A Comparison of Universal and Mean-Variance Efficient Portfolios

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1 Summary

A portfolio is an allocation of wealth among investment opportunities, or assets. In this presentation, we will consider two portfolio strategies, as summarized below:

- **Universal Portfolios [3]:**
  - *Objective*: To "track" the return of the best possible constant percentage allocation of wealth (a.k.a. constant rebalanced portfolio (CRP)) chosen with knowledge of future asset returns
  - *Algorithm*: Return-weighted average of all CRPs
  - *Main Assumptions:* No statistical assumptions are made on asset returns; asymptotic analysis, however, requires several sequence-based assumptions on asset returns
  - *Performance:* Yields average return of all CRPs; return is greater than the geometric mean of stock returns (i.e. value line index); asymptotically tracks the return of the best CRP under many sequence-based assumptions
- **Mean-Variance Efficient Portfolios [8],[9],[5]**:
  - *Objective*: To minimize the variance of portfolio returns, while maintaining a desired average return
  - *Algorithm*: A solution to a constrained optimization problem
  - *Main Assumptions*: Assumes asset returns from different periods are independent and identically distributed (IID) random vectors; requires estimates of the distribution's mean and covariance matrix.
  - *Performance:* Tracks objective portfolio if IID assumptions hold; otherwise, no guarantee

Both strategies use asset returns from previous investment periods to build the portfolio for the current investment period.

2 Investment Dynamics

A portfolio describes a strategy for allocating and re-allocating wealth among M assets over a particular time horizon, to attain certain objectives. The time horizon and objectives vary from investor to investor depending on their preferences. In this summary, we assume that the investor falls into one of two categories:

- The investor wants a portfolio whose return tracks the return of the best constant percentage allocation of wealth given future stock returns.
- The investor wants a desired average return, but wishes to minimize the return variance.

Universal portfolios might be more suited to the first type of investor, while mean-variance efficient portfolios to the latter.

We simplify matters by dividing the time horizon into N periods. Let $p_{n,m}$ and $a_{n,m}$ be the price and amount owned, respectively, of asset m at the *beginning* of period n. The wealth of the investor at the beginning of period n is

$$h_n = a_{n,1}p_{n,1} + \cdots + a_{n,M}p_{n,M} \tag{1}$$

We define the factor by which the wealth increased from the beginning to the end of period n as $R_n = h_{n+1}/h_n$, assuming that the price of each asset at the end of a period is equal to its price at the beginning of the next period. Furthermore, if we assume that no new wealth is added to the portfolio, the wealth at the end of a period must equal the wealth at the beginning of the next, i.e.

$$\begin{aligned} h_{n+1} &= a_{n+1,1}p_{n+1,1} + \cdots + a_{n+1,M}p_{n+1,M} && (2) \\ &= a_{n,1}p_{n+1,1} + \cdots + a_{n,M}p_{n+1,M} && (3) \end{aligned}$$

Notice that from (3)

$$\begin{aligned} R_n &= \frac{a_{n,1}p_{n+1,1}}{h_n} + \cdots + \frac{a_{n,M}p_{n+1,M}}{h_n} && (4) \\ &= \frac{a_{n,1}p_{n,1}}{h_n}\frac{p_{n+1,1}}{p_{n,1}} + \cdots + \frac{a_{n,M}p_{n,M}}{h_n}\frac{p_{n+1,M}}{p_{n,M}} && (5) \\ &= b_{n,1}X_{n,1} + \cdots + b_{n,M}X_{n,M} && (6) \\ &= b_n^t X_n && (7) \end{aligned}$$

where we define

$$b_{n,m} = \frac{a_{n,m}p_{n,m}}{h_n} \tag{8}$$

as the fraction of wealth in asset m at the *beginning* of period n,

$$X_{n,m} = \frac{p_{n+1,m}}{p_{n,m}} \tag{9}$$

as the factor by which asset m increased in price from the *beginning* to the *end* of period n, $b_n^t = (b_{n,1} \dots b_{n,M})$, and $X_n^t = (X_{n,1} \dots X_{n,M})$.

A constant-rebalanced portfolio (CRP) is an investment strategy where the percentage allocation of wealth in each asset is constant over time, i.e. $b_1 = b_2 = \dots = b_N = b$. A buy-and-hold (B&H) portfolio is a constant rebalanced portfolio that places and leaves all the initial wealth in a single asset, i.e. $b = e_m = (0, 0, \dots, 0, 1, 0, \dots, 0)^t$ is the m-th standard Euclidean basis vector.

We define the factor by which the wealth has increased from the *beginning* of the first period to the *end* of the N-th period as

$$
\begin{aligned}
S_N &= \frac{h_{N+1}}{h_1} && (10)\\
&= \left(\frac{h_{N+1}}{h_N}\right)\left(\frac{h_N}{h_{N-1}}\right)\cdots\left(\frac{h_2}{h_1}\right) && (11)\\
&= \prod_{n=1}^{N} R_n && (12)\\
&= \prod_{n=1}^{N} b_n^t X_n && (13)
\end{aligned}
$$

Assuming that $b_n^t X_n > 0$ for $n = 1, \dots, N$, we can further manipulate the total return S_N to give

$$
\begin{aligned}
S_N &= \exp\left(\sum_{n=1}^{N} \log b_n^t X_n\right) && (14)\\
&= \exp\left(N W_N\right) && (15)
\end{aligned}
$$

where log() denotes the natural logarithm, and

$$W_N = \frac{1}{N}\sum_{n=1}^{N} \log b_n^t X_n \tag{16}$$

is called the “doubling” rate of the portfolio.

3 Universal Portfolios

3.1 Investment Objective

The goal of the universal portfolio [3] is to “track” the return of the best constant rebalanced portfolio (CRP) chosen after future asset outcomes are revealed. That is, the universal portfolio tries to track

$$S_N^* = \max_{b \in B} S_N(b) = \max_{b \in B} W_N(b) \tag{17}$$

where $B = \{b \in \mathcal{R}^M : b_m \geq 0, m = 1, 2, \dots, M; \sum_{m=1}^{M} b_m = 1\}$ is the set of allowable CRPs. Denote a CRP that returns S_N^* as b_N^*. Notice that the restriction $b_m \geq 0$ prohibits the use of short sales, i.e. selling an asset before buying it.

3.2 Universal Portfolio Algorithm

The universal portfolio algorithm is the return-weighted average of all CRPs [3]:

$$\text{Initialize:} \quad \hat{b}_1 = \left(\tfrac{1}{M}, \cdots, \tfrac{1}{M}\right)^t$$

$$\text{For } n > 1: \quad \hat{b}_n = \frac{\int_B bS_{n-1}(b)db}{\int_B S_{n-1}(b)db}$$

The universal portfolio is the weighted average of all CPRs, with greater emphasis placed on those portfolios with larger returns.

3.3 Performance

The universal portfolio's return is the average of all CRPs' returns [3]:

$$\hat{S}_N = \prod_{n=1}^{N} \hat{b}_n^t X_n = \frac{\int_B S_N(b)db}{\int_B db} \tag{18}$$

As a consequence of Jensen's inequality, it is also greater than the geometric mean of stock returns (i.e. value line index) [3]:

$$\hat{S}_N \geq \left(\prod_{m=1}^{M} S_N(e_m)\right)^{1/M} \tag{19}$$

Under many assumptions, the returns of the universal and best CRP portfolio have the same asymptotic growth rate in the exponent [3]:

$$\frac{\hat{S}_N}{S_N^*} \sim \left(\sqrt{\frac{2\pi}{N}}\right)^{M-1} \frac{(M-1)!}{|J^*|^{1/2}} \tag{20}$$

in the sense that the ratio of the left and right hand sides equals one for large N. Here, J_N^* is a "sensitivity matrix" defined as the curvature (Hessian) of $W_N(b)$ at its maximum, and J^* is this curvature for large N. The asymptotic property in (20) holds if the stock sequence $X_1, X_2, \ldots,$ satisfies:

- $a \leq X_{n,m} \leq q$, $m = 1, \ldots, M$,$n = 1, 2, \ldots$, for some $0 < a \leq q < \infty$
- The sensitivity matrix $J_N^* \to J^*$ for some positive definite matrix J^*
- The best CRP $b_N^* \to b^*$ for some b^* in the interior of the portfolio simplex B

and there exists a function $W(b)$ such that

- The doubling rate $W_N(b) \nearrow W(b)^*$ (i.e. converges uniformly and monotonically) for any portfolio $b \in B$
- $W(b)$ is strictly concave
- $W(b)$ has bounded third partial derivatives
- $W(b)$ achieves its maximum at b^* in the interior of the portfolio simplex B

For proofs of these performance results, see [3].

3.4 Implementation

Evaluating the integral in the universal portfolio algorithm is exponential in the number of stocks [7]. Blum and Kalai [1] proposed an approximate implementation based on the uniform random sampling of the portfolio simplex. In the worst case, this method is also exponential in the number of stocks. Empirically, however, this algorithm seems to be efficient, at least for two stocks [3],[4]. Matlab source code for this method is presented in the appendix.

Kalai and Vempala in [7] propose an implementation based on non-uniform random sampling of the portfolio simplex. Unlike the uniform random implementation, this method is polynomial in the number of stocks and samples.

Cover in [4] presents a recursive implementation similar to the recursive generation of the binomial coefficients.

4 Mean-Variance Efficient Portfolios

4.1 Investment Objective

The goal of the mean-variance efficient portfolio is to minimize the variance of the portfolio returns while maintaining a desired average return. This method implies an underlying probability distribution on the asset returns, and requires estimating their means and covariances.

Let $\bar{X}_n$ and V_n denote causal estimators of the mean, $E(X_n)$, and covariance matrix, $E[(X_n - E(X_n))(X_n - E(X_n))^t]$, of the asset return sequence for the n-th period. One possible choice for these estimators is

$$\bar{X}_n = \frac{1}{\Lambda} \sum_{k=1}^{n-1} \lambda^{n-k-1} X_k \tag{21}$$

$$V_n = \frac{1}{\Lambda} \sum_{k=1}^{n-1} \lambda^{n-k-1} (X_k - \bar{X}_n)(X_k - \bar{X}_n)^t \tag{22}$$

where $0 < \lambda \leq 1$ is a factor to exponentially "forget" past data, and $\Lambda = \sum_{k=1}^{n-1} \lambda^{n-k-1}$ is the sum of the forgetting factors. Setting $\lambda = 1$ gives the usual arithmetic mean. Estimators for the mean and variance of the portfolio return for the n-th period are then $b_n^t \bar{X}_n$ and $b_n^t V_n b_n$,respectively.

The mean-variance efficient portfolio for the n-th period is the solution to quadratic programming problem:

$$\begin{array}{ll} \text{Minimize:} & b_n^t V_n b_n \\ \\ \text{Subject to:} & 1)\ \sum_{m=1}^{M} b_{n,m} = 1 \\ \\ & 2)\ b_n^t \bar{X}_n = \bar{R}_n \end{array}$$

where $\bar{R}_n$ is the desired average return for the n-th period. Notice that this mean-variance efficient portfolio allows short sales, i.e. it does not restrict $b_{n,m}$ to be non-negative. Not allowing short sales complicates matters, and is discussed in [6].

4.2 Mean-Variance Efficient Portfolio Algorithm

Using Lagrange multipliers and assuming that V_n is non-singular, the solution to this quadratic programming problem is ([10],[5]):

$$b_n = w_0(1 - \bar{R}_n) + w_1 \bar{R}_n \tag{23}$$

where w_0 and w_1 are portfolios that yield on average 0% and 100% returns, respectively:

$$w_0 = \frac{1}{D}\left(c_1 V_n^{-1} 1 - c_2 V_n^{-1} \bar{X}_n\right) \tag{24}$$

$$w_1 = \frac{1}{D}\left(c_1 V_n^{-1} 1 - c_2 V_n^{-1} \bar{X}_n\right) + \frac{1}{D}\left(c_3 V_n^{-1} \bar{X}_n - c_2 V_n^{-1} 1\right) \tag{25}$$

Here:

$$1^t = (1, \ldots, 1) \tag{26}$$

$$c_1 = \bar{X}_n^t V_n^{-1} \bar{X}_n \tag{27}$$

$$c_2 = \bar{X}_n^t V_n^{-1} 1 \tag{28}$$

$$c_3 = 1^t V_n^{-1} 1 \tag{29}$$

$$D = c_1 c_3 - c_2^2 \tag{30}$$

4.3 Performance

If the asset returns are independent and identically distributed random vectors according to some unknown distribution, then the estimates $\bar{R}_n$ and V_n approach the true mean and covariance by the law of large numbers (with $\lambda = 1$). Under this assumption, the above algorithm asymptotically yields the minimum variance portfolio for a desired average return.

5 Examples

In this section, we present several examples illustrating the performance of the previously discussed portfolio strategies. All examples use daily adjusted closing price data from January 2, 1991 to August 8, 2001 (2,679 days) obtained from the historical quotes service on finance.yahoo.com.

5.1 Universal Portfolios

Figures 1 through 3 show constant rebalanced portfolios and the universal portfolio for two stocks. Notice that the universal portfolio does not always outperform the best buy-and-hold portfolios (Figures 2 and 3). Furthermore, the best CRP might be a buy-and-hold portfolio (Figure 3).

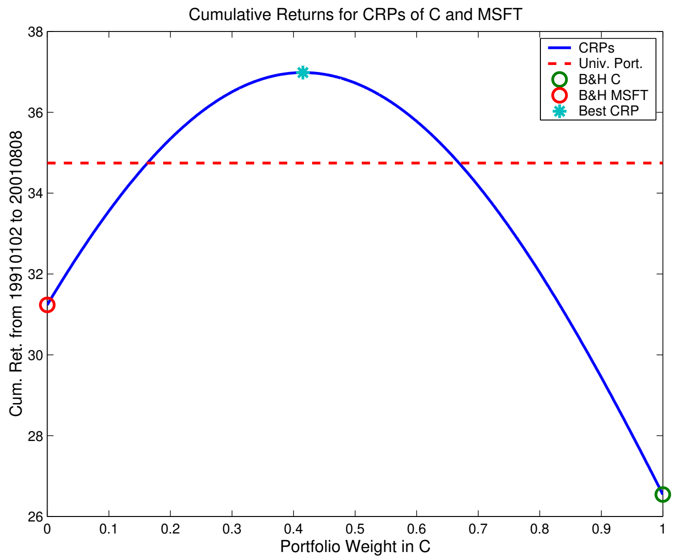

Figure 1: Constant rebalanced portfolios and universal portfolio for Citigroup (C) and Microsoft (MSFT). The universal portfolio performs better than either buy-and-hold portfolio.

5.2 Mean-Variance Efficient Portfolios

Figure 4 shows the set of mean-variance efficient portfolios for different desired average returns. The portfolios are composed of eight large U.S. companies (Citigroup (C), General Electric (GE), International Business Machine (IBM), Microsoft (MSFT), Oracle (ORCL), Pfizer (PFE), Walmart (WMT), Exxon (XOM)) and Vanguard's S&P 500 Index Fund (VFINX). The mean and co-

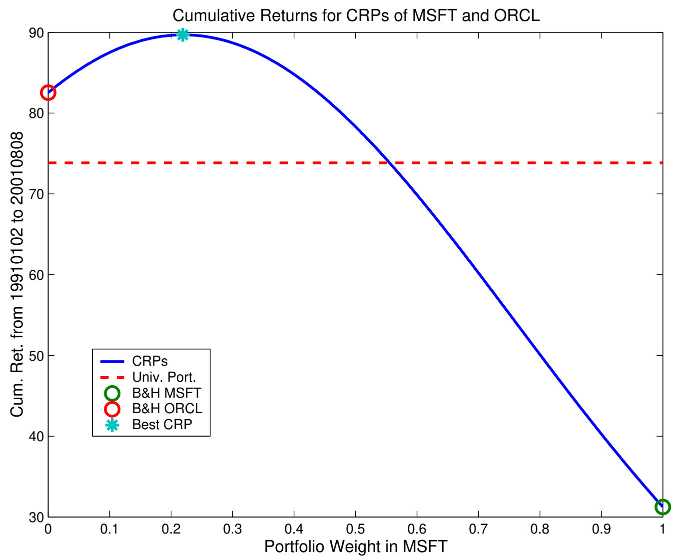

Figure 2: Constant rebalanced portfolios and universal portfolio for MSFT and Oracle (ORCL). The universal portfolio performs better than the buy-and-hold portfolio for MSFT, but not ORCL.

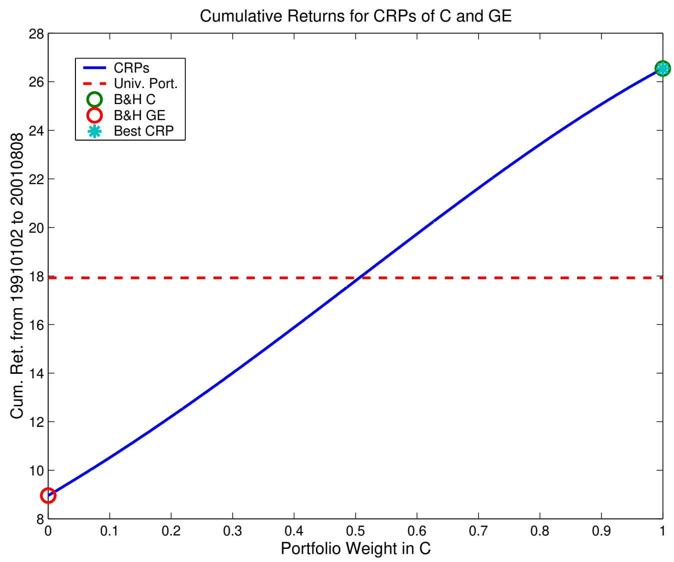

Figure 3: Constant rebalanced portfolios and universal portfolio for C and General Electric (GE). The best CRP in this example is the buy-and-hold portfolio in C.

variance matrix of the asset returns were calculated using (21) and (22) with $\lambda = 1$.

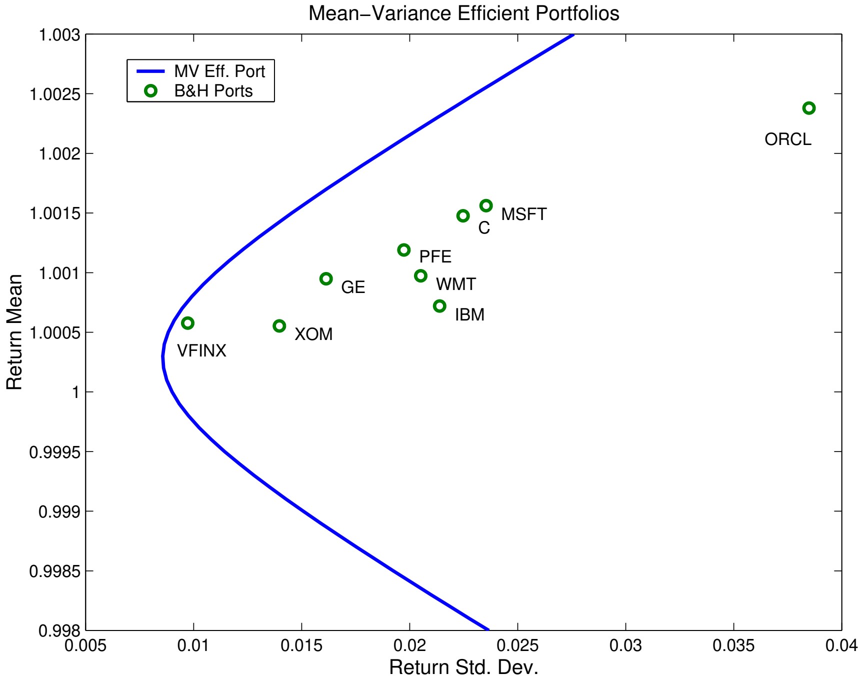

Figure 4: A plot of the mean-variance efficient frontier and buy-and-hold portfolios for C, GE, IBM, MSFT, ORCL, PFE, VFINX, WMT, and XOM. The mean and covariance matrix of the asset returns was estimated using (21) and (22) with $\lambda = 1$. Notice that the Vanguard S&P 500 Index (VFINX) lies near the efficient frontier as suggested by the Capital Asset Pricing Model (CAPM) [2].

5.3 Comparison of Universal and Mean-Variance Efficient Portfolios

In this section, we will try to compare the performance of the universal and mean-variance efficient portfolios. One should be careful not to draw general conclusions such as "the universal portfolio is better than the mean-variance efficient portfolio," without specifically defining what "better" means and considering the constraints on each portfolio. For example, as presented in this summary, we have not allowed short sales in universal portfolios, while we have allowed them in mean-variance efficient portfolios.

Furthermore, we have not considered trading costs. Buying and holding the Vanguard S&P 500 Index fund is much cheaper than trading every day to maintain a CRP.

Deciding which portfolio is "better" is a lot like deciding which is better, steak or lobster?[1] The answer depends on a person's preferences, much like choosing a portfolio depends on an investor's preferences. Is the investor young and single with 40 years to work before retirement? Or does the investor have a family and is approaching retirement? These factors might influence what "better" means for each individual investor.

One final disclaimer: "past performance does not necessarily indicate future returns."

Figure 5 shows the daily cumulative returns (13) of the universal portfolio, best CRP, mean-variance efficient portfolio, Vanguard S&P 500 Index Fund (a good benchmark) , and equally weighted CRP from January 2, 1991 to August 8, 2001. The universal and equally-weighted portfolio perform very similarly for this set of data, yielding higher average returns than the Vanguard S&P 500 Index Fund. The mean-variance efficient portfolio yielded a slightly lower daily average (0.10%) than its target average (0.11%).

Varying the forgetting factor λ did not seem to improve performance much, and in some cases made it worse. Making the forgetting factor too small causes the mean-variance efficient portfolio to rely too heavily on the recent data.

6 Conclusions

Universal and mean-variance efficient portfolios are two possible investment strategies. The universal portfolio attempts to track the best constant rebalanced portfolio, chosen after future asset returns are revealed. The mean-variance efficient portfolio attempts to minimize the variance of portfolio returns while maintaining a desired average return. Both portfolios have their advantages and disadvantages, and the question of which portfolio is "better," is not obvious.

References

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[2] Z. Bodie, A. Kane, and A.J. Marcus. *Investments*. Irwin/Mcgraw-Hill, 4th edition, 1999. Chapter 9.

[3] T.M. Cover. Universal portfolios. *Mathematical Finance*, 1(1):1–29, January 1991.

[1] Then again, it is sometimes like deciding between steak and spam.

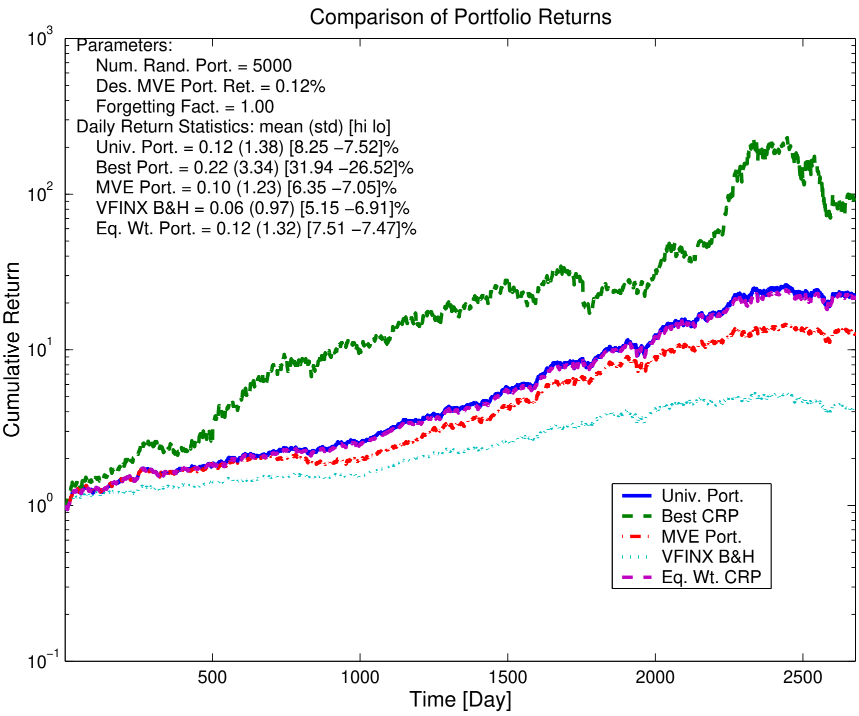

Figure 5: A comparison of universal portfolio, best CRP, mean-variance efficient portfolio, Vanguard S&P 500 Index Fund, and equally weighted CRP returns.

[4] T.M. Cover and E. Ordentlich. Universal portfolios with side information. *IEEE Transactions on Information Theory*, 42(2), March 1996.

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A Matlab Source Code

A.1 Universal Portfolio Algorithm

```
function [b_hat,b,S] = univ_port_rand(x,num_port)

%    desc:  calculates the universal porfolios for a MxN stock matrix x
%           for M assets by Monte Carlo averaging
%
%  syntax: [b_hat,b,S] = univ_port_rand(x,num_port)
%
%  inputs: x = MxN stock matrix for two assets over N periods;
%              x(:,n) = gross return for two assets during perdiod n
%          num_port = number of portfolios to average over
% outputs: b_hat = universal portfolio weights
%              b(:,n) = porfolio for the beginning of period n
%          b = portfolios averaged over
%              b(:,k) = k-th porfolio
%          S = cumulative returns of each portfolio
%              S(k,:) = returns of b(:,k)
%
%     ref: Kalai and Vempala, "Efficient Algorithms for Universal Portfolios"

%   prog:  Shane M. Haas
%   date:  Sept 2001

%  parameters
N = size(x,2);  % number of periods
M = size(x,1);  % number of stocks

% create uniformly random + B&H portfolios
b_rand = rand(M-1,num_port);
b_sort = sort(b_rand,1);
b = [b_sort(1,:); diff(b_sort); 1 - b_sort(end,:)];
b = [eye(M) b];
num_port = size(b,2);

% calculate returns of each portfolio
S = cumprod(b'*x,2);
```

```
S_norm = S./(ones(num_port,1)*sum(S,1));

% calculate universal portfolio
b_hat = b*S_norm;

% add initial equal weight portfolio
b_hat = [1/M*ones(M,1) b_hat(:,1:end-1)];
```

A.2 Mean-Variance Efficient Portfolio Algorithm

```
function [w_opt,var_opt,w0,w1] = mv_eff_port(cov_mat,mean_ret,des_ret)

%    desc:  Calculates the mean-variance efficient portfolio, i.e.
%
%                  minimize:  w'*cov_mat*w
%                subject to:  1) sum(w) = 1
%                             2) w'*mean_ret = des_ret
%
%    syntax: [w_opt,w0,w1] = mv_eff_port(cov_mat,mean_ret,desired_ret)
%
%    inputs: cov_mat = covariance matrix of stock returns (non-singular)
%            mean_ret = mean return of stocks
%            des_ret = desired return of portfolio
%
%   outputs: w_opt = optimal portfolio that minimizes return variance
%                    yet returns desired_ret on average
%            var_opt = return variance of w_opt portfolio
%            w0 = minimum variance portfolio that yields 0 expected return
%            w1 = minimum variance portfolio that yields 100% expected return
%
%     note:  w_opt = w0 + (w1 - w0)*desired_ret
%
%      ref:  Wang, "15.407 Finance Course Notes"
%            Bertsekas, "Nonlinear Programming," pg. 267

% constants
num_stocks = length(mean_ret);
num_port = length(des_ret);
one = ones(num_stocks,1);
inv_V = inv(cov_mat);

% make column vectors
mean_ret = mean_ret(:);
des_ret = des_ret(:);

% some auxillary variables
a1 = mean_ret'*inv_V*mean_ret;
a2 = mean_ret'*inv_V*one;
a3 = one'*inv_V*one;
D = a1*a3 - a2^2;

% calculate w0 and w1
w0 = 1/D*(a1*inv_V*one - a2*inv_V*mean_ret);
w1 = 1/D*(a1*inv_V*one - a2*inv_V*mean_ret + a3*inv_V*mean_ret - a2*inv_V*one);

% calculate optimal portfolios
w_opt = w0*ones(1,num_port) + ((w1 - w0)*ones(1,num_port)).*(ones(num_stocks,1)*des_ret');

% calculate variance of optimal portfolios
var_opt = diag(w_opt'*cov_mat*w_opt);
```

A.3 Mean-Variance Efficient Portfolio for Data

```
function [mv_port,mean_ret,cov_mat] = calc_mv_eff_port(stock_ret,des_ret,ff,data_skip)
```

```
%   desc:  Calculates the portfolio at time n based on the data
%          stock_ret(1:n-1) that tries to minimizes return variance while
%          returning des_ret on average.
%
%  syntax: mv_port = calc_mv_eff_port(stock_ret,des_ret,ff,data_skip)
%
%  inputs: stock_ret(:,n) = stock returns of the n-th period
%          des_ret(n) = desired return for the n-th period
%          ff = forgetting factor to exponentially discount past data
%               when calculating mean and covariance matrix
%          data_skip = index of period to calculate first portfolio
%                note: data_skip > num_stocks necessary for non-singular cov
%
% outputs: mv_port(:,n) = mean-var eff port for the n-th period

% constants
num_per = size(stock_ret,2);
num_stocks = size(stock_ret,1);

% initialize
mv_port = zeros(num_stocks,num_per);
mean_ret = zeros(num_stocks,num_per);
cov_mat = zeros(num_stocks,num_stocks,num_per);

if length(des_ret) == 1
   des_ret = des_ret*ones(1,num_per);
end

% calculate mean-variance efficient portfolio
h_wait = waitbar(0,'Calculating MVE Portfolio');

% intialize
mv_port(:,1:data_skip-1) = 1/num_stocks*ones(num_stocks,data_skip-1);
sign_ret = sign(prod(diag(mv_port(:,1:data_skip-1)'*stock_ret(:,1:data_skip-1))));

for n = data_skip:num_per

   if n == 273
      %keyboard
   end
   % select window of data
   dat_win = 1:n-1;

   % calculate forgetting factor matrix
   ff_vect = fliplr(ff.^[0:n-2]);
   sum_ff = sum(ff_vect);
   ff_mat = diag(ff_vect);

   % calculate mean and covariance matrix
   mean_ret(:,n) = 1/sum_ff*sum((ones(num_stocks,1)*ff_vect).*stock_ret(:,dat_win),2);
   no_mean_stock = stock_ret(:,dat_win) - mean_ret(:,n)*ones(1,n-1);
   cov_mat(:,:,n) = 1/sum_ff*no_mean_stock*ff_mat*no_mean_stock';

   % calculate mean-var efficient portfolio
   mv_port(:,n) = mv_eff_port(squeeze(cov_mat(:,:,n)),mean_ret(:,n),sign_ret*des_ret(n));

   % if cumulative return is negative, want negative des_ ret
   sign_ret = sign_ret*sign(mv_port(:,n)'*stock_ret(:,n));

   if sign_ret < 0
      %keyboard
   end
   waitbar(n/num_per,h_wait)

end
close(h_wait)
```