

13 Small Gain Theorem

This lecture discusses statement, application, and generalizations of the Small Gain Theorem for systems.

13.1 Basic Small Gain Theorem

The basic formulation of the Small Gain Theorem concerns an *interconnection* of two systems as shown on Figure 13.1, where $\mathcal{G}$ and Δ are two i/o system models: $\mathcal{G}$ has its input w and output v partitioned according to

$$w(t) = \begin{bmatrix} w_0(t) \\ w_1(t) \end{bmatrix}, \quad v(t) = \begin{bmatrix} v_0(t) \\ v_1(t) \end{bmatrix}, \quad (13.1)$$

and Δ has input v_1 and output w_1 .

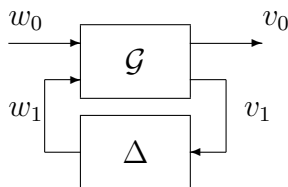


Figure 13.1: Small Gain Theorem Setup

The block diagram of Figure 13.1 defines a new system $\mathcal{G}_\Delta$, sometimes denoted as $\mathcal{G} \star \Delta$, as the set of all pairs (w_0, v_0) for which there exists $(v_1, w_1) \in \Delta$ such that $(w, v) \in \mathcal{G}$ for w, v satisfying (13.1). Note that $\mathcal{G}_\Delta$ is not necessarily an i/o model.

The statement of the Small Gain Theorem below provides a bound on the L2 gain of system $\mathcal{G}_\Delta$, defined as the maximal lower bound of those $\gamma > 0$ for which

$$\inf_{T>0} \int_0^T \{\gamma^2 |w_0(t)|^2 - |v_0(t)|^2\} dt > -\infty \quad (13.2)$$

for all $(w_0, v_0) \in \mathcal{G}_\Delta$.

Theorem 13.1 *If L2 gain of $\mathcal{G}$ is not larger than 1 and L2 gain of Δ is smaller than 1 then L2 gain of $\mathcal{G}_\Delta$ is not larger than 1.*

Proof. Since L2 gain of Δ is smaller than 1, there exists $\gamma_0 \in (0, 1)$ such that for every $\gamma \in (\gamma_0, 1)$ i/o pair $(v_1, w_1) \in \Delta$ there exists a constant $C_1 \in \mathbb{R}$ such that the inequality

$$\int_0^T \{\gamma^2 |v_1(t)|^2 - |w_1(t)|^2\} dt \geq C_1 \quad (13.3)$$

is satisfied for all $T > 0$. Since L2 gain of $\mathcal{G}$ is not larger than 1, for every i/o pair

$$\left(\begin{bmatrix} w_0 \\ w_1 \end{bmatrix}, \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right) \in \mathcal{G}$$

there exists a constant $C_2 \in \mathbb{R}$ such that the inequality

$$\int_0^T \{\gamma^{-2}(|w_0(t)|^2 + |w_1(t)|^2) - |v_0(t)|^2 - |v_1(t)|^2\} dt \geq C_2 \quad (13.4)$$

is satisfied for all $T > 0$. Multiplying (13.3) by γ^{-2} and adding (13.4) yields

$$\int_0^T \{\gamma^{-2}|w_0(t)|^2 - |v_0(t)|^2\} dt \geq \gamma^{-2}C_1 + C_2,$$

which means that L2 gain of $\mathcal{G}_\Delta$ is not larger than γ^{-2} . Since γ can be made arbitrarily close to 1, L2 gain of $\mathcal{G}_\Delta$ is not larger than 1. ■

The interconnection in Figure 13.1 is called *well posed* if $\mathcal{G}_\Delta$ is an input-output model, i.e. if every input w_0 corresponds to at least one output. Since well-posedness of an interconnection is an important abstraction serving as a necessary condition of “good behavior” of feedback systems, it is important to understand that Theorem 13.1 does not claim well-posedness of $\mathcal{G}_\Delta$.

Example 13.1 The nonlinear system Δ mapping $v_1 \in \mathcal{L}$ to

$$w_1(t) = \begin{cases} 2v_1(t), & \int_0^t |v_1(\tau)|^2 d\tau < 1, \\ 0, & \text{otherwise,} \end{cases}$$

has zero L2 gain. The LTI system $\mathcal{G}$ mapping $w = [w_0; w_1]$ to $v = [v_0; v_1]$, where $v_0 = v_1 = 0.5(w_0 + w_1)$, has L2 gain 1. Nevertheless the interconnection in Figure 13.1 is not well posed for these $\mathcal{G}, \Delta$ because every pair $(w_0, v_0) \in \mathcal{G}_\Delta$ will satisfy the condition $w_0(t) = 0$ for all t such that

$$\int_0^t |w_1(\tau)|^2 d\tau < 4.$$

Let $G : \mathbb{R}^{d+m} \mapsto \mathbb{R}^{k+n}$ and $\delta : \mathbb{R}^n \mapsto \mathbb{R}^m$ be two functions with gain not larger than 1. Let $G_\Delta : \mathbb{R}^d \mapsto \mathbb{R}^k$ be a function such that for every $f_0 \in \mathbb{R}^d$ there exist $y_1 \in \mathbb{R}^n$ satisfying the equation

$$\begin{bmatrix} G_\Delta(f_0) \\ y_1 \end{bmatrix} = G \left(\begin{bmatrix} f_0 \\ \Delta(y_1) \end{bmatrix} \right).$$

The “memoryless” version of the small gain theorem stated in one of the first lectures claims that the gain of $G_\Delta : \mathbb{R}^d \mapsto \mathbb{R}^k$ will be not larger than 1. In contrast, the very similar Theorem 13.1 requires Δ to have L2 gain strictly less than 1. This is for a good reason, as relaxing the constraint would make the conclusion invalid.

Example 13.2 Let G be the memoryless linear system mapping input $w = [w_0; w_1]$ to $v = [v_0; v_1]$ according to

$$\begin{bmatrix} v_0(t) \\ v_1(t) \end{bmatrix} = \begin{bmatrix} (1+t)^{-1/2}w_1(t) \\ w_1(t) \end{bmatrix}.$$

Let Δ be the first order LTI system mapping input v_1 to output w_1 according to

$$w_1(t) = x(t), \quad \dot{x}(t) = -x(t) + v_1(t).$$

The L2 gain of both systems equals 1. However, while the interconnection is well posed, the resulting system $\mathcal{G}_\Delta$ contains i/o pair $w_0(t) \equiv 0$, $v_0(t) = (1+t)^{-1/2}$. Since this v_0 has infinite energy, L2 gain of $\mathcal{G}_\Delta$ equals $+\infty$.

13.1.1 Example: Pure Delay Robustness of Servo Systems

Consider the task of analyzing stability of the servo feedback system shown on Figure 13.2, where P (“plant”) and K (“controller”) are given transfer functions, $\tau > 0$ is the pure delay in the open loop, and signals q , r , u , and f represent controlled output, reference input, control action, and plant disturbance respectively.

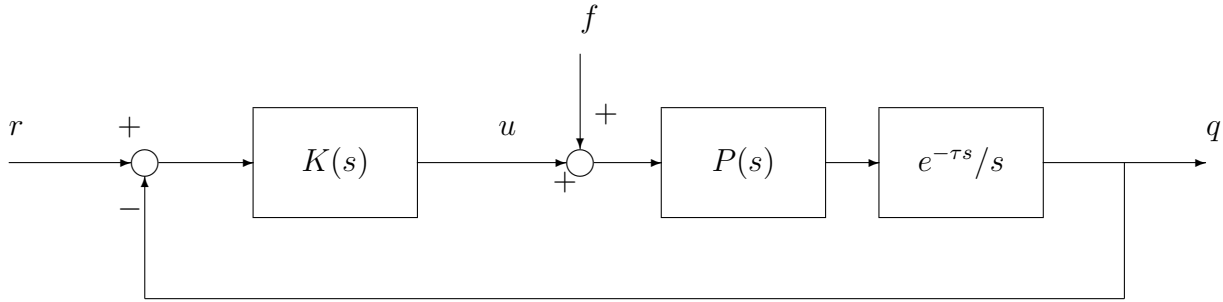


Figure 13.2: Servo System With Delay

Assuming the product $P(s)K(s)/s$ has no unstable zero/pole cancellation, stability of the feedback system is equivalent to negativity of the real part of all solutions $s \in \mathbb{C}$ of the “transcendental” equation

$$se^{\tau s} + P(s)K(s) = 0,$$

which could be relatively difficult to compute, especially when $\tau > 0$ is not given precisely but only an upper bound $\tau < T$. The Small Gain Theorem provides an easy sufficient criterion of stability for this setup.

First note that stability is equivalent to finiteness of the L2 gain for the closed loop system with input $[r; f]$ and output $[q; u]$. Second, note that transfer function $\exp(-\tau s)/s$ can be represented in the form

$$\frac{e^{-\tau s}}{s} = \frac{1}{s} + T\Delta(s), \quad \Delta(s) = \frac{e^{-\tau s} - 1}{Ts},$$

where L2 gain of the LTI system defined by transfer function $\Delta(s)$ is less than 1 for all $\tau \in [0, T)$. Accordingly the block diagram on Figure 13.2 can be re-written in the form shown on Figure 13.3, where $\epsilon > 0$ is a small parameter.

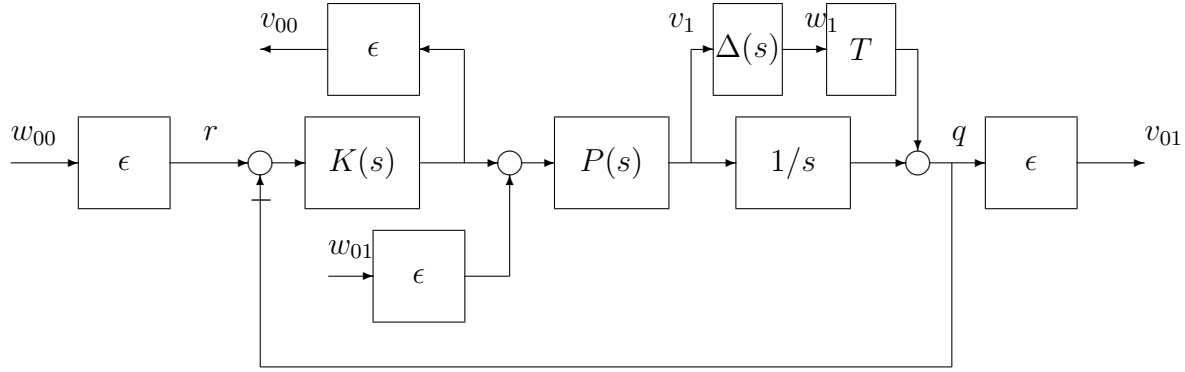


Figure 13.3: Modified Servo System Diagram

The feedback system on Figure 13.3 can be viewed as interconnection of the two LTI systems: the one with input v_1 and output w_1 defined by transfer matrix $\Delta(s)$, and the one with input w and output v , where

$$w = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}, \quad w_0 = \begin{bmatrix} w_{00} \\ w_{01} \end{bmatrix}, \quad v = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}, \quad v_0 = \begin{bmatrix} v_{00} \\ v_{01} \end{bmatrix},$$

with transfer matrix

$$G(s) = \frac{1}{1 + K(s)P(s)/s} \begin{bmatrix} \epsilon^2 K(s) & \epsilon^2 K(s)P(s)/s & -\epsilon TK(s) \\ \epsilon^2 K(s)P(s)/s & \epsilon^2 P(s)/s & \epsilon T \\ \epsilon K(s)P(s) & \epsilon P(s) & TK(s)P(s) \end{bmatrix}.$$

Since L2 gain of the LTI system defined by Δ is less than 1, a sufficient condition of stability is $\|G\|_\infty \leq 1$. When $\epsilon > 0$ is small enough, this will be guaranteed by the inequality

$$T\|H\|_\infty < 1, \quad H(s) = \frac{K(s)P(s)}{1 + K(s)P(s)/s}.$$