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Space-Time Block Coding

To improve the performance of a wireless transmission system in which the channel quality fluctuates, researchers suggested that the receiver be provided with multiple received signals generated by the same underlying data. These suggestions are refered to as diversity which exists in different forms including temporal diversity, frequency diversity, and antenna diversity.

Temporal diversity includes channel coding in conjunction with time interleaving which involve redundancy in time domain. Frequency diversity refers to transmission on different frequencies which provides redundancy in frequency domain. Antenna diversity can be viewed as redundancy in spatial domain and implemented by using multiple antennae at both the transmit side (base station) and the receive side (mobile units).

Space-time coding refers to channel coding techniques for transmission with multiple transmit and receive antennae. This summary discusses the work in [TSC98] and [TJC99] which contribute to understanding systems with multiple transmit antennae. The following parameters describe a simple analytical model in [TSC98] and [TJC99] on which we shall concentrate.

- n: number of transmit antennae.
- m: number of receive antennae.
- $\alpha_{i,j}$: path gain from transmitter i to receiver, $1 \leq i \leq n$, $1 \leq j \leq m$. Assume they are independent and have Gaussian distribution with mean zero and variance 1/2 per (real) dimension.
- l: length of block codes.
- c_t^i: transmitted signal at time t by transmit antenna i, $1 \leq t \leq l$, $1 \leq i \leq n$.
- r_t^j: received signal at time t by receive antenna j, $1 \leq t \leq l$, $1 \leq j \leq m$.
- η_t^j: additive white Gaussian noise with mean zero and variance $1/SNR$ per dimension.

We shall concentrate on the analysis based on the assumption of block fading and perfect channel information, i.e. $\alpha_{i,j}$'s are fixed throughout the block length l and are known to the receiver.

A codeword $\mathbf{c}$ in a block code is described by a vector $\mathbf{c} = (c_1^1, ..., c_l^1, c_1^2, ..., c_l^2, ..., c_1^n, ..., c_l^n)$. At time t, the receiver obtains

$$r_t^j = \sum_{i=1}^{n} \alpha_{i,j} c_t^i + \eta_t^j,$$

and select the codeword that minimizes the distance

$$\sum_{t=1}^{l} \sum_{j=1}^{m} \left| r_t^j - \sum_{i=1}^{n} \alpha_{i,j} c_t^i \right|^2 .$$

The pairwise probability of incorrectly decoding $\mathbf{e}$ when $\mathbf{c}$ is transmitted is denoted by $P(\mathbf{c} \to \mathbf{e})$. Given a particular set of $\alpha_{i,j}$'s, using the standard bound $\mathcal{Q}(x) \leq \exp(-x^2/2)$,

$$\begin{aligned} P(\mathbf{c} \to \mathbf{e} | \alpha_{i,j}) &\leq \exp[-d^2(\mathbf{c}, \mathbf{e}) SNR/2], \quad \text{where} \\ d^2(\mathbf{c}, \mathbf{e}) &= \sum_{j=1}^{m} \sum_{t=1}^{l} \left| \sum_{i=1}^{n} \alpha_{i,j} c_t^i - \sum_{i=1}^{n} \alpha_{i,j} e_t^i \right|^2 . \end{aligned}$$

Define $\Omega_j = (\alpha_{i,j}, ..., \alpha_{n,j})$ and $A_{i,i'} = (c_1^i - e_1^i, ..., c_l^i - e_l^i) \cdot (c_1^{i'} - e_1^{i'}, ..., c_l^{i'} - e_l^{i'})$. Then it follows that $d^2(\mathbf{c}, \mathbf{e}) = \sum_{j=1}^{m} \Omega_j A \Omega_j^*$.[1]

Let $\lambda_1, ..., \lambda_n$ denote (possibly zero) eigenvalues of A. Since A is Hermatian, λ_i's are real and nonnegative. Let r be the rank of A. By averaging the upper bound on $P(\mathbf{c} \to \mathbf{e} | \alpha_{i,j})$ and with some algebra,

$$P(\mathbf{c} \to \mathbf{e}) \leq \left(\prod_{i=1}^{r} \lambda_i \right)^{-m} \left(\frac{SNR}{2} \right)^{-rm} .$$

The exponent rm is refered to as diversity advantage or diversity order, while the product $(\prod_{i=1}^{r} \lambda_i)^{1/r}$ is refered to as coding advantage. Define

$$B(\mathbf{c}, \mathbf{e}) = \begin{bmatrix} e_1^1 - c_1^1 & e_2^1 - c_2^1 & \cdots & e_l^1 - c_l^1 \\ e_1^2 - c_1^2 & e_2^2 - c_2^2 & \cdots & e_l^2 - c_l^2 \\ \vdots & \vdots & & \vdots \\ e_1^n - c_1^n & e_2^n - c_2^n & \cdots & e_l^n - c_l^n \end{bmatrix} .$$

It follows that $B(\mathbf{c}, \mathbf{e})$ is a square root matrix of $A(\mathbf{c}, \mathbf{e})$, i.e. $A = BB^*$, and their ranks are equal. The above upper bound expression for $P(\mathbf{c} \to \mathbf{e})$ gives rise to two criteria for space-time coding.

[1] A^* denotes the conjugate transpose of A.

- The rank criterion: for maximum diversity order nm, $B(\mathbf{c}, \mathbf{e})$ has to be full rank for all pairs of codewords $\mathbf{c}$ and $\mathbf{e}$.
- The coding advantage criterion: given diversity order rm, try to maximize the minimum of $(\prod_{i=1}^{r} \lambda_i)$ over all pairs of codewords.

We shall concentrate on the rank criterion for space-time block code designs using results from the subject of orthogonal designs.

Real Orthogonal Designs

A real orthogonal design of size n is an $n \times n$ orthogonal matrix whose rows are permutations of real numbers $\pm x_1, ..., \pm x_n$. Without loss of generality, the first row can be assigned as $(x_1, ..., x_n)$. For example, real orthogonal designs for $n = 2$ and $n = 4$ are

$$\begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix}, \qquad \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2 & x_1 & -x_4 & x_3 \\ -x_3 & x_4 & x_1 & -x_2 \\ -x_4 & -x_3 & x_2 & x_1 \end{bmatrix}.$$

The existence of real orthogonal designs for different values of n is known as the Hurwitz-Radon problem in mathematics. It was shown that real orthogonal designs exist if and only if $n = 2, 4$, or 8.

A space-time block code based on real orthogonal designs of size n ($n = 2, 4$, or 8) can be constructed as follow. The encoder takes in a block of nb bits. For each i, $1 \leq i \leq n$, the encoder select a symbol s_i from a real constellation $\mathcal{A}$ of size 2^b. The encoder then use $s_1, ..., s_n$ to build an orthogonal matrix $\mathcal{O}(s_1, ..., s_n)$ based on real orthogonal designs of size n. At time t, the n antennae transmit the t^{th} row of $\mathcal{O}(s_1, ..., s_n)$.

Due to orthogonality, the diversity order of this code is nm. The rate of is code is b bits/s/Hz, which is shown to be optimal under the diversity order nm for the constellation size 2^b.

Orthogonality also simplifies the minimum-distance decoding rule. Let $\delta_k(i)$ be the sign of x_i in the k^{th} row of $\mathcal{O}$, and $\epsilon_k(p)$ be the column in row p where $\pm x_k$ is. It follows that the receiver only needs to combine received signals linearly

$$R_i = \sum_{t=1}^{n}\sum_{j=1}^{m} r_t^j \alpha_{\epsilon_t(i),j}\delta_t(i), \quad 1 \le i \le n,$$

and decide on s_i such that

$$s_i = \arg\min_{s\in\mathcal{A}} |R_i - s|^2 + \left(-1 + \sum_{k,l} |\alpha_{k.l}|^2\right) |s|^2.$$

However, real orthogonal designs only exist for $n = 2, 4$, or 8. By developing generalized real orthogonal designs, space-time block codes for other numbers of antennae can be obtained.

Before discussing generalized real orthogonal designs, it is worth mentioning that a more immediate generalization is linear processing orthogonal designs in which entries in orthogonal matrices $\mathcal{O}(x_1, ..., x_n)$ are linear combinations of $x_1, ..., x_n$ (instead of permutations of $\pm x_1, ..., \pm x_n$ as before). However, it is shown that allowing linear processing of $x_1, ..., x_n$ does not increase the values of n such that orthogonal designs exist.

Generalized Real Orthogonal Designs

A generalized real orthogonal design of size n is an $p \times n$ matrix $\mathcal{G}$ with entries $0, \pm x_1, ..., \pm x_k$ such that $\mathcal{G}^*\mathcal{G} = D$, a $p \times p$ diagonal matrix whose i^{th} diagonal entry is of the form $l_1^i x_1^2 + ... + l_k^i x_k^2$ with $l_1^i = ... = l_k^i$ being strictly positive integers. However, without loss of generality, one can assume that $\mathcal{G}^*\mathcal{G} = I(x_1^2 + ... + x_k^2)$.

A space-time block code based on generalized real orthogonal designs of size n can be constructed as follow. The encoder takes in a block of kb bits. For each i, $1 \le i \le k$, the encoder select a symbol s_i from a real constellation $\mathcal{A}$ of size 2^b. The encoder then use $s_1, ..., s_k$ to build matrix $\mathcal{G}(s_1, ..., s_k)$ based on generalized real orthogonal designs of size n.

The result diversity order is nm. Since the maximum rate is b bits/s/Hz and the block length is p, the rate of this code is defined as ratio between the actual transmitted bits and the maximum transmitted bits, which is $(kb)/(pb) = k/p$.

Define $A(R, n)$ as the minimum p such that there exists a $p \times n$ generalized orthogonal design with rate at least R. (If no such p exists, assign $A(R, n) = \infty$.) Finding the values of $A(R, n)$ is considered the fundamental question of generalized orthogonal design theory. The most interesting case is finding $A(1,n)$ which involves codes with optimal rates.

It is shown that for any R, $A(R,n) < \infty$. In particular, for $R = 1$,

$$\begin{aligned} A(1,n) &= \min_{(c,d)\in\mathcal{U}} 2^{4c+d}, \quad \text{where} \\ \mathcal{U} &= \{(c,d) : 0 \le c, 0 \le d \le 4, 8c + 2^d \ge n\}. \end{aligned}$$

Therefore, we can construct space-time coding for any number of transmit antennae. However, for values of n other than 2, 4, or 8, we do not have the block length p equal to the number of antennae n. We next describe the code construction in details.

The code construction is based on a Hurwitz-Radon family of matrices. A set of $n \times n$ real matrices $\{B_1, ..., B_k\}$ is called a size k Hurwitz-Radon family of matrices if

$$\begin{aligned} &B_i^* B_i = I, \; B_i^* = -B_i, \quad 1 \le i \le k, \\ &B_i B_j = -B_j B_i, \qquad\qquad 1 \le i, j \le k. \end{aligned}$$

For any $n = 2^a b$, where b is odd and $a = 4c + d$ with $0 \le d < 4$ and $0 \le c$, Radon showed that any Hurwitz-Radon family of matrices contains less than $\rho(n) = 8c + 2^d \le n$ matrices. Furthermore, for any n, by explicit construction, there exists a Hurwitz-Radon family of matrices with $\rho(n) - 1$ members which are all integer matrices (all entries are 0 or ± 1).

To construct the space-time block code of length p. Choose a Hurwitz-Radon family of integer matrices with $\rho(p) - 1$ members $\{A_1, A_2, ..., A_{\rho(p)-1}\}$. Let $A_0 = I$ and denote $X = (x_1, ..., x_p)$. We can construct a $p \times n$ generalized real orthogonal design $\mathcal{G}$ by setting the j^{th} column of $\mathcal{G}$ to be $A_{j-1} X^*$. It follows that $\mathcal{G}$ has full rank and thus yields diversity order nm as desired.

Using this method, generalized real orthogonal designs of size 3, 5, 6, and 7 are explicted constructed. Therefore, block codes for any number of antennae between 2 and 8 are available.

Examples of orthogonal designs of size 3 and 5 are given below.

$$\mathcal{G}_3 = \begin{bmatrix} x_1 & x_2 & x_3 \\ -x_2 & x_1 & -x_4 \\ -x_3 & x_4 & x_1 \\ -x_4 & x_3 & x_2 \end{bmatrix}, \quad \mathcal{G}_5 = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ -x_2 & x_1 & x_4 & -x_3 & x_6 \\ -x_3 & -x_4 & x_1 & x_2 & x_7 \\ -x_4 & x_3 & -x_2 & x_1 & x_8 \\ -x_5 & -x_6 & -x_7 & -x_8 & x_1 \\ -x_6 & x_5 & -x_8 & x_7 & -x_2 \\ -x_7 & x_8 & x_5 & -x_6 & -x_3 \\ -x_8 & -x_7 & x_6 & x_5 & -x_4 \end{bmatrix}.$$

Complex Orthogonal Designs

We shall end by briefly mentioning key results in complex orthogonal designs. A complex orthogonal design $\mathcal{O}_c$ of size n is an orthogonal matrix whose rows are permutations of $\pm x_1, ..., \pm x_n$, their conjugates $\pm x_1^*, ..., \pm x_n^*$, or multiples of these indeterminates by $\pm\sqrt{-1}$.

It is shown that complex orthogonal designs exist if and only if $n = 2$. Again, allowing linear processing of transmit signals does not increase the set of n for which the designs exist.

To extend the set of n for complex orthogonal designs, generalized complex orthogonal designs are defined in an analogous fashion as generalized real orthogonal designs. However, block codes are guaranteed to exist for any value of n only for rate $R \leq 1/2$. For rate $R > 1/2$, by allowing linear processing of transmit signals, block codes of rate $3/4$ for $n = 3$ and 4 are shown to exist by explicit construction. The problem of complex orthogonal designs of rate $R > 1/2$ is still not well understood.

References

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[TJC99] V. Tarokh, H. Jafarkhani, and A.R. Calderbank, "Space-time block codes for orthogonal designs," *IEEE Transactions on Information Theory*, Vol.45, No.5, July 1999, pp.1456-

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