\,\text{yr}^{-1}.$$

Integrating a rectangle of that height over the the range $t = 0 \ldots 2\,\text{yr}$ gives:

$$N_{\text{babies}} \sim \underbrace{4 \cdot 10^6\,\text{yr}^{-1}}_{\text{height}} \times \underbrace{2\,\text{years}}_{\text{infancy}} = 8 \cdot 10^6.$$



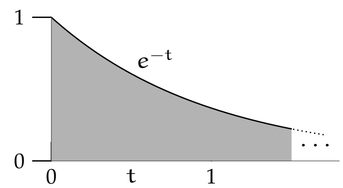
The true number is almost exactly the same as the preceding estimate. As often happens when making approximations, the two errors canceled.

Problem 4.2 Landfill volume
Estimate the landfill volume used each year by disposable diapers (nappies).

Problem 4.3 Cost
Estimate the annual revenue of the US diaper industry.

4.2 Simple integrations

In the number-of-babies example, discretization helped integrate an unknown function (or, rather, a function that required a lot of work to determine). Integration is a difficult operation, so discretization can be useful even with known functions.



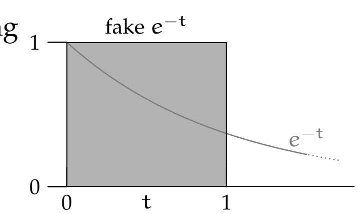
Consider the following integral:

$$\int_0^\infty e^{-t}\, dt.$$

Instead of dividing the area into thin vertical rectangles – the calculus method – replace it with one rectangle. For its height, a natural choice is the maximum height of e^{-t}, namely 1.

Its width is harder to choose. If the rectangle is too wide, it overestimates the area under the curve, which lies under the rectangle. If is too narrow, it underestimates the area by excluding too large a region from the rectangle. In the first case, the curve has fallen too much by the time it escapes the rectangle; in the second case, the curve has not fallen enough. The happy medium is to require that the curve has fallen 'significantly' when it leaves the rectangle. With luck, the overestimate in area from using the curve's maximum height as the rectangle's height will compensate for the underestimate in neglecting the region outside the rectangle.

A reasonable criterion for significance is falling by a factor of 2. This change happens when t increases by one half-life or $\ln 2$. An alternative criterion is less familiar but it compensates with simplicity: falling by a factor of e. With $f(t) = e^{-t}$, this change happens when t goes from t to $t+1$. The 'fall by a factor of e' criterion makes the rectangle's width 1. The resulting rectangle is a unit square, and its area exactly matches the integral:



$$\int_0^\infty e^{-t}\, dt = 1.$$

Problem 4.4
Discretize to find

$$\int_0^\infty e^{-at}\,dt.$$

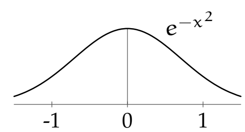
Check your answer using dimensions and easy cases.

Problem 4.5 Cone free-fall distance
For the falling cones of Section 3.4, the analysis computed only the terminal velocity, and the home experiment involved dropping the cones from a height of 1 or 2 m. Estimate how far a cone falls before it reaches a significant fraction of its terminal velocity. Is it a significant fraction of the fall height of 1 or 2 m?

4.3 Full width at half maximum

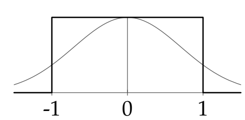
For the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$



Section 3.1.2 explained the polar-coordinates trick to show that it is $\sqrt{\pi}$. That particular trick works only for an infinite integration range (either $-\infty \ldots \infty$ or $0 \ldots \infty$). Otherwise, the integral cannot be evaluated in closed form. In short, the infinite-range Gaussian integral is doable with calculus – but only barely. Discretization provides a less accurate but general-purpose alternative.

▶ *Use the recipe of Section 4.2 to estimate the Gaussian integral.*



As before, replace the area under the curve with a single rectangle. What are its height and width? The general recipe is to choose as the height the maximum of the function and to choose as the width the distance until the function falls significantly. In the exponential-decay example from Section 4.2, a significant change meant a factor of e. The maximum of e^{-x^2} is at $x = 0$ when $e^{-x^2} = 1$, so the approximating rectangle has unit height. The function falls by a factor of e at $x = \pm 1$, so the approximating rectangle extends over the range $x = -1 \ldots 1$. The rectangle therefore has width 2 and area 2. This estimate is quite accurate. The exact area is $\sqrt{\pi} \approx 1.77$, so the error is roughly 13%: a reasonable return for a one-line derivation.

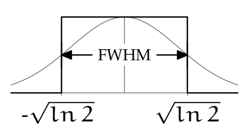
Another recipe, worth knowing because it can be more accurate, arose in the ancient days of spectroscopy. Spectroscopes measure the wavelengths or frequencies at which a molecule absorbs radiation and measures the corresponding absorption strengths. These data provided an early probe into the structure of atoms and molecules, and was essential to the development of quantum theory [19].

A similar investigation occurs in today's particle accelerators such as SLAC in California and CERN and in Geneva. Particles, perhaps protons and neutrons, collide at high energies, showering fragments that carry information about the structure of the original particles. Analogously, to understand how a finely engineered wristwatch works, hammer it and see what the flying shards and springs reveal.

The spectroscope, fortunately, was a less destructive tool. A chart recorder plotted the absorption strength as the spectroscope swept through the wavelength range. The area of the peaks was the important datum, and whole books such as [20] are filled with those measurements. Almost a half century before digital chart recorders and numerical integration , how did one compute these areas?

The recipe was the FWHM: the full width at half maximum. The FWHM approximation replaces the peak with a rectangle whose height is the peak height and whose width is the 'full width at half the maximum':

1. **M**. Find the **m**aximum value (the peak value).
2. **HM**. Find one-half of the maximum value, which is the **h**alf **m**aximum.
3. **FWHM**. Find the two wavelengths – above and below the peak – where the absorption has fallen to one-half of the maximum value. The **f**ull **w**idth is the difference between these two wavelengths.



Try this recipe on the Gaussian integral and compare the estimate with the estimate from the recipe of finding where the function changed by a factor of e. The Gaussian has maximum height 1 at $x = 0$. The half maximum is 1/2, which happens when $x = \pm\sqrt{\ln 2}$. The full width is then $2\sqrt{\ln 2}$, and the area of the rectangle – which estimates the original integral – is $2\sqrt{\ln 2}$. Here, side by side, are the estimate and the exact integral:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \begin{cases} \sqrt{\pi} = 1.7724\ldots & \text{(exact)},\\ 2\sqrt{\ln 2} = 1.6651\ldots & \text{(estimate)}.\end{cases}$$

The FWHM estimate is accurate to 6%, one-half the error of the previous recipe. It too is better than one has a right to expect for such little work.

Problem 4.6 Trying the FWHM heuristic

Estimate these integrals using the FWHM heuristic to choose the rectangle. How accurate is each estimate?

a. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$. [Exact answer: π.]

b. $\int_{-\infty}^{\infty} e^{-x^4}\,dx$. [Exact answer: $\Gamma(1/4)/2 \approx 1.813$.]

4.4 Stirling's approximation

The FWHM recipe accurately estimates one of the most useful functions in applied mathematics, the factorial function:

$$n! \equiv n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

Stirling's approximation is

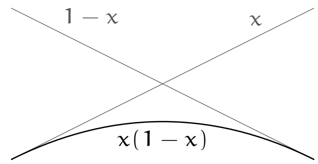
$$n! \approx n^n e^{-n} \sqrt{2\pi n}.$$

There are many ways to derive this approximation. Here we discretize an integral whose value is $n!$:

$$n! = \int_0^{\infty} t^n e^{-t}\,dt.$$

Problem 4.7 Checking the integral
Use integration by parts to show that

$$\int_0^{\infty} t^n e^{-t}\,dt = n!$$



To approximate the integral with a rectangle, first sketch the integrand $t^n e^{-t}$. It is the product of the increasing function t^n with the decreasing function e^{-t}. Such products almost always have a maximum. A familiar example is the product of the increasing function x with the decreasing function $1-x$ (over the range $x \in [0,1]$ where both functions are positive). The product $x(1-x)$ rises from zero and then falls back to zero, with a peak at $x = 1/2$.

To check that the product $t^n e^{-t}$ has a peak, look at its behavior in two extreme cases: the short run $t = 0$ and the long run $t \to \infty$. When $t = 0$, the exponential is 1, but the polynomial factor t^n multiplies it by zero. When $t \to \infty$, the polynomial factor t^n pushes the product toward infinity while the exponential factor e^{-t} pushes it toward zero. Who wins the struggle?

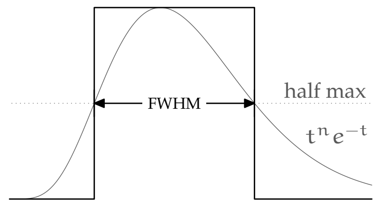
The short answer is that an exponential beats any polynomial. To see the struggle in slow motion, compare e^t with t^n as $t \to \infty$. The Taylor series for e^t contains all powers of t, so it is an infinite-degree polynomial. So e^t/t^n contains positive powers of t, no matter how large n is. Therefore, e^t/t^n goes to infinity when t gets sufficiently large. Similarly, its reciprocal $t^n e^{-t}$ goes to zero as $t \to \infty$.

Being zero at the extremes $t = \infty$ and $t = 0$, the product $t^n e^{-t}$ has a peak (unless it is zero everywhere). In fact, it has exactly one peak. (How can you show that?)

Problem 4.8 Effect of n on peak location
As n grows, does the peak of $t^n e^{-t}$ move left, stay in the same location, or move right? Give an intuitive argument without using calculus.

▶ *What is the FWHM rectangle that approximates this peak?*

With $t^n e^{-t}$ having a peak, the FWHM recipe can approximate its area. The recipe requires finding the height (the maximum of the function) and the width (the FWHM) of the approximating rectangle. To find these parameters, slurp the t^n into the exponent:



$$t^n e^{-t} = e^{n \ln t} e^{-t} = e^{n \ln t - t}.$$

To maximize $t^n e^{-t}$, maximize its exponent $f(t) \equiv n \ln t - t$ by setting $f'(t_{\text{peak}}) = 0$:

$$f'(t_{\text{peak}}) = \frac{n}{t_{\text{peak}}} - 1 = 0.$$

Therefore $t_{\text{peak}} = n$.

The height of the peak is one parameter of the approximating rectangle. At the peak, $f(t)$ is $f(n) = n \ln n - n$, so the original integrand, which is $e^{f(t)}$, is

$$e^{f(t_{\text{peak}})} = e^{f(n)} = e^{n \ln n - n} = \frac{n^n}{e^n} = \left(\frac{n}{e}\right)^n.$$

To find the width, look at how $f(t)$ behaves near the peak $t = n$ by writing it as a Taylor series around the peak:

$$f(t) = f(n) + f'(n)(t-n) + \frac{1}{2} f''(n)(t-n)^2 + \cdots.$$

The first derivative is zero at $t = n$ because at $t = n$ is a maximum; so the second term in the Taylor series vanishes. To evaluate the third term, compute $f''(n)$:

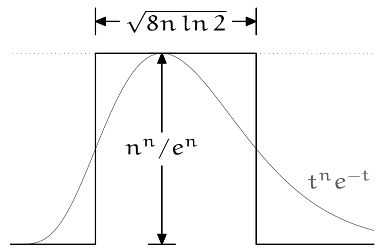
$$f''(n) = -\frac{n}{t^2} = -\frac{1}{n}.$$

So

$$f(t) = \underbrace{n \ln n - n}_{f(n)} + \frac{1}{2} \times \underbrace{\left(-\frac{1}{n}\right)}_{f''(n)} (t-n)^2 + \cdots.$$

The first term gives the height of the peak – which we already computed. The second term says how the height falls as t moves away from n. The result is an approximation for the integrand:

$$e^{f(t)} = \left(\frac{n}{e}\right)^n e^{-(t-n)^2/2n}.$$



The first factor is a constant, the peak height. The second factor is the familiar Gaussian. This one is centered at $t = n$ and contains $1/2n$ in the exponent but otherwise it's just a Gaussian. It falls by a factor of 2 when $(t-n)^2/2n = \ln 2$, which is when

$$t_\pm = n \pm \sqrt{2n \ln 2}.$$

The FWHM is $t_+ - t_-$, which is $\sqrt{8n \ln 2}$. The approximate area under $e^{f(t)}$, which is $n!$, is then

$$n! \approx \left(\frac{n}{e}\right)^n \times \sqrt{8n \ln 2}.$$

This approximation reproduces the most important factors of Stirling's approximation: the n^n in the numerator and the e^n in the denominator. Stirling's approximation contains $\sqrt{2\pi}$ instead of $\sqrt{8 \ln 2}$ – a change of only 6%.

Problem 4.9 Coincidence?
The FWHM approximation for the area under a Gaussian (Section 4.3) was also accurate to 6%. Coincidence?

Problem 4.10 More accurate constant factor
Where does the more accurate constant factor of $\sqrt{2\pi}$ come from?

4.5 Pendulum period

Is it coincidence that g, in units of meters per second squared, is 9.81, very close to $\pi^2 \approx 9.87$? Their proximity suggests a connection. Indeed, they are connected through the original definition of the meter. It was proposed by the the Dutch scientist and engineer Christian Huygens (science and engineering were not separated in the 17th century) – called 'the most ingenious watchmaker of all time' by the great physicist Arnold Sommerfeld