

Lecture 5

The Linear Quadratic Regulator (LQR) Problem

Lecture Overview: The LQR Problem

- **Finite Horizon Problem: Review**
- **Infinite Horizon Problem**
 - **Controllability, Observability, and their Implications**
 - **Stability**
- **Algebraic Riccati Equation**
- **Existence of Solutions, Properties**

Formulation of the Linear Quadratic Regulator (LQR) Problem

$$[0, T]$$

$$x_{t+1} = A_t x_t + B_t u_t + w_t$$

$$c_t(x_t, u_t) = x_t^\top Q_t x_t + u_t^\top R_t u_t$$

$$c_T(x_T) = x_T^\top Q_T x_T$$

where $x_t \in \mathbb{R}^n$, $w_t \sim \mathcal{N}(0, \Sigma)$, i.i.d. white noise

- **Objective:**

$$\min_{\pi} \mathbb{E} \left[c_T(x_T) + \sum_{t=0}^{T-1} c_t(x_t, u_t) \right]$$

- **Assumptions:**

Q_t is positive semi-definite

R_t is positive definite

Structure of Optimal Solution: Finite Horizon

$$\mu_t(x_t) = -G_t^{-1} H_t x_t = -G_t^{-1} B_t^\top K_{t+1} A_t x_t = -L_t x_t$$

$$V_t(x) = x^\top K_t x + \text{const.}$$

$$K_t = A_t^\top \left(K_{t+1} - K_{t+1} B_t G_t^{-1} B_t^\top K_{t+1} \right) A_t + Q_t$$

$$G_t = R_t + B_t^\top K_{t+1} B_t, \quad K_T = Q_T.$$

Latter known as Riccati Equation

Assume Stationary System (LTI)

Equations

$$K_T = Q_T$$

$$K_t = A^\top \left(K_{t+1} - K_{t+1} B G_t^{-1} B^\top K_{t+1} \right) A + Q$$

$$G_t = R + B^\top K_{t+1} B$$

$$L_t = -G_t^{-1} B^\top K_{t+1} A$$

Asymptotic Behavior: Convergence Under Controllability

What happens asymptotically?

$$K_t \rightarrow K, \quad K = A^\top (K - K B G^{-1} B^\top K) A + Q$$

$$G_t \rightarrow G = R + B^\top K B$$

$$L_t \rightarrow L = -G^{-1} B^\top K A$$

Need to show

- Convergence of Riccati Equation
- Stability of the linear system under the control law

More on linear systems

$$x_{k+1} = Ax_k + Bu_k,$$

$$A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m},$$

$$y_k = Cx_k,$$

$$C \in \mathbb{R}^{p \times n},$$

$$x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^m, y_k \in \mathbb{R}^p.$$

Output $y_k = Cx_k$ may not be the whole state

More on linear systems: Reachability

Reachability: A system is reachable if there exists a control law that takes every initial state x_0 to every final state x_f in finite time

Reachability Condition

$$\text{rank} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n \quad \Longleftrightarrow \quad \sum_{i=0}^{n-1} A^i B B^\top (A^i)^\top = n$$

More on linear systems: Reachability

Reachability: A system is reachable if there exists a control law that takes every initial state x_0 to every final state x_f in finite time

Define the reachability Gramian:

$$P = \sum_{i=0}^{\infty} A^i B B^{\top} (A^i)^{\top} \quad \text{When is this well defined}$$

Then:

$$A P A^{\top} - P = -B^{\top} B$$

System is reachable iff:

$$P \succ 0$$

Reachability: Example

Reachability: A system is reachable if there exists a control law that takes every initial state x_0 to every final state x_f in finite time

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\mathcal{R} = [B, AB, A^2B] = \begin{bmatrix} b_1 & b_2 & 0 \\ b_2 & 0 & 0 \\ b_3 & 0 & 0 \end{bmatrix}.$$

Since the third column is zero, $\text{rank}(\mathcal{R}) \leq 2 < 3$. Hence (A, B) is *not reachable*.

More on linear systems: Observability

Observability: A system is observable if there are no initial state x_0 for which the output is identically equal to zero.

Observability Condition

$$\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \iff \text{rank} \left(\sum_{i=0}^{n-1} (A^i)^\top C^\top C A^i \right) = n$$

More on linear systems: Observability

Observability: A system is observable if there are no initial state for which the output is identically equal to zero.

Define the Observability Gramian:

$$Q = \sum_{i=0}^{\infty} (A^i)^{\top} C^{\top} C A^i$$

Then:

$$A^{\top} Q A - Q = -C^{\top} C$$

System is Observable iff:

$$Q \succ 0$$

Observability: Example

Observability: A system is observable if there are no initial state for which the output is identically equal to zero.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad C = [1 \quad 0 \quad 0].$$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}.$$

Compute each term:

$$C = [1 \quad 0 \quad 0], \quad CA = [1 \quad 0 \quad 0], \quad CA^2 = [1 \quad 0 \quad 0].$$

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Stability dynamical systems

Consider the system

$$x_{k+1} = f(x_k), \quad f(0) = 0.$$

- **Stable (Lyapunov stable):** For every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|x_0\| < \delta \implies \|x_k\| < \varepsilon \quad \forall k \geq 0.$$

- **Asymptotically stable:** The equilibrium is stable, and in addition

$$\|x_0\| < \delta \implies \lim_{k \rightarrow \infty} x_k = 0.$$

- **Exponentially stable:** There exist $c > 0$, $\lambda \in (0, 1)$ such that

$$\|x_k\| \leq c\lambda^k \|x_0\|, \quad \forall k \geq 0.$$

Lyapunov Stability

Consider the system

$$x_{k+1} = f(x_k), \quad f(0) = 0.$$

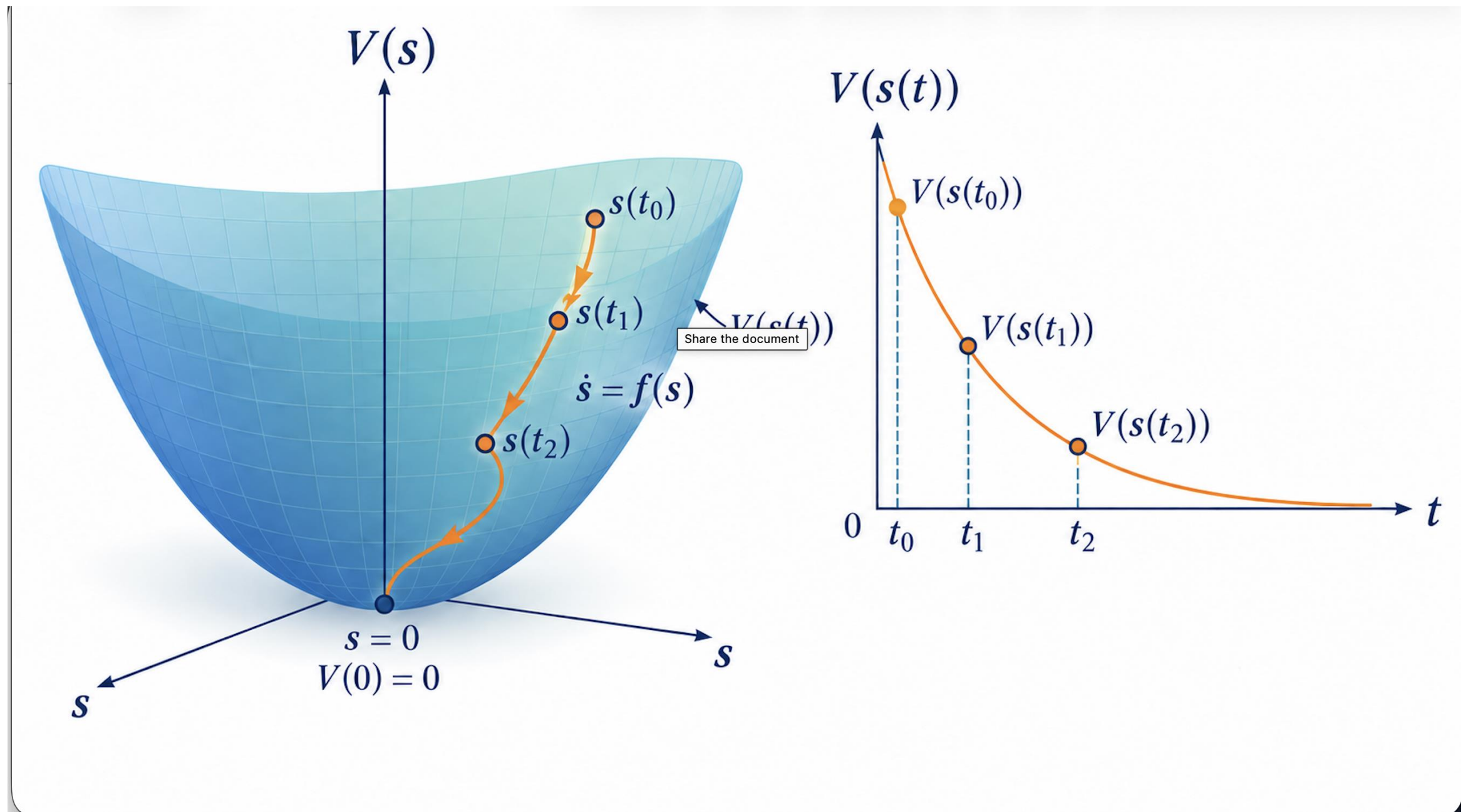
A function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is called a **Lyapunov function** if:

- $V(x) > 0$ for all $x \neq 0$, and $V(0) = 0$.
- With

$$\Delta V(x) = V(f(x)) - V(x),$$

we have:

- $\Delta V(x) \leq 0 \implies$ the equilibrium is Lyapunov stable.
- $\Delta V(x) < 0 \implies$ the equilibrium is asymptotically stable.



How about linear systems

For the linear system $x_{k+1} = Ax_k$, a quadratic candidate is

$$V(x) = x^\top Px, \quad P \succ 0,$$

with

$$\Delta V(x) = x^\top (A^\top PA - P)x.$$

If there exists $P \succ 0$ such that $A^\top PA - P \prec 0$, the equilibrium is asymptotically stable.

LTI Case:

In a time-invariant system, $\Phi(t, \tau) = A^{t-\tau}$. Stability is guaranteed if:

$$A^t \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty$$

which occurs if and only if all eigenvalues of A lie strictly inside the unit circle (i.e., have magnitude less than 1).

Linear Systems: Stationary Policies

$$x_{t+1} = Ax_t + Bu_t$$

$$y = Cx_t$$

Under the optimal control law: $(A + BL)$

$$x_{t+1} = (A + BL)x_t + w_t$$

Recall: Stability is equivalent to having the eigenvalues of $(A + BL)$ inside the unit disc!

Stability and Reachability

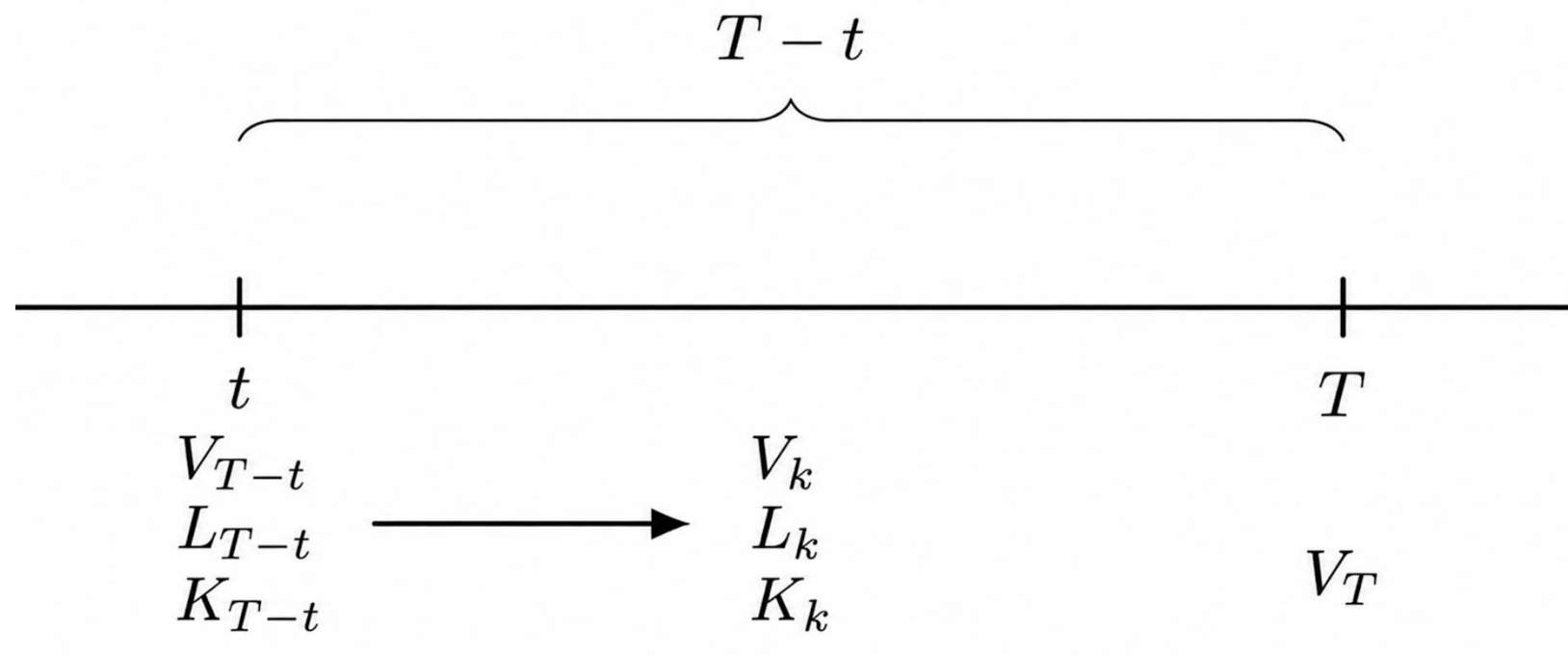
If the system is reachable, then there always exists a feedback gain to stabilize the system

Pole Placement Theorem: If (A, B) is reachable, then given any n^{th} order polynomial $q(z)$, there exists a feedback matrix L such that $\det(zI - A - BL) = q(z)$.

Homework!

What are the implications of this result?

Asymptotic Behavior: Convergence



Using the change of variables ($T - t \rightarrow k$) The cost of the remaining k steps is given by:

$$x^\top \bar{K}_k x$$

Riccati Iteration

$$\bar{K}_{k+1} = A^\top (\bar{K}_k - \bar{K}_k B (B^\top \bar{K}_k B + R)^{-1} B^\top \bar{K}_k) A + Q$$

Assume zero terminal cost:

$$\bar{K}_0 = 0$$

Gain Matrix

$$\bar{L}_{k+1} = - (B^\top \bar{K}_k B + R)^{-1} B^\top \bar{K}_k A$$

Asymptotic Behavior: Convergence

Show Convergence:

$$\lim_{k \rightarrow \infty} \bar{K}_k = \bar{K}$$

Show boundedness and Monotonicity!

Convergence follows

Asymptotic Behavior: Boundedness

Reachability implies

$$x^T K_t x \leq c \|x\|^2 \quad \forall t$$

How do we show this?

Asymptotic Behavior: Monotonicity

$$x^\top \bar{K}_k x \leq x^\top \bar{K}_{k+1} x$$

Proof: Use the same strategy $u_0, \dots, u_{k+1}$ and ignore the last step.

Asymptotic Behavior: Stability

- Show closed loop system is stable

- Closed loop dynamics:

$$L = -(R + B^\top K B)^{-1} B^\top K A, \quad A_{cl} = A + B L$$

- Construct a Lyapunov function!
- What is a good candidate for a Lyapunov function?

Asymptotic Behavior: Positivity

If (Q, A) is observable, then $K_t > 0$.

Interpret Q as an output!

Proof of positivity

- Assume (Q, A) is observable.
- Suppose $x_0^\top K_n x_0 = 0$, for some x_0 .
- This implies that the optimal policy yields zero cost.
- Since $R > 0$, this implies that $u_t = 0$ for all $t = 1, \dots, n - 1$.
- Then $x_t = A^t x_0$.
- But we can compute the cost directly:

$$x_0^\top \left(Q + A^\top Q A + (A^\top)^2 Q A^2 + \dots + (A^\top)^{n-1} Q A^{n-1} \right) x_0$$

- If cost is zero, this implies the system is *not observable*.

Asymptotic Stability

$$x_{k+1} = Ax_k + Bu_k$$

$$K = Q + A^\top KA - A^\top KB(R + B^\top KB)^{-1}B^\top KA$$

$$L = -(R + B^\top KB)^{-1}B^\top KA, \quad A_{\text{cl}} = A + BL$$

Asymptotic Stability

Propose the following function as a Lyapunov function candidate:

$$V(x) = x^\top K x$$

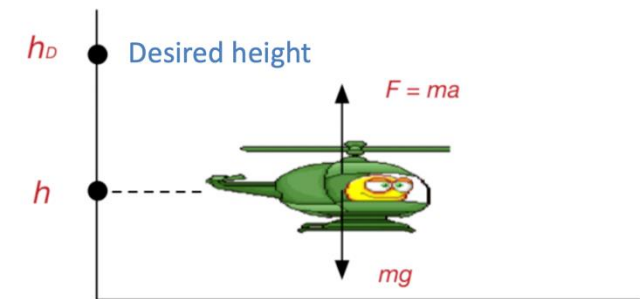
$$\Delta V = V(x_{k+1}) - V(x_k) = x_k^\top (A_{\text{cl}}^\top K A_{\text{cl}} - K) x_k$$

$$A_{\text{cl}}^\top K A_{\text{cl}} - K = -(Q + L^\top R L)$$

$$\Delta V = -(x_k^\top Q x_k + u_k^\top R u_k) \leq 0$$

Thus, $V(x)$ is a Lyapunov function, and the closed-loop system $x_{k+1} = (A - BL)x_k$ is asymptotically stable.

Example



For $m = 1$, $T_s = 0.1$ s,

$$A_d = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0.005 \\ 0.1 \end{bmatrix}.$$

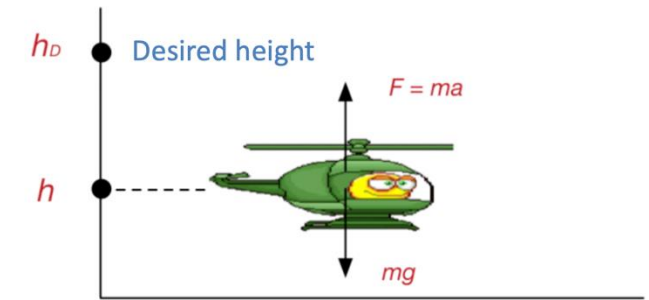
We minimize the infinite-horizon cost

$$J = \sum_{k=0}^{\infty} \left[(x_k - x_{\text{ref}})^{\top} Q (x_k - x_{\text{ref}}) + u_k^{\top} R u_k \right], \quad (1)$$

where

$$x_{\text{ref}} = \begin{bmatrix} h_D \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 20 & 0 \\ 0 & 2 \end{bmatrix}, \quad R = [0.1]. \quad (2)$$

Example



The Riccati equation is

$$K = A_d^\top K A_d - A_d^\top K B_d (R + B_d^\top P B_d)^{-1} B_d^\top K A_d + Q.$$

The solution is

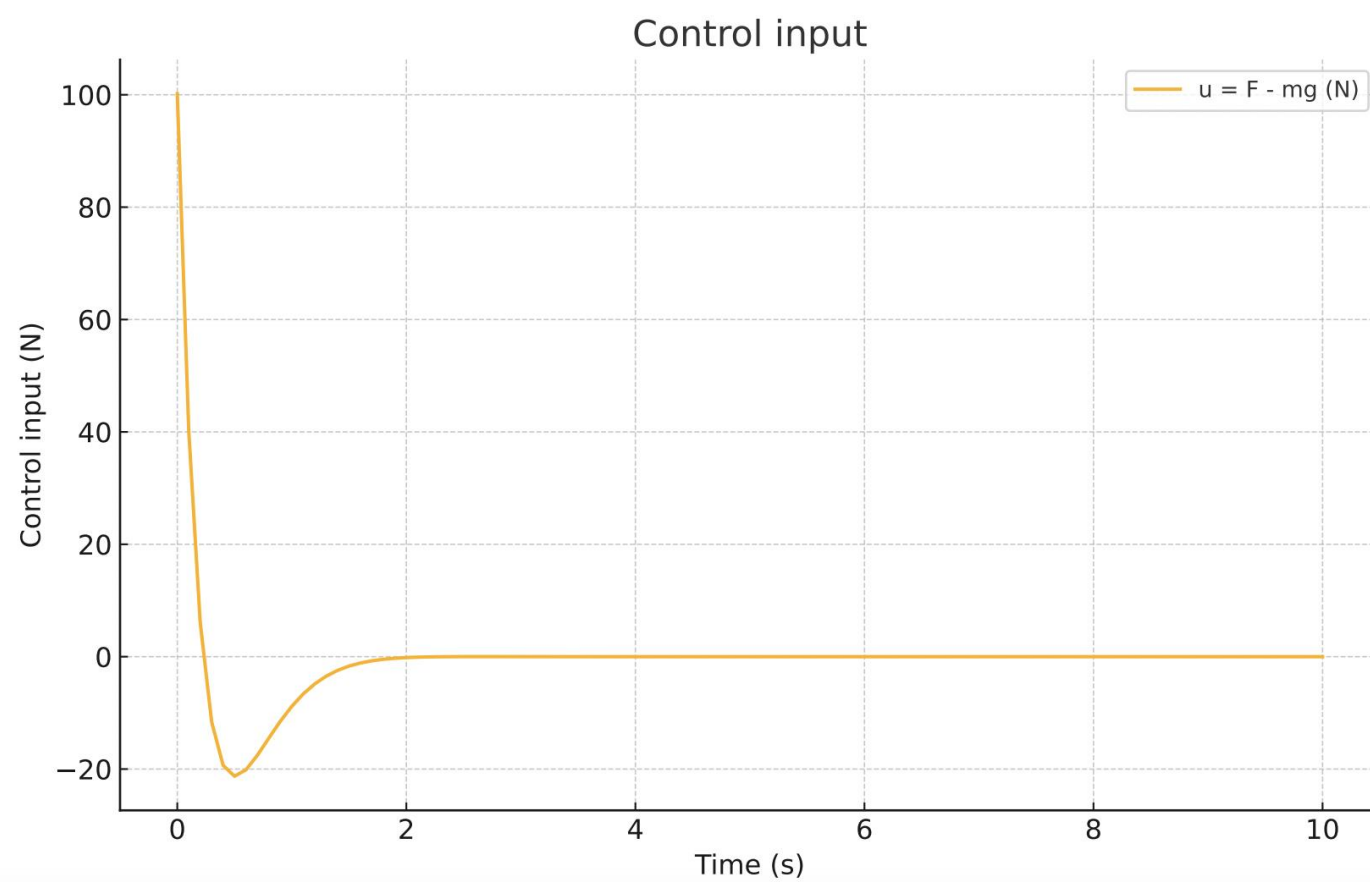
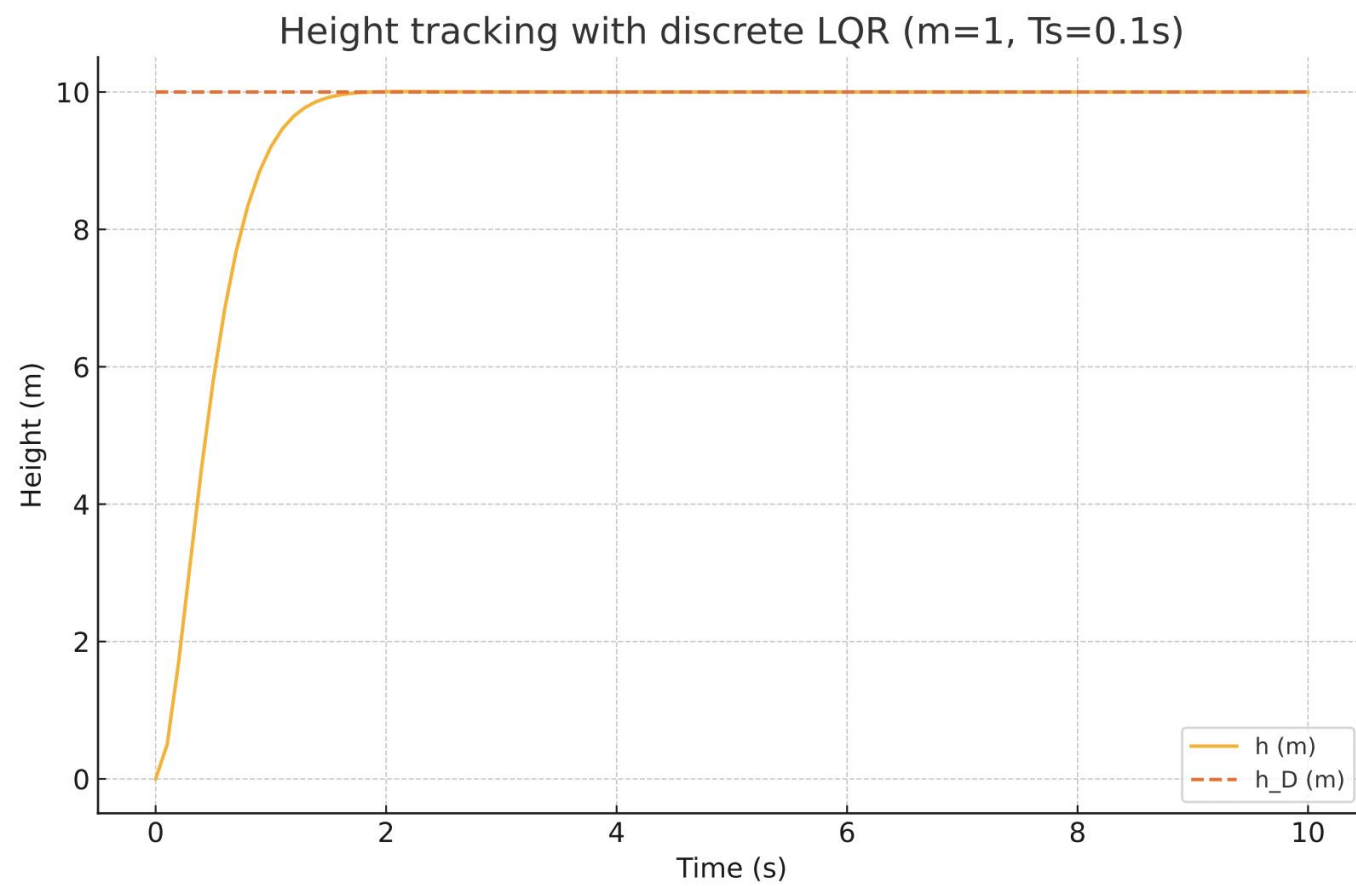
$$K = \begin{bmatrix} 109.481409 & 14.491377 \\ 14.491377 & 8.208113 \end{bmatrix}.$$

The optimal LQR gain is

$$L = (R + B_d^\top K B_d)^{-1} B_d^\top K A_d = [10.017306 \quad 5.483544].$$

Control Law

$$u_k = -L(x_k - x_{\text{ref}}), \quad F_k = mg + u_k.$$



Example

