Infinite horizon problems

tl;dr: Dynamic programming still works

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6.7920 Reinforcement Learning: Foundations and Methods

Readings

1. DPOC2 1.1-1.2, 1.5, 2.1-2.3

Outline

1. Infinite horizon Markov Decision Processes
2. Value iteration
3. Policy iteration

Outline

Example: The Amazing Goods Company

- *Description.* At each month t, a warehouse contains s_t *items* of a specific goods and the demand for that goods is D (stochastic). At the end of each month the manager of the warehouse can *order* a_t more items from the supplier.

- The cost of maintaining an inventory of s is $h(s)$.
- The cost to order a items is $C(a)$.
- The income for selling q items if $f(q)$.
- If the demand $d \sim D$ is bigger than the available inventory s, customers that cannot be served leave.
- The value of the remaining inventory at the end of the year is $g(s)$.
- Constraint: the store has a maximum capacity C.

Recall: Markov Decision Process

Definition (Markov decision process)

A **Markov decision process** (MDP) is defined as a tuple $M = (S, A, P, r, \gamma)$ where

- S is the *state* space,
- A is the *action* space,

$\Big\}$ often simplified to finite

- $P(s'|s,a)$ is the transition probability with

$$P(s'|s,a) = \mathbb{P}(s_{t+1} = s'|s_t = s, a_t = a)$$

- $r(s,a,s')$ is the immediate reward at state s upon taking action a,

$\Big\}$ sometimes simply $r(s)$

- $\gamma \in [0,1)$ is the discount factor.

Example: The Amazing Goods Company

- Discount: $\gamma = 0.95$. A dollar today is worth more than a dollar tomorrow.

Recall: Markov Decision Process

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Example: The Amazing Goods Company

- Objective: $V(s_0; a_0, \ldots) = \sum_{t=0}^{\infty} \gamma^t r_t$. This corresponds to the cumulative reward, including the value of the remaining inventory at "the end."
- The "horizon" of the problem is 12 (12 months in 1 year), i.e. $\mathrm{r}_{12} = \mathrm{g}(\mathrm{s}_{12})$; $r_t = 0, t > 12$.

Optimization Problem

- Our goal: solve the MDP

Definition (Optimal policy and optimal value function)

The solution to an MDP is an optimal policy π^* satisfying

$$\pi^* \in \arg\max_{\pi \in \Pi} V^{\pi}$$

where Π is some policy set of interest.

The corresponding value function is the optimal value function

$$V^* = V^{\pi^*}$$

State Value Function

- Given a policy $\pi = (d_1, d_2, \dots)$ (deterministic to simplify notation)
  - Infinite time horizon with discount: the problem never terminates but rewards which are closer in time receive a higher importance.

$$V^\pi(s) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t, \pi_t, (h_t)) | s_0 = s; \pi\right]$$

  with discount factor $0 \leq \gamma < 1$:
    - Small = short-term rewards, big = long-term rewards
    - For any $\gamma \in [0, 1)$ the series always converges (for bounded rewards)

  - Used when: there is uncertainty about the deadline and/or an intrinsic definition of discount.

State Value Function

- Given a policy $\pi = (d_1, d_2, \dots)$ (deterministic to simplify notation)
  - Finite time horizon T: deadline at time T, the agent focuses on the sum of the rewards up to T.

$$V^\pi(t,s) = \mathbb{E}\left[\sum_{\tau=t}^{T-1} r\big(s_\tau, \pi_\tau, (h_\tau)\big) + R(s_T) | s_t = s; \pi = (\pi_t, \dots, \pi_T)\right]$$

    where R is a value function for the final state.

  - Used when: there is an intrinsic deadline to meet.

State Value Function

- Given a policy $\pi = (d_1, d_2, \ldots)$ (deterministic to simplify notation)
  - Stochastic shortest path: the problem never terminates but the agent will eventually reach a termination state.

$$V^{\pi}(s) = \mathbb{E}\left[\sum_{t=0}^{T} r\big(s_t, \pi_t, (h_t)\big) | s_0 = s; \pi\right]$$

  where T is the first (random) time when the termination state is achieved.

  - Used when: there is a specific goal condition.

State Value Function

- Given a policy $\pi = (d_1, d_2, \dots)$ (deterministic to simplify notation)
  - Infinite time horizon with average reward: the problem never terminates but the agent only focuses on the (expected) average of the rewards.

$$V^\pi(s) = \lim_{T \to \infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T-1} r\big(s_t, \pi_t, (h_t)\big) | s_0 = s; \pi\right]$$

  - Used when: the system should be constantly controlled over time.

Notice

From now on we mostly work in the *discounted infinite horizon* setting (except part of Lecture 6).

Most results (*not always so smoothly*) extend to other settings.

DP applies to infinite horizon problems, too!

- Finite horizon stochastic and Markov problems (e.g. driving, robotics, games)

$$V_T^*(s_T) = r_T(s_T) \quad \text{for all } s_T \quad \text{(terminal reward)}$$
$$V_t^*(s_t) = \max_{a_t \in A} r_t(s_t, a_t) + \mathbb{E}_{s_{t+1} \sim P(\cdot|s_t, a_t)}[V_{t+1}^*(s_{t+1})]$$
$$\text{for all } s_t, \text{and } t = \{T-1, \ldots, 0\}$$

- From finite to (discounted) infinite horizon problems?

- Infinite horizon stochastic problems (e.g. package delivery over months or years, long-term customer satisfaction, control of autonomous vehicles)

$$V^*(s) = \max_{a \in A} r(s, a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V^*(s')] \quad \text{for all } s$$

Really?

- Infinite horizon stochastic problems (e.g. package delivery over months or years, long-term customer satisfaction, control of autonomous vehicles)

$$V^*(s) = \max_{a \in A} r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V^*(s')] \quad \text{for all } s$$

☞This is called the optimal Bellman equation.

- An optimal policy is such that:

$$\pi^*(s) = \arg\max_{a \in A} \left[r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V^*(s')] \right] \quad \text{for all } s$$

- **Discuss**: Any difficulties with this new algorithm?

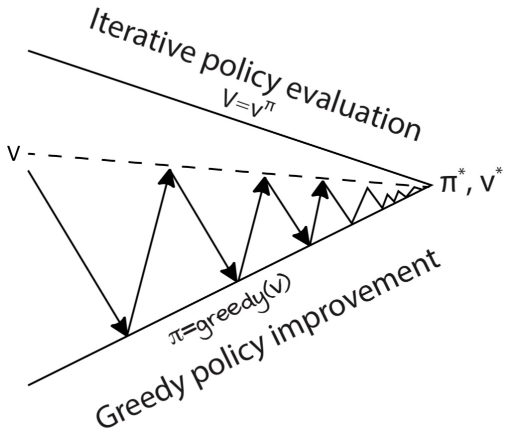
Outline

Value iteration algorithm

1. Let $V_0(s)$ be any function $V_0 : S \to \mathbb{R}$. [Note: not stage 0, but iteration 0.]
2. Apply the principle of optimality so that given V_i at iteration i, we compute
$$V_{i+1}(s) = \mathcal{T}V_i(s) = \max_{a \in A} r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} [V_i(s')] \quad \text{for all } s$$
3. Terminate when V_i stops improving, e.g. when $\max_s |V_{i+1}(s) - V_i(s)|$ is small.
4. Return the greedy policy: $\pi_K(s) = \arg\max_{a \in A} r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V_K(s')$

☞ A key result: $V_i \to V^*$, as $i \to \infty$.

☞ Helpful properties
- Markov process
- Contraction in max-norm
- Cauchy sequences
- Fixed point



Adapted from Morales, Grokking Deep Reinforcement Learning, 2020.

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Definition (Optimal Bellman operator)

For any $W \in \mathbb{R}^{|S|}$, the optimal Bellman operator is defined as
$$\mathcal{T}W(s) = \max_{a \in A} r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} W(s') \quad \text{for all } s$$

☞ Then we can write the algorithm step 2 concisely:
$$V_{i+1}(s) = \mathcal{T}V_i(s) \quad \text{for all } s$$

Key question: Does $V_i \to V^*$?

The Optimal Bellman Equation

Bellman's Principle of Optimality (Bellman (1957)):

"An optimal policy has the property that, whatever the initial state and the initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

The Optimal Bellman Equation

Theorem (Optimal Bellman Equation)

The optimal value function V^* (i.e. $V^* = \max_{\pi} V^{\pi}$) is the solution to the optimal Bellman equation:

$$V^*(s) = \max_{a \in A} \left[r(s,a) + \gamma \sum_{s'} p\,(s'|s,a)\, V^*(s') \right]$$

And any optimal policy is such that:

$$\pi^*(a|s) \geq 0 \Leftrightarrow a \in \arg\max_{a' \in A} \left[r(s,a') + \gamma \sum_{s'} p\,(s'|s,a)\, V^*(s') \right]$$

- Or, for short: $V^* = \mathcal{T} V^*$
- Proof: In a bit. First, let's apply it to see why this is important to prove.

☞ There is always a deterministic policy (see: Puterman, 2005, Chapter 7)

Properties of Bellman Operators

Proposition

1. Contraction in L_∞-norm: for any $W_1, W_2, \in \mathbb{R}^N$

$$\|\mathcal{T}W_1 - \mathcal{T}W_2\|_\infty \leq \gamma \|W_1 - W_2\|_\infty$$

2. Fixed point: V^* is the unique fixed point of $\mathcal{T}$, i.e. $V^* = \mathcal{T}V^*$.

Proof: value iteration

- From contraction property of $\mathcal{T}$, $V_k = \mathcal{T}\mathrm{V}_{\mathrm{k}-1}$, and optimal value function $V^* = \mathcal{T}V^*$:

$$\begin{aligned}
&\|V^* - V_{k+1}\|_\infty \\
&\quad = \|\mathcal{T}V^* - \mathcal{T}V_k\|_\infty && \text{[optimal Bellman eq. and value iteration]} \\
&\quad \leq \gamma \|V^* - V_k\|_\infty && \text{[contraction]} \\
&\quad \leq \gamma^{k+1} \|V^* - V_0\|_\infty && \text{[recursion]} \\
&\quad \to 0 \\
&V_k \to V^* && \text{[fixed point]}
\end{aligned}$$

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2. Fixed point: V^* is the unique fixed point of $\mathcal{T}$, i.e. $V^* = \mathcal{T}V^*$.

Proof: value iteration

- Convergence rate. Let $\epsilon > 0$ and $\|r\|_\infty \le r_{\max}$, then after at most

$$\|V^* - V_k\|_\infty \le \gamma^k \|V^* - V_0\|_\infty < \epsilon \implies K > \frac{\log\left(\frac{r_{\max}}{(1-\gamma)\epsilon}\right)}{\log(\frac{1}{\gamma})}$$

Proof: Contraction of the Bellman Operator

For any $s \in S$

$$|\mathcal{T}W_1(s) - \mathcal{T}W_2(s)|$$

$$= \left| \max_a \left[r(s,a) + \gamma \sum_{s'} p(s'|s,a)\, W_1(s') \right] - \max_{a'} \left[r(s,a') + \gamma \sum_{s'} p(s'|s,a')\, W_2(s') \right] \right|$$

$$\le \max_a \left| \left[r(s,a) + \gamma \sum_{s'} p(s'|s,a)\, W_1(s') \right] - \left[r(s,a) + \gamma \sum_{s'} p(s'|s,a)\, W_2(s') \right] \right|$$

$$= \gamma \max_a \sum_{s'} p(s'|s,a)\, |W_1(s') - W_2(s')|$$

$$\le \gamma \|W_1 - W_2\|_\infty \max_a \sum_{s'} p(s'|s,a) \; = \gamma \|W_1 - W_2\|_\infty$$

$$\max_x f(x) - \max_{x'} g(x') \le \max_x (f(x) - g(x))$$

■

The Optimal Bellman Equation

Theorem (Optimal Bellman Equation)

The optimal value function V^* (i.e. $V^* = \max_\pi V^\pi$) is the solution to the optimal Bellman equation:

$$V^*(s) = \max_{a \in A} \left[r(s,a) + \gamma \sum_{s'} p\left(s' | s, a\right) V^*(s') \right]$$

And any optimal policy is such that:

$$\pi^*(a|s) \geq 0 \Leftrightarrow a \in \arg\max_{a' \in A} \left[r(s,a') + \gamma \sum_{s'} p\left(s' | s, a\right) V^*(s') \right]$$

Or, for short: $V^* = \mathcal{T} V^*$

☞ There is always a deterministic policy (see: Puterman, 2005, Chapter 7)

Proof: The Optimal Bellman Equation

For any policy $\pi = (a, \pi')$ (possibly non-stationary),

$$V^*(s) = \max_{\pi} \mathbb{E}\left[\sum_{t\geq 0}^{\infty} \gamma^t r(s_t, \pi(s_t)) \mid s_0 = s; \pi\right]$$

[value function]

$$= \max_{(a,\pi')} \left[r(s,a) + \gamma \sum_{s'} p(s'|s,a) V^{\pi'}(s') \right]$$

[Markov property & change of "time"]

$$= \max_{a} \left[r(s,a) + \gamma \sum_{s'} p(s'|s,a) \max_{\pi'} V^{\pi'}(s') \right]$$

$$= \max_{a} \left[r(s,a) + \gamma \sum_{s'} p(s'|s,a) V^*(s') \right]$$

[value function]

Proof: Line 2 (also, the Bellman Equation)

For simplicity, consider any stationary policy $\pi = (\pi, \pi, \dots)$,

$$V^{\pi}(s) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t, \pi(s_t)) \,|\, s_0 = s; \pi\right]$$ [value function]

$$= r(s, \pi(s)) + \mathbb{E}\left[\sum_{t=1}^{\infty} \gamma^t r(s_t, \pi(s_t)) \,|\, s_0 = s; \pi\right]$$ [Markov property]

$$= r(s, \pi(s)) + \gamma \sum_{s'} \mathbb{P}(s_1 = s' | s_0 = s; \pi(s_0)) \ \mathbb{E}\left[\sum_{t=1}^{\infty} \gamma^{t-1} r(s_t, \pi(s_t)) \,|\, s_1 = s'; \pi\right]$$

[MDP and change of "time"]

$$= r(s, \pi(s)) + \gamma \sum_{s'} p(s' | s, \pi(s)) \ \mathbb{E}\left[\sum_{t'=0}^{\infty} \gamma^{t'} r(s_{t'}, \pi(s_{t'})) \,|\, s_{0'} = s'; \pi\right]$$

$$= r(s, \pi(s)) + \gamma \sum_{s'} p(s' | s, \pi(s)) \, V^{\pi}(s')$$ [value function]

Proof: Line 3

For the =, we have:

$$\max_{\pi'} \sum_{s'} p(s'|s,a)V^{\pi'}(s') \leq \sum_{s'} p(s'|s,a) \max_{\pi'} V^{\pi'}(s')$$

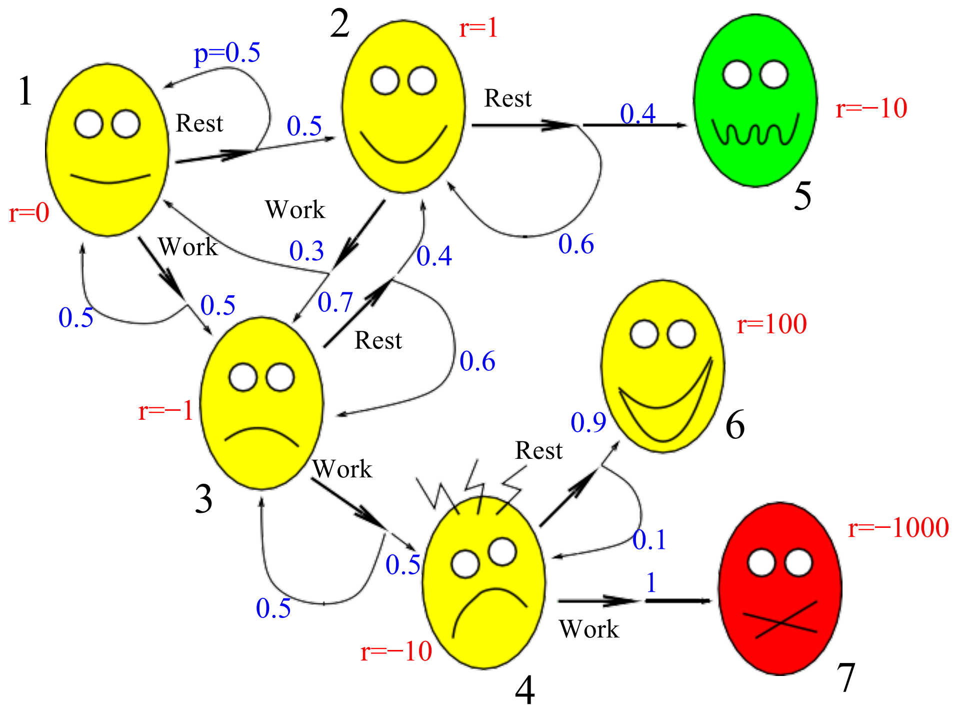
But, let $\bar{\pi}(s') = \operatorname{argmax}_{\pi'} V^{\pi'}(s')$

$$\sum_{s'} p(s'|s,a) \max_{\pi'} V^{\pi'}(s') \leq \sum_{s'} p(s'|s,a)\, V^{\bar{\pi}}(s') \leq \max_{\pi'} \sum_{s'} p(s'|s,a)V^{\pi'}(s')$$

■

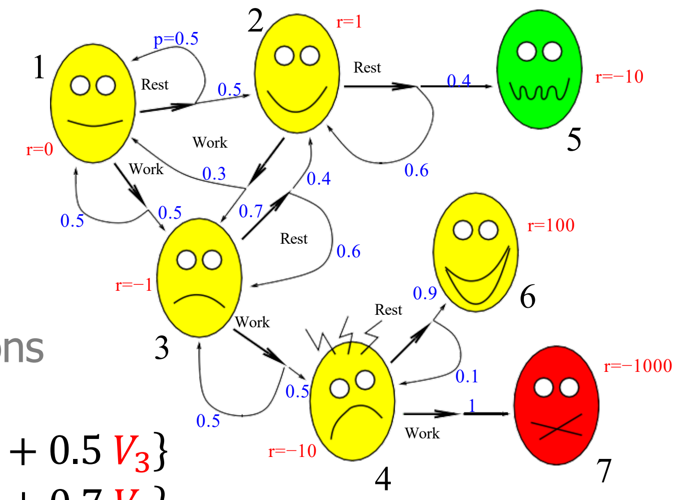
The student dilemma

- *Model*: all the transitions are Markov, states s_5, s_6, s_7 are terminal.
- *Setting*: infinite horizon with terminal states.
- *Objective*: find the policy that maximizes the expected sum of rewards before achieving a terminal state.
- *Notice*: Not a discounted infinite horizon setting. But the Bellman equations hold unchanged.
- **Discuss**: What kind of problem setting is this? (Hint: value function.)



The student dilemma

$$V^*(s) = \max_{a \in A}\left[r(x,a) + \gamma \sum_y p\,(y|x,a)\,V^*(y)\right]$$



System of equations

$$\begin{cases} V_1 = & \max\{0 + 0.5\,V_1 + 0.5\,V_2; 0 + 0.5\,V_1 + 0.5\,V_3\} \\ V_2 = & \max\{0 + 0.4\,V_5 + 0.6\,V_2; 0 + 0.3\,V_1 + 0.7\,V_3\} \\ V_3 = & \max\{-1 + 0.4\,V_2 + 0.6\,V_3; -1 + 0.5\,V_4 + 0.5\,V_3\} \\ V_4 = & \max\{-10 + 0.9\,V_6 + 0.1\,V_4; -10 + V_7\} \\ V_5 = & -10 \\ V_6 = & 100 \\ V_7 = & -1000 \end{cases}$$

Sanity check: Where is the difficulty in solving this system of equations?

System of Equations

The optimal Bellman equation:

$$V^*(s) = \max_{a \in A} \left[r(s,a) + \gamma \sum_{s'} p\left(s' | s, a\right) V^*(s') \right]$$

Is a non-linear system of equations with N unknowns and N non-linear constraints (i.e. the max operator).

Value iteration algorithm

1. Let $V_0(s)$ be any function $V_0 : S \to \mathbb{R}$. [Note: not stage 0, but iteration 0.]
2. Apply the principle of optimality so that given V_i at iteration i, we compute
$$V_{i+1}(s) = \mathcal{T}V_i(s) = \max_{a \in A} r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} [V_i(s')] \quad \text{for all } s$$
3. Terminate when V_i stops improving, e.g. when $\max_s |V_{i+1}(s) - V_i(s)|$ is small.
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☞ A key result: $V_i \to V^*$, as $i \to \infty$.

☞ Helpful properties
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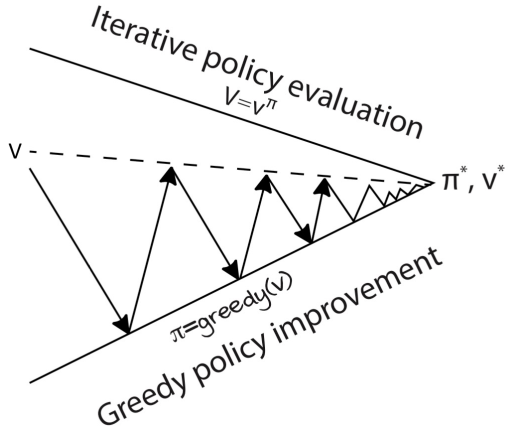


Adapted from Morales, Grokking Deep Reinforcement Learning, 2020.

Value Iteration: the Complexity

Time complexity

- Each iteration takes on the order of S^2A operations.

$$V_{k+1}(s) = \mathcal{T}V_k(s) = \max_{a\in A}\left[r(s,a) + \gamma\sum_{s'} p(s'|s,a)V_k(s')\right]$$

- The computation of the greedy policy takes on the order of S^2A operations.

$$\pi_K(s) \in \arg\max_{a\in A}\left[r(s,a) + \gamma\sum_{s'} p(s'|s,a)V_K(s')\right]$$

- Total time complexity on the order of KS^2A.

Space complexity

- Storing the MDP: dynamics on the order of S^2A and reward on the order of SA.
- Storing the value function and the optimal policy on the order of S.

Value Iteration: Extensions and Implementations

Asynchronous VI:

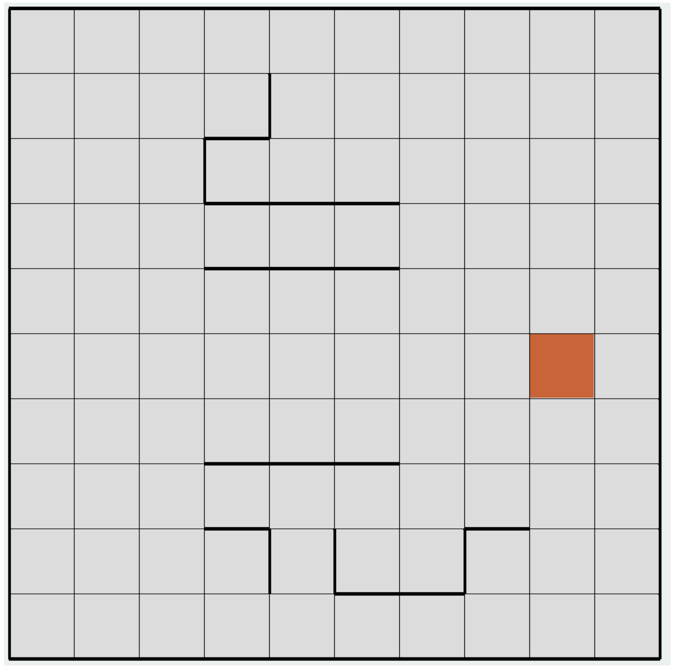
1. Let V_0 be any vector in R^N
2. At each iteration $k = 1, 2, \dots., K$
   - Choose a state s_k
   - Compute $V_{k+1}(s_k) = \mathrm{T}V_k(s_k)$
3. Return the greedy policy

$$\pi_K(s) \in \arg\max_{a \in A} \left[r(s,a) + \gamma \sum_{s'} p(s'|s,a) V_K(s') \right]$$

Comparison

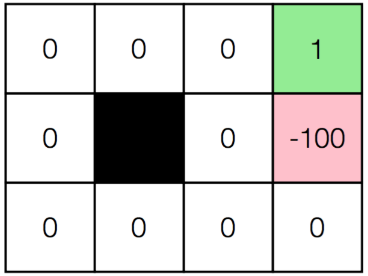
- Reduced time complexity to O(SA)
- Using round-robin, number of iterations increased by at most O(KS) but much smaller in practice if states are properly prioritized
- Convergence guarantees if no starvation

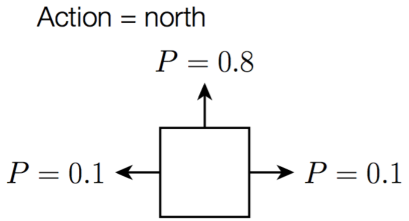
The Grid-World Problem



Example: Winter parking (with ice and potholes)

- Simple grid world with a *goal state* (green, desired parking spot) with reward (+1), a *"bad state"* (red, pothole) with reward (-100), and all other states neural (+0).
- *Omnidirectional vehicle (agent)* can head in any direction. Actions move in the desired direction with probably 0.8, in one of the perpendicular directions with.
- Taking an action that would bump into a wall leaves agent where it is.

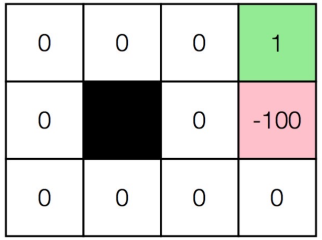




[Source: adapted from Kolter, 2016]

Example: value iteration

Running value iteration with $\gamma = 0.9$



Original reward function

(a)

Recall value iteration algorithm:

$$\mathrm{V}_{\mathrm{i+1}}(s) = \max_{a \in A} r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V_i(s') \quad \text{for all } s$$

Let's arbitrarily initialize V_0 as the reward function, since it can be any function.

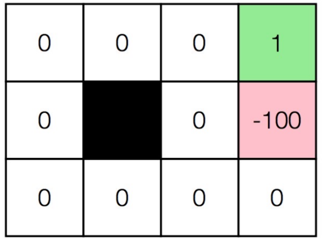
Example update (red state):

$$\mathrm{V}_1(\text{red}) = -100 + \gamma \max\{ \begin{array}{ll} 0.8V_0(\text{green}) + 0.1V_0(\text{red}) + 0, & \text{[up]} \\ 0 + 0.1V_0(\text{red}) + 0, & \text{[down]} \\ 0 + 0.1V_0(\text{green}) + 0, & \text{[left]} \\ 0.8V_0(\text{red}) + 0.1V_0(\text{green}) + 1 \ \} & \text{[right]} \end{array}$$

$= -100 + 0.9(0.1 * 1) = -99.91$ [best: go left]

Example: value iteration

Running value iteration with $\gamma = 0.9$



0	0	0	1
0	■	0	-100
0	0	0	0

Original reward function

(a)

Recall value iteration algorithm:

$$\mathrm{V}_{i+1}(s) = \max_{a \in A} r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V_i(s') \quad \text{for all } s$$

Let's arbitrarily initialize V_0 as the reward function, since it can be any function.

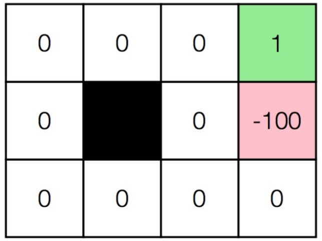
Example update (green state):

$$\begin{aligned} \mathrm{V}_1(\text{green}) = 1 \quad + \gamma \max\{ \; & 0.8V_0(\text{green}) + 0.1V_0(\text{green}), && \text{[up]} \\ & 0.8V_0(\text{red}) + 0.1V_0(\text{green}), && \text{[down]} \\ & 0 + 0.1V_0(\text{green}) + 0.1V_0(\text{red}), && \text{[left]} \\ & 0.8V_0(\text{red}) + 0.1V_0(\text{green}) + 0 \quad \} && \text{[right]} \end{aligned}$$

$= 1 + 0.9(0.9 * 1) = 1.81$ [best: go up]

Example: value iteration

Running value iteration with $\gamma = 0.9$

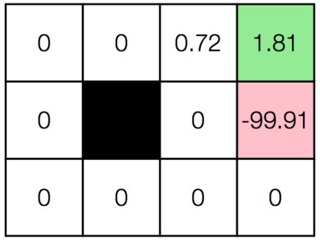


0	0	0	1
0	■	0	-100
0	0	0	0

Original reward function

(a)

Running value iteration with $\gamma = 0.9$



0	0	0.72	1.81
0	■	0	-99.91
0	0	0	0

$\hat{V}$ at one iteration

(b)

Recall value iteration algorithm:

$$V_{i+1}(s) = \max_{a \in A} r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V_i(s') \quad \text{for all } s$$

Let's arbitrarily initialize V_0 as the reward function, since it can be any function.

Need to also do this for all the "unnamed" states, too.

Example: value iteration

Running value iteration with $\gamma = 0.9$

0	0	0	1
0	■	0	-100
0	0	0	0

Original reward function

(a)

Running value iteration with $\gamma = 0.9$

0	0	0.72	1.81
0	■	0	-99.91
0	0	0	0

$\hat{V}$ at one iteration

(b)

Running value iteration with $\gamma = 0.9$

0.809	1.598	2.475	3.745
0.268	■	0.302	-99.59
0	0.034	0.122	0.004

$\hat{V}$ at five iterations

(c)

Running value iteration with $\gamma = 0.9$

2.686	3.527	4.402	5.812
2.021	■	1.095	-98.82
1.390	0.903	0.738	0.123

$\hat{V}$ at 10 iterations

(d)

Running value iteration with $\gamma = 0.9$

5.470	6.313	7.190	8.669
4.802	■	3.347	-96.67
4.161	3.654	3.222	1.526

$\hat{V}$ at 1000 iterations

(e)

Running value iteration with $\gamma = 0.9$

→	→	→	↑
↑	■	←	←
↑	←	←	↓

Resulting policy after 1000 iterations

(f)

Outline

1. Infinite horizon Markov Decision Processes
2. Value iteration
3. **Policy iteration**
   a. Generalized policy iteration
   b. Bellman equation, and properties
   c. Convergence
   d. Geometric interpretations

More generally…

Value iteration:

1. $V_{i+1}(s) = \max_{a \in A} r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V_i\ (s')]$ for all s
2. $\pi_K(s) = \arg\max_{a \in A} r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}\, V_K(s')$

Related Operations:

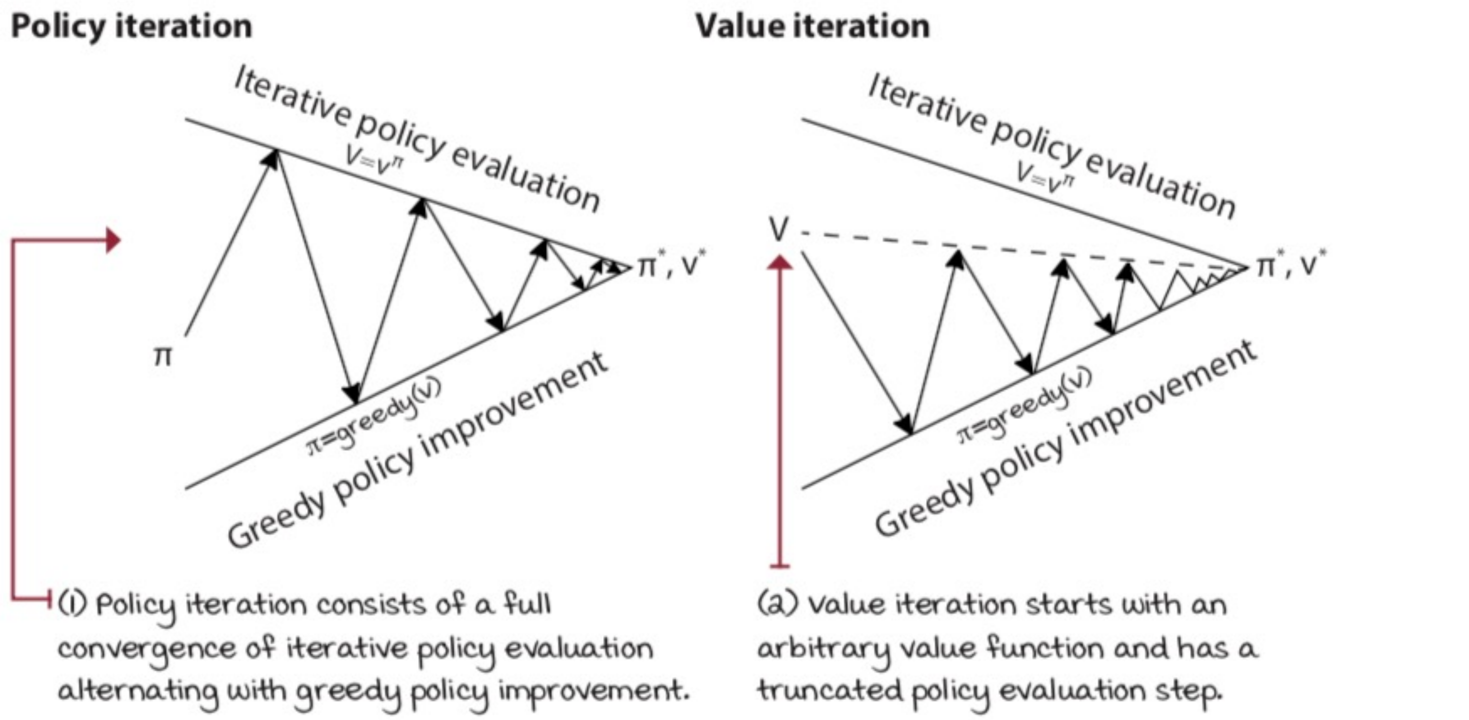
- Policy evaluation: $V_{i+1}(s) = r\big(s, \pi_i(s)\big) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,\pi_i(s))}\big[V_i\ \ (s')\big]$ for all s
- Policy improvement: $\pi_i(s) = \arg\max_{a \in A} r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}\, V_i\ \ (s')$

☞ Generalized Policy Iteration:

- Repeat:
  1. Policy evaluation for N steps
  2. Policy improvement

- Value iteration: $N = 1$; Policy iteration: $N = \infty$

In pictures



(1) Policy iteration consists of a full convergence of iterative policy evaluation alternating with greedy policy improvement.

(2) Value iteration starts with an arbitrary value function and has a truncated policy evaluation step.

Adapted from Morales, Grokking Deep Reinforcement Learning, 2020.

Policy Iteration: the Idea

1. Let π_0 be any stationary policy

2. At each iteration $k = 1, 2, \dots, K$
    - Policy evaluation: given π_k, compute V^{π_k}
    - Policy improvement: compute the greedy policy

$$\pi_{k+1}(s) \in \arg\max_{a \in A} \left[r(s,a) + \gamma \sum_{s'} p(s'|s,a) V^{\pi_k}(s') \right]$$

3. Stop if $V^{\pi_k} = V^{\pi_{k-1}}$

4. Return the last policy π_K

Policy Iteration: the Guarantees

Proposition

The policy iteration algorithm generates a sequence of policies with non-decreasing performance

$$V^{\pi_{k+1}} \geq V^{\pi_k}$$

and it converges to π^* in a finite number of iterations.

The Bellman Equation

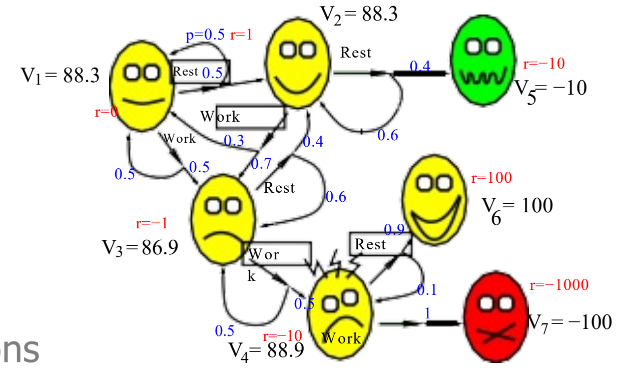
Theorem (Bellman equation)

For any stationary policy $\pi = (\pi, \pi, \dots)$, at any state $s \in S$, the state value function satisfies the Bellman equation:

$$V^{\pi}(s) = r\big(s, \pi(s)\big) + \gamma \sum_{s' \in S} p(s' | s, \pi(s)) V^{\pi}(s')$$

The student dilemma

- **Discuss**: How to solve this system of equations?



$$V^{\pi}(x) = r\big(x, \pi(x)\big) + \gamma \sum_{y} p(y|x, \pi(x)) V^{\pi}(y)$$

System of equations

$$\begin{cases} V_1 = & 0 + 0.5\, V_1 + 0.5\, V_2 \\ V_2 = & 1 + 0.3\, V_1 + 0.7\, V_3 \\ V_3 = & -1 + 0.5\, V_4 + 0.5\, V_3 \\ V_4 = & -10 + 0.9 V_6 + 0.1\, V_4 \\ V_5 = & -10 \\ V_6 = & 100 \\ V_7 = & -1000 \end{cases} \Longrightarrow$$

$$V, R \in \mathbb{R}^7, P^{\pi} \in \mathbb{R}^{7\times 7}$$

$$V = R + PV$$

$$\Downarrow$$

$$V = (I - P)^{-1} R$$

Recap: The Bellman Operators

Notation. w.l.o.g. a discrete state space $|S| = N$ and $V^{\pi} \in \mathbb{R}^N$ (analysis extends to include $N \to \infty$)

Definition

For any $W \in \mathbb{R}^N$, the Bellman operator $\mathrm{T}^{\pi}: \mathbb{R}^N \to \mathbb{R}^N$ is

$$\mathrm{T}^{\pi} W(s) = r\big(s, \pi(s)\big) + \gamma \sum_{s'} p(s'|s, \pi(s)) W(s')$$

And the optimal Bellman operator (or dynamic programming operator) is

$$\mathrm{T} W(s) = \max_{a \in A} \left[r(s,a) + \gamma \sum_{s'} p(s'|s,a) W(s) \right]$$

The Bellman Operators

Proposition

Properties of the Bellman operators

1. Monotonicity: For any $W_1, W_2 \in \mathbb{R}^N$, if $W_1 \leq \mathrm{W}_2$ component-wise, then

$$\mathrm{T}^\pi W_1 \leq \mathrm{T}^\pi W_2$$

$$\mathrm{T} W_1 \leq \mathrm{T} W_2$$

2. Offset: For any scalar $c \in \mathbb{R}$,

$$\mathrm{T}^\pi (W + cI_N) = \mathrm{T}^\pi W + \gamma c I_N$$

$$\mathrm{T}(W + cI_N) = \mathrm{T}W + \gamma c I_N$$

The Bellman Operators

Proposition

3. Contraction in L_∞-norm: For any $W_1, W_2 \in \mathbb{R}^N$

$$\|\mathrm{T}^\pi W_1 - \mathrm{T}^\pi W_2\|_\infty \leq \gamma \|W_1 - W_2\|_\infty$$
$$\|\mathrm{T} W_1 - \mathrm{T} W_2\|_\infty \leq \gamma \|W_1 - W_2\|_\infty$$

4. Fixed point: For any policy π,

V^π is the unique fixed point of T^π

V^* is the unique fixed point of T

- For any $W \in \mathbb{R}^N$ and any stationary policy π

$$\lim_{k \to \infty} (\mathrm{T}^\pi)^k W = V^\pi$$
$$\lim_{k \to \infty} (\mathrm{T})^k W = V^*$$

Policy Iteration: the Idea

1. Let π_0 be any stationary policy
2. At each iteration $k = 1, 2, \dots., K$
   - Policy evaluation: given π_k, compute V^{π_k}
   - Policy improvement: compute the greedy policy

$$\pi_{k+1}(s) \in \arg\max_{a \in A}\left[r(s,a) + \gamma \sum_{s'} p(s'|s,a) V^{\pi_k}(s')\right]$$

3. Stop if $V^{\pi_k} = V^{\pi_{k-1}}$
4. Return the last policy π_K

Policy Iteration: the Guarantees

Proposition

The policy iteration algorithm generates a sequence of policies with non-decreasing performance

$$V^{\pi_{k+1}} \geq V^{\pi_k}$$

and it converges to π^* in a finite number of iterations.

Proof: Policy Iteration

- From the definition of the Bellman operators and the greedy policy π_{k+1}

$$V^{\pi_k} = \mathcal{T}^{\pi_k} V^{\pi_k} \leq \mathcal{T} V^{\pi_k} = \mathcal{T}^{\pi_{k+1}} V^{\pi_k}$$

- and from the monotonicity property of $\mathcal{T}^{\pi_{k+1}}$, it follows that

$$V^{\pi_k} \leq \mathcal{T}^{\pi_{k+1}} V^{\pi_k}$$
$$\mathcal{T}^{\pi_{k+1}} V^{\pi_k} \leq (\mathcal{T}^{\pi_{k+1}})^2 V^{\pi_k}$$
$$\dots$$
$$(\mathcal{T}^{\pi_{k+1}})^{n-1} V^{\pi_k} \leq (\mathcal{T}^{\pi_{k+1}})^n V^{\pi_k}$$
$$\dots$$

- Joining all inequalities in the chain, we obtain

$$V^{\pi_k} \leq \lim_{n\to\infty} (\mathcal{T}^{\pi_{k+1}})^n V^{\pi_k} = V^{\pi_{k+1}}$$

- Then $(V^{\pi_k})_k$ is a non-decreasing sequence.

Policy Iteration: the Guarantees

Since a finite MDP admits a finite number of policies, then the termination condition is eventually met for a specific k.

Thus, proposition holds with an equality and we obtain

$$V^{\pi_k} = \mathcal{T} V^{\pi_k}$$

and $V^{\pi_k} = V^*$ which implies that π_k is an optimal policy.

$\blacksquare$

Policy Iteration: Complexity

Notation. For any policy π the reward vector is $r^\pi(x) = r(x, \pi(x))$ and the transition matrix is $[P^\pi]_{x,y} = p(y|x, \pi(x))$

- Policy Evaluation Step
  - Direct computation: For any policy π compute
$$V^\pi = (I - \gamma P^\pi)^{-1} r^\pi$$
    Complexity: O(S³).
  - Iterative policy evaluation: For any policy π
$$\lim_{n\to\infty} \mathrm{T}^\pi V_0 = V^\pi$$
    Complexity: An ϵ-approximation of V^π requires $O\left(S^2 \frac{\log\left(\frac{1}{\epsilon}\right)}{\log\left(\frac{1}{\gamma}\right)}\right)$ steps.
  - Monte-Carlo simulation: In each state s, simulate n trajectories $\left(\left(s_t^i\right)_{t\geq 0}\right)_{1\leq i\leq n}$ following policy π and compute
$$\hat{V}^\pi(s) \simeq \frac{1}{n}\sum_{i=1}^{n}\sum_{t\geq 0}\gamma^t r\left(s_t^i, \pi\left(s_t^i\right)\right)$$
    Complexity: In each state, the approximation error is $O\left(\frac{r_{\max}}{1-\gamma}\sqrt{\frac{1}{n}}\right)$.

Policy Iteration: Complexity

- Policy Improvement Step
  - Complexity O(S^2A)
- Number of Iterations
  - At most $O\left(\frac{SA}{1-\gamma}\log\left(\frac{1}{1-\gamma}\right)\right)$
  - Other results exist that do not depend on γ

Another application of Bellman operators

Lemma: Performance Loss in Value Iteration

Let V_K be the function computed after K iterations by value iteration, then the greedy policy

$$\pi_K(s) \in \arg\max_{a \in A} \left[r(s,a) + \gamma \sum_{s'} p(s'|s,a) V_K(s') \right]$$

is such that

$$\underbrace{\|V^* - V^{\pi_K}\|_\infty}_{\text{performance loss}} \le \frac{2\gamma}{1-\gamma} \underbrace{\|V^* - V_K\|_\infty}_{\text{approx. error}}$$

Furthermore, there exists $\epsilon > 0$ such that if $\|V^* - V_K\|_\infty \le \epsilon$, then π_K is optimal.

Proof: Performance Loss

- **Note 1**: We drop the K everywhere.

- **Note 2**: π is the greedy policy corresponding to V, and V^π is the value function evaluated with π.

$$\begin{aligned} \|V^* - V^\pi\|_\infty &\le \|\mathrm{T}V^* - \mathrm{T}^\pi V\|_\infty + \|\mathrm{T}^\pi \mathrm{V} - \mathrm{T}^\pi V^\pi\|_\infty \\ &\le \|\mathrm{T}V^* - \mathrm{T}V\|_\infty + \gamma\|\mathrm{V} - V^\pi\|_\infty \\ &\le \gamma\|\mathrm{V}^* - V\|_\infty + \gamma(\|\mathrm{V} - V^*\|_\infty + \|\mathrm{V}^* - V^\pi\|_\infty) \\ &\le \frac{2\gamma}{1-\gamma}\|\mathrm{V}^* - V\|_\infty \end{aligned}$$

■

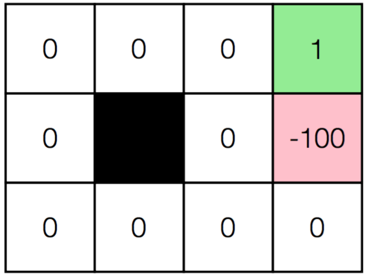
Comparison between Value and Policy Iteration

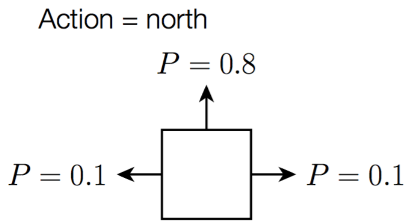
- Value Iteration
  - Pros: each iteration is computationally efficient.
  - Cons: convergence is only asymptotic.

- Policy Iteration
  - Pros: converge in a finite number of iterations (often small in practice).
  - Cons: each iteration requires a full policy evaluation and it might be expensive.

Example: Winter parking (with ice and potholes)

- Simple grid world with a *goal state* (green, desired parking spot) with reward (+1), a *"bad state"* (red, pothole) with reward (-100), and all other states neural (+0).
- *Omnidirectional vehicle (agent)* can head in any direction. Actions move in the desired direction with probably 0.8, in one of the perpendicular directions with.
- Taking an action that would bump into a wall leaves agent where it is.

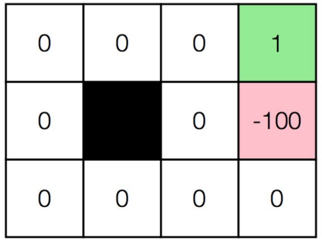




[Source: adapted from Kolter, 2016]

Example: value iteration

Running value iteration with $\gamma = 0.9$

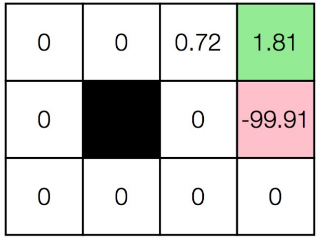


0	0	0	1
0	■	0	-100
0	0	0	0

Original reward function

(a)

Running value iteration with $\gamma = 0.9$

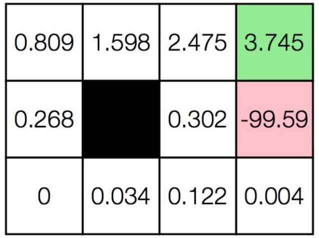


0	0	0.72	1.81
0	■	0	-99.91
0	0	0	0

$\hat{V}$ at one iteration

(b)

Running value iteration with $\gamma = 0.9$

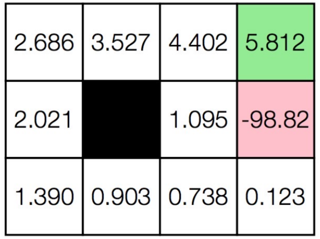


0.809	1.598	2.475	3.745
0.268	■	0.302	-99.59
0	0.034	0.122	0.004

$\hat{V}$ at five iterations

(c)

Running value iteration with $\gamma = 0.9$

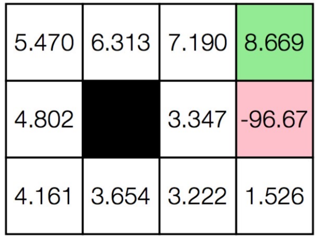


2.686	3.527	4.402	5.812
2.021	■	1.095	-98.82
1.390	0.903	0.738	0.123

$\hat{V}$ at 10 iterations

(d)

Running value iteration with $\gamma = 0.9$

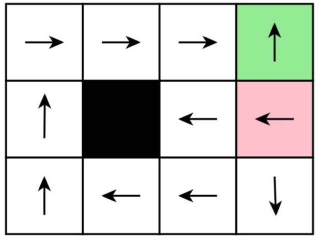


5.470	6.313	7.190	8.669
4.802	■	3.347	-96.67
4.161	3.654	3.222	1.526

$\hat{V}$ at 1000 iterations

(e)

Running value iteration with $\gamma = 0.9$

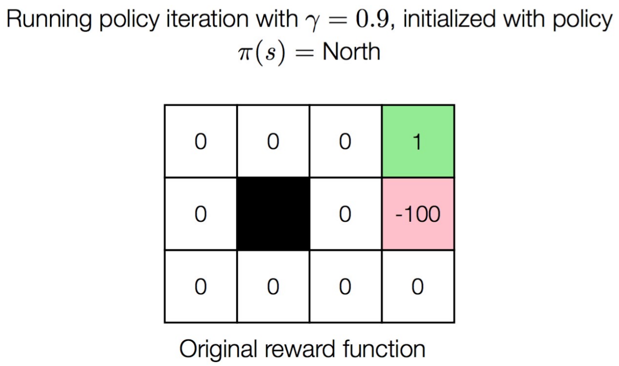


→	→	→	↑
↑	■	←	←
↑	←	←	↓

Resulting policy after 1000 iterations

(f)

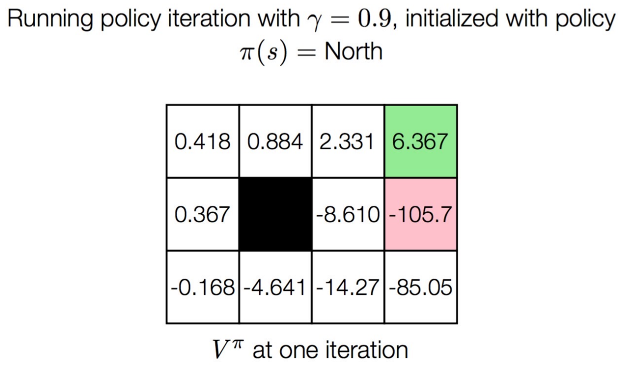
Example: policy iteration


Running policy iteration with $\gamma = 0.9$, initialized with policy $\pi(s) =$ North

0	0	0	1
0	■	0	-100
0	0	0	0

Original reward function

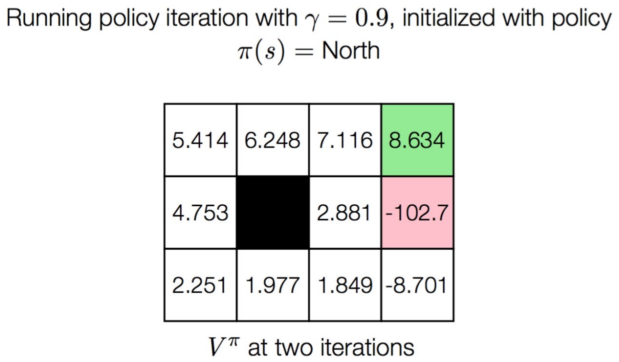
(a)


Running policy iteration with $\gamma = 0.9$, initialized with policy $\pi(s) =$ North

0.418	0.884	2.331	6.367
0.367	■	-8.610	-105.7
-0.168	-4.641	-14.27	-85.05

V^π at one iteration

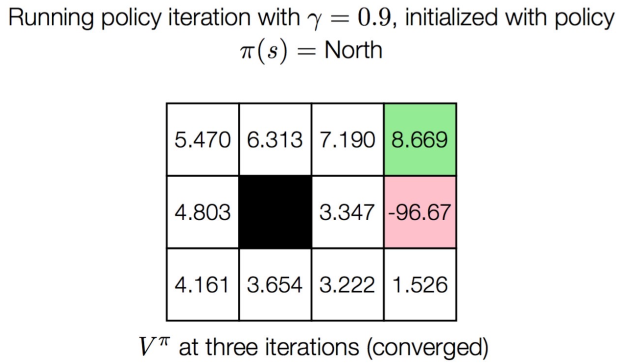
(b)


Running policy iteration with $\gamma = 0.9$, initialized with policy $\pi(s) =$ North

5.414	6.248	7.116	8.634
4.753	■	2.881	-102.7
2.251	1.977	1.849	-8.701

V^π at two iterations

(c)

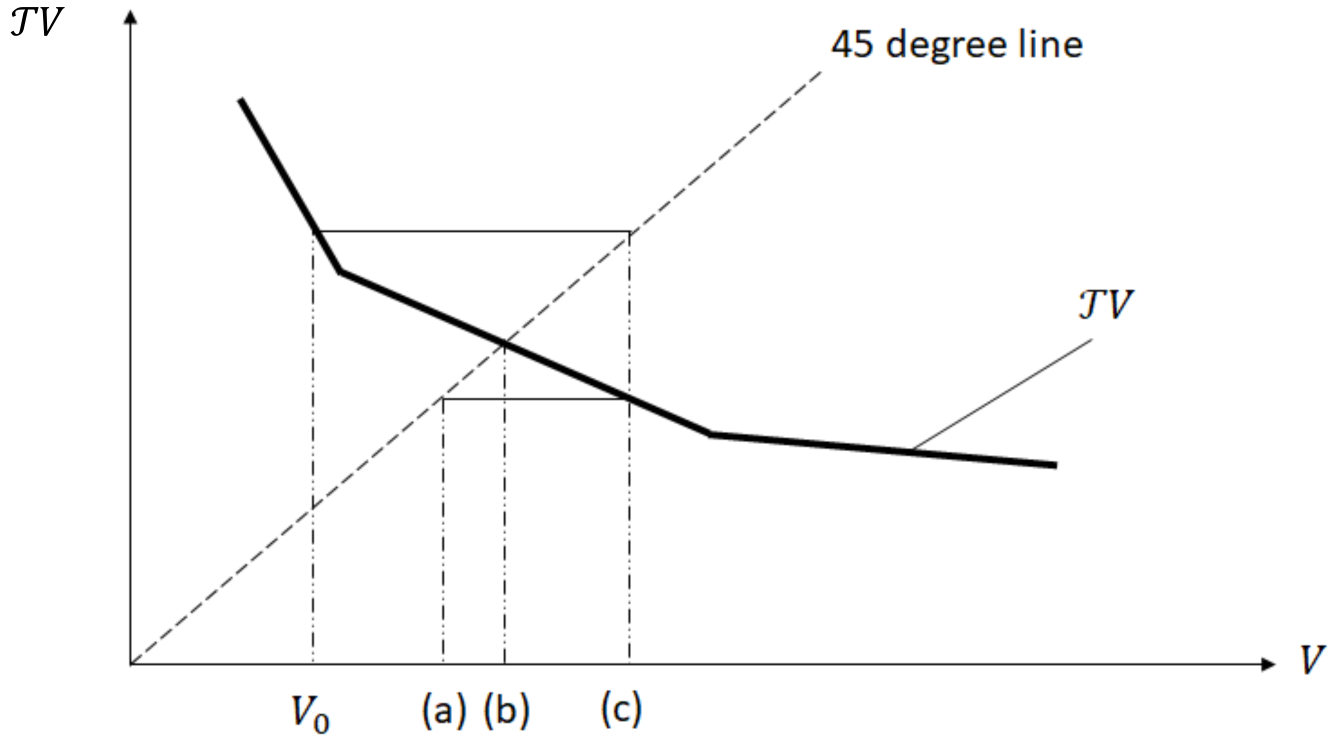

Running policy iteration with $\gamma = 0.9$, initialized with policy $\pi(s) =$ North

5.470	6.313	7.190	8.669
4.803	■	3.347	-96.67
4.161	3.654	3.222	1.526

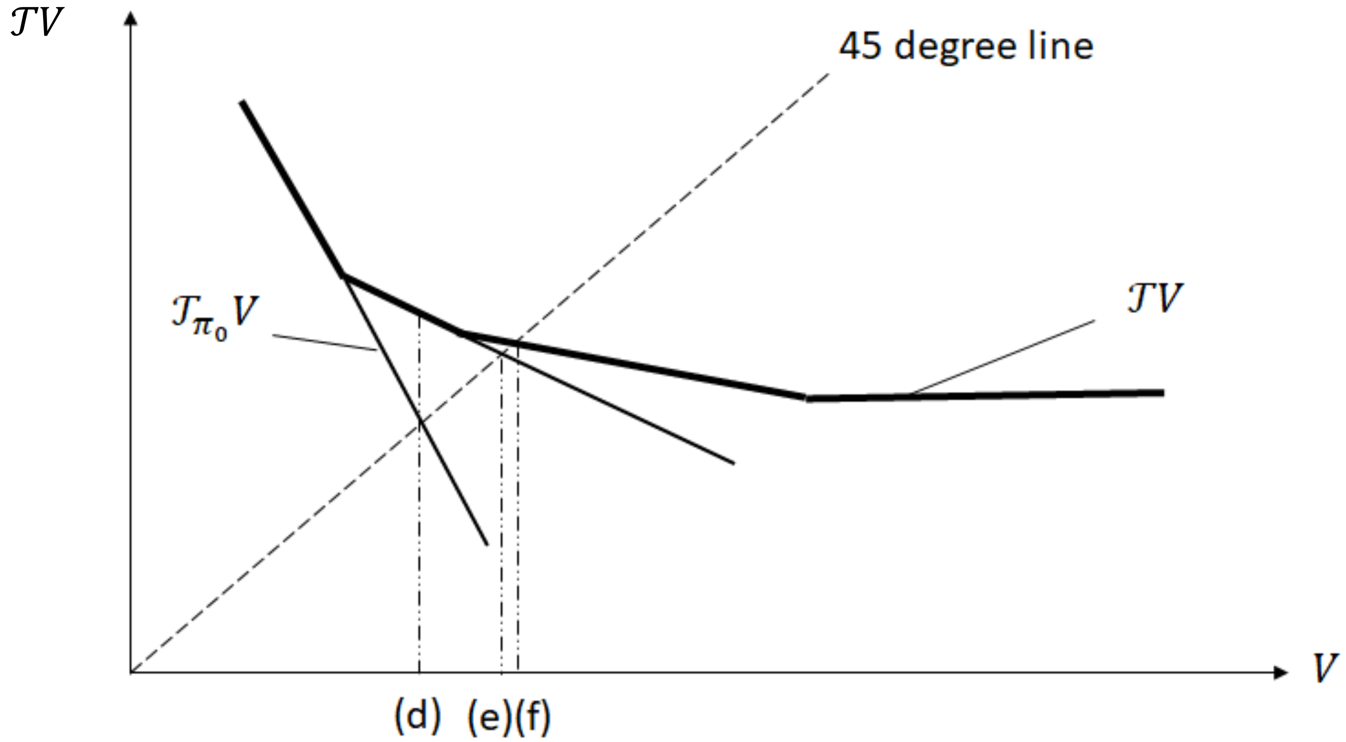
V^π at three iterations (converged)

(d)

Value iteration: geometric Interpretation



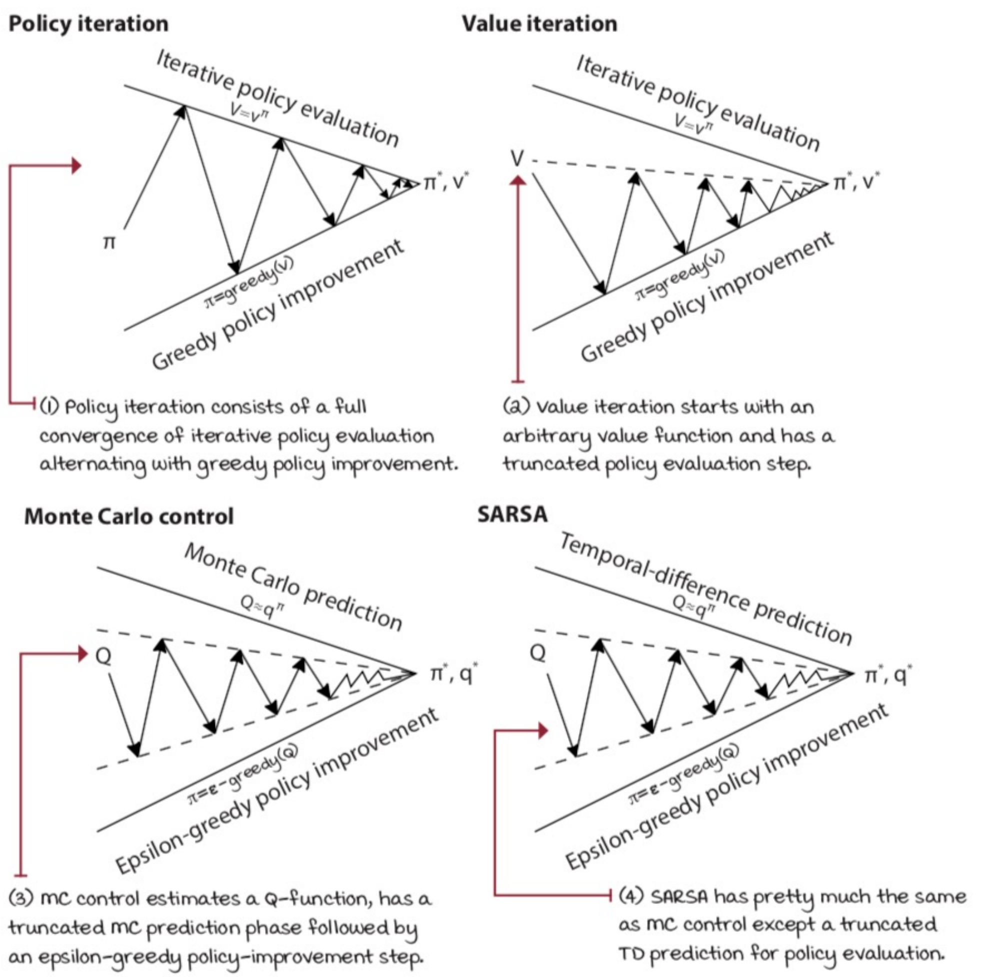
Policy iteration: geometric Interpretation



Demo: value iteration vs policy iteration

[REINFORCEjs]

More variations



(1) Policy iteration consists of a full convergence of iterative policy evaluation alternating with greedy policy improvement.

(2) Value iteration starts with an arbitrary value function and has a truncated policy evaluation step.

(3) MC control estimates a Q-function, has a truncated MC prediction phase followed by an epsilon-greedy policy-improvement step.

(4) SARSA has pretty much the same as MC control except a truncated TD prediction for policy evaluation.

Adapted from Morales, Grokking Deep Reinforcement Learning, 2020.

Summary & Takeaways

- When specifying a sequential problem, care should be taken to select an appropriate type of policy and value function, depending on the use case.
- The ideas from **dynamic programming**, namely the **principle of optimality**, carry over to infinite horizon problems.
- The **value iteration** algorithm solves discounted infinite horizon MDP problems by leveraging results of **Bellman operators**, namely the **optimal Bellman equation**, **contractions**, and **fixed points**.
- **Generalized policy iteration** methods include policy iteration and value iteration.
- **Policy iteration** algorithm additionally leverages **monotonicity** and **Bellman equation**.
- The update mechanism for VI and PI differ and thus their convergence in practice depends on the **geometric structure** of the optimal value function.

References

1. With many slides adapted from Alessandro Lazaric and Matteo Pirotta.

2. Dimitri P. Bertsekas. Dynamic Programming and Optimal Control. Volume 2. 4th Edition. (2012). Chapters 1-2: Discounted Problems.

3. R. E. Bellman. Dynamic Programming. Princeton University Press, Princeton, N.J., 1957.