ESSENTIALS OF INTRODUCTORY CLASSICAL MECHANICS

Sixth Edition

CHAPTER 1

SPACE, TIME AND SCIENCE

OVERVIEW

In this chapter we discuss the scientific approach to the study of natural phenomena and introduce the fundamental concepts of space and time. We will see how the Euclidean nature of space and the absolute scale of time allow us to construct mathematical relations between time of observation and position of the observed object in space which can be used to describe and predict the motion of a particle in simple situations.

When you have completed this chapter you should:

- ✔ appreciate the difference between vector and scalar quantities and be able to manipulate vector quantities in equations;
- ✔ understand the concepts of position, distance, time, speed, velocity, and acceleration and their mathematical interrelations;
- ✔ understand what is meant by the dimensions of a physical quantity, and know how to check the dimensional consistency of an equation;
- ✔ understand how to interpret the units of a physical quantity, and know how to convert units when necessary;
- ✔ be able to use what you have learned in describing and predicting the motion of a point particle, including uniform circular motion and the motion of a particle with constant acceleration, such as a projectile.

ESSENTIALS

Physics is an attempt to understand and predict natural phenomena using the scientific method. The scientific method uses the results of observation and/or experiment to construct theories which can be used to predict the results of further experiments. Supplementary Notes

To formalize the results of observation and experiment we need to introduce simple concepts or quantities which can be expressed numerically and represented by mathematical symbols. Predictive theories consist of relations between these mathematical symbols, i.e. mathematical equations. The most important step in constructing a good theory is choosing the right basic concepts. Supplementary Notes.

The first two fundamental concepts that are needed for classical mechanics are ***time***, and ***position in space***. These are necessary to identify a particular event we have observed. It is a basic assumption of classical mechanics that we can in principle specify the position of an object at a given time to arbitrarily high precision. Supplementary Notes.

Time is said to be ***absolute*** if the passage of time is unaffected by the position or motion of the observer. Time is found to be absolute to a very good approximation. The basic SI unit of time is the ***second***. (SI stands for ***Système International***, the standard international system of metric units.) The second was originally defined as 1/86400 of the Earth's rotational period (one day), but this was not precise enough for modern laboratory measurements, so it has been redefined by specifying that the period of the radiation emitted in the transition between the two lowest energy levels of the ^{133}Ce atom is (1/9,192,631,770) s. This is the basis of atomic clocks.

Space is said to be ***Euclidean*** if it obeys all the axioms of Euclidean geometry, and consequently also obeys the Pythagorean theorem. To a very good approximation space is found to be Euclidean. Space is said to be ***three-dimensional***, meaning that three numbers are necessary to identify a point in space unambiguously. The basic SI unit for measuring distances in space is the ***meter***, which is defined by specifying that the speed of light in a vacuum is precisely 299,792,458 m/s. (This may seem an odd way to define a standard length, but experimentally it turns out that this definition can be implemented with greater precision than the definition based on a standard platinum-iridium bar, adopted in 1889, or the 1960 definition based on the wavelength of krypton-86 radiation. The current definition was adopted in 1983, but the platinum-iridium bar from 1889 is still preserved by the International Bureau of Weights and Measures.) Supplementary Notes.

Absolute time and Euclidean space are the basic concepts used to construct classical mechanics. Both of these idealizations are extraordinarily accurate under normal everyday circumstances, and both seem so obvious that it often seems pedantic to discuss them; it is hard to imagine how they could possibly be violated. Nonetheless, since the early part of the twentieth century physicists have become convinced that neither of these idealizations is a completely accurate picture of reality.

The violation of the principle of absolute time was introduced by Albert Einstein in his theory of special relativity (1905). Specifically, the theory proposed, and experiments have confirmed, that all clocks slow down if they travel at high speeds, comparable to that of light (3.0×10^8 m/s). Since all clocks slow in exactly the same way, it is fair to say that time itself slows down for high speed observers. While it is difficult to accelerate a wristwatch to near-light speeds, the effect can be seen easily by using unstable subatomic particles as clocks. Particles called muons, for example, decay with a half-life of 1.5×10^{-6} second when they are at rest. Muons produced in the upper atmosphere by cosmic ray collisions, however, typically travel at 99.9% of the speed of light, and are found to have a half-life roughly 20 times longer than the value for stationary muons. Clocks also run slower in very strong gravitational fields, such as those in the vicinity of black holes, and Einstein's theory of general relativity (1916) tells us that in such conditions space is not Euclidean, either. Indeed, on sufficiently large scales the whole universe may be non-Euclidean.

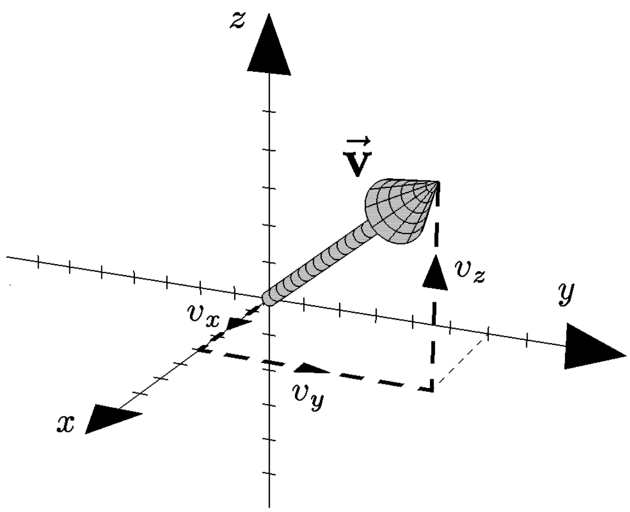
Finally, the assumption that we can in principle specify the position and velocity of an object to arbitrarily high precision breaks down at the atomic scale, where quantum mechanics must be used. Our understanding of the physics of atoms relies on the proposition that the trajectory of an electron in an atom is not only unknown, but cannot even be defined. The mathematical framework of quantum mechanics describes an electron that truly behaves as if it is in many places at once.

Despite these failures in extreme conditions, the assumptions of classical mechanics remain "true" in everyday life—quantum and relativistic effects are unmeasurably small. The laws and techniques we introduce in this book are therefore applicable to a wide range of real-life situations.

The main goal of classical mechanics is to understand motion: why do objects move in the way they do, and how can we predict their motion? Before we can discuss the underlying causes of motion, however, we have to first develop the mathematical machinery needed to simply ***describe*** how things move. Chapter 1 will be devoted to the description of motion, a subtopic of classical mechanics known as ***kinematics***. The study of forces and why things move the way they do, which will be the subject of much of the remainder of the book, is known as ***dynamics***.

To specify position in space, relative to a chosen origin, we need both a distance and a direction. The mathematical entities possessing both magnitude (size) and direction are called ***vectors***. Positions in space are therefore represented mathematically by vectors. Problems 1B

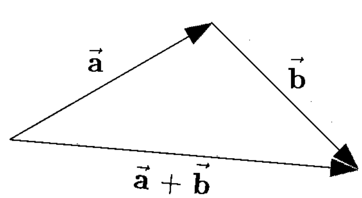
To emphasize the difference between a vector and a number, in this book we will denote vectors by boldfaced symbols with an arrow on top, such as $\vec{\mathbf{v}}$. The simplest way to specify a vector $\vec{\mathbf{v}}$ in three dimensions is to choose a coordinate system and then give the components of the vector along the x, y, and z axes, as shown in the diagram:



In this book we will express vectors explicitly by writing their three components in brackets:

$$\boxed{\vec{\mathbf{v}} \equiv [v_x, v_y, v_z]\ .}$$

The operation of ***vector addition*** can be defined graphically by placing the tail of the second vector on the head of the first. The sum is then the vector that extends from the tail of the first to the head of the second:



In component language, one simply adds the components:

$$\begin{aligned}\vec{\mathbf{a}} + \vec{\mathbf{b}} &= [a_x, a_y, a_z] + [b_x, b_y, b_z] \\ &= [a_x + b_x, a_y + b_y, a_z + b_z] \ .\end{aligned}$$

The negative of a vector is defined by negating all the components, or equivalently by reversing the direction and leaving the magnitude fixed. One subtracts a vector by adding its negative.

In many textbooks you will find $\vec{\mathbf{v}}$ written using vector addition as

$$\vec{\mathbf{v}} \equiv v_x \hat{\boldsymbol{x}} + v_y \hat{\boldsymbol{y}} + v_z \hat{\boldsymbol{z}} \ ,$$

where $\hat{\boldsymbol{x}}$ represents the ***unit vector*** (vector of magnitude 1) in the x-direction, and so on, or as

$$\vec{\mathbf{v}} \equiv v_x \hat{\boldsymbol{\imath}} + v_y \hat{\boldsymbol{\jmath}} + v_z \hat{\boldsymbol{k}} \ ,$$

where $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ are just a different notation for $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$.

The graphical picture of vector addition makes it clear that ***vectors do not have a definite position in space***. The vector $\vec{\mathbf{b}}$, moved so that its tail lies on the head of $\vec{\mathbf{a}}$, has not become a different vector because it has moved from its original position. A vector is defined by its magnitude and its direction, not its location.

Many physical quantities, such as mass, time, or temperature, have only magnitude and not direction (except perhaps for a plus (+) or minus (−) sign). These quantities are called ***scalars***, and each can be represented by a single ordinary number and manipulated according to the familiar rules of algebra and arithmetic. The magnitude of a vector, $v = |\vec{\mathbf{v}}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$, is an example of a scalar. Each component of a vector is represented by a single, ordinary number, but technically speaking they are not scalars, since they depend on the direction in which the coordinate system is oriented.

Multiplication of a vector by a positive scalar is defined as the multiplication of the vector's magnitude by the scalar, leaving the direction unchanged. Multiplication of a vector by a negative scalar results in a vector of the opposite direction, with the magnitude given by the product of the original magnitude and the absolute value of the scalar. In terms of components, we multiply a vector $\vec{\mathbf{v}}$ by a scalar s by simply multiplying each component by s:

$$s\,[v_x, v_y, v_z] = [s\,v_x, s\,v_y, s\,v_z] \ .$$

To divide a vector by a scalar s, one divides each component by s. There are two standard methods of multiplying a vector by a vector, known as the dot product and cross product, which we will introduce when we need them in Chapters 4 and 9, respectively. Please note that other operations are not defined—don't add a scalar to a vector, and don't try to divide by a vector!

Position, velocity, and acceleration are all vectors, while time is a scalar.

The ***position vector*** of one point in space relative to another is often called the ***displacement***. The magnitude of the displacement is the ***distance*** between the points. Problems 1B

Velocity is the rate of change of position with time.

Acceleration is the rate of change of velocity with time. Problems 1A

Rates of change are represented mathematically by ***derivatives***. The derivative of a vector with respect to a scalar is a vector, so velocity and acceleration are both vectors. If the displacement vector of an object is given as a function of time by $\vec{\mathbf{r}}(t)$, then the velocity of the object is defined by

$$\boxed{\vec{\mathbf{v}}(t) \equiv \frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t} \equiv \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t} \ .}$$

This is equivalent to differentiating each component of the vector independently. That is, if $\vec{\mathbf{r}}(t) \equiv [x(t), y(t), z(t)]$, then

$$\vec{\mathbf{v}}(t) = \left[\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right] \ .$$

(Note that this is different from differentiating the magnitude of $\vec{\mathbf{r}}(t)$, which is NOT the right way to find the velocity.) The acceleration of the object is defined by

$$\boxed{\vec{\mathbf{a}}(t) \equiv \frac{\mathrm{d}\vec{\mathbf{v}}(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2\vec{\mathbf{r}}(t)}{\mathrm{d}t^2} \ ,}$$

or in component form

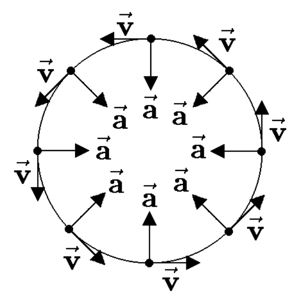
$$\vec{\mathbf{a}}(t) = \left[\frac{\mathrm{d}v_x}{\mathrm{d}t}, \frac{\mathrm{d}v_y}{\mathrm{d}t}, \frac{\mathrm{d}v_z}{\mathrm{d}t}\right] = \left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \frac{\mathrm{d}^2y}{\mathrm{d}t^2}, \frac{\mathrm{d}^2z}{\mathrm{d}t^2}\right] \ .$$

The magnitude of the velocity is called the ***speed***. Note that ***distance*** and ***speed*** are both scalars, as they are the magnitudes of the vectors ***displacement*** and ***velocity***, respectively.

Note that the dimensions of velocity are [length]/[time], and of acceleration [length]/[time]^2. In solving problems it is good practice to check that the dimensions of the result are what you expect them to be. The product of an acceleration and a time, for example, will always have dimensions [length]/[time], the dimensions of a velocity. Therefore, if you find that a distance is calculated as the product of an acceleration and a time, then you would know that you made an algebraic error.

Where a numerical value is attached to a physical quantity, we must consider not only its dimensions, but also its ***units***: a speed of 1 m/s is not the same as 1 mile per hour. A general method for converting units is demonstrated in the solution to Problem 1A.4. Problem 1A.4

A change in the velocity of an object does not necessarily require a change in speed. A good example of this is ***uniform circular motion***. A particle moving with constant speed v in a circle of radius r does not have constant velocity (because the ***direction*** of the velocity is changing). In fact the particle has an acceleration of constant magnitude Problems 1D



$$|\vec{\mathbf{a}}| = \frac{v^2}{r}$$

directed towards the center of the circle. This is called ***centripetal acceleration*** (where the word "centripetal" means "pointing towards the center"). The eight dots in the diagram at the right represent the same particle at different times, as it travels around the circle at constant speed.

In the case where the acceleration is ***constant***, the velocity $\vec{\mathbf{v}}$ and position $\vec{\mathbf{r}}$ of a body at time t are given by

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}_0 + \vec{\mathbf{a}}\,t$$

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}_0 + \vec{\mathbf{v}}_0\,t + \frac{1}{2}\vec{\mathbf{a}}\,t^2 \ ,$$

where $\vec{\mathbf{v}}_0$ and $\vec{\mathbf{r}}_0$ are its velocity and position at time $t = 0$. These equations can be used in ***any*** case of constant acceleration. They are especially easy to use if a body is confined to one dimension (i.e. can move only back and forth along a given line). In this situation there is no practical difference between vectors and ordinary numbers, as the concept of direction reduces to a plus or minus sign (backwards or forwards along the line). In this context the above equations reduce to

$$v^2 = v_0^2 + 2a(x - x_0) \ .$$

Problems 1A

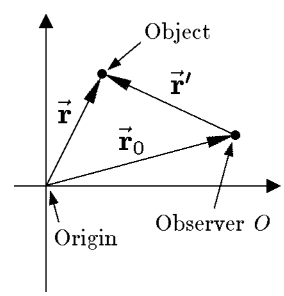
To solve problems involving more than one dimension, it is often helpful to decompose the vector quantities into their components in a convenient coordinate system. In the simplest cases each component equation forms a ***separate*** one-dimensional problem.

Problems 1C

An important example in which the component equations can be treated separately is the motion of freely falling bodies. If we neglect air resistance (as we will until Chapter 6), then near the Earth's surface a freely falling body has a constant acceleration $\vec{\mathbf{g}}$ directed downwards. The magnitude g of $\vec{\mathbf{g}}$, to two significant figures, is 9.8 $\mathrm{m/s^2}$. If the falling body initially has no horizontal velocity component, then the motion is purely vertical and hence one-dimensional. If the object is not simply dropped, but instead is launched with velocity $\vec{\mathbf{v}}$ at some angle to $\vec{\mathbf{g}}$, then we call it a ***projectile***, and we decompose the motion into horizontal and vertical components. The horizontal acceleration is zero, and therefore the horizontal velocity is constant. The vertical component of the motion behaves exactly as in the previous case, when there was no horizontal velocity. Projectile motion is two-dimensional.

Problems 1C.2, 1C.3, and 1C.4

In many cases it is useful to consider the position or velocity of an object as seen by an observer who is not located at the origin of the coordinate system, and/or is not stationary. If an object is at position $\vec{\mathbf{r}}$, then its position vector $\vec{\mathbf{r}}'$ ***relative*** to an observer O at position $\vec{\mathbf{r}}_0$ is



$$\vec{\mathbf{r}}' = \vec{\mathbf{r}} - \vec{\mathbf{r}}_0 \ .$$

Similarly, if the velocity of an object is $\vec{\mathbf{v}}$, then its velocity ***relative*** to an observer O whose velocity is $\vec{\mathbf{v}}_0$ is

$$\vec{\mathbf{v}}' = \vec{\mathbf{v}} - \vec{\mathbf{v}}_0 \ .$$

Problems 1E

One can also introduce a new coordinate system—also called a new ***frame of reference***—with the observer O as the origin. The position and velocity of the object in this new frame of reference are $\vec{\mathbf{r}}'$ and $\vec{\mathbf{v}}'$, respectively, as calculated from the formulas above.

Vectors (or scalars) measured with respect to a particular observer, stationary or moving, are said to be measured in a particular ***frame of reference***. Changing frames of reference—i.e., changing the point of view from which you observe the motion—can often be a useful tool in solving problems.

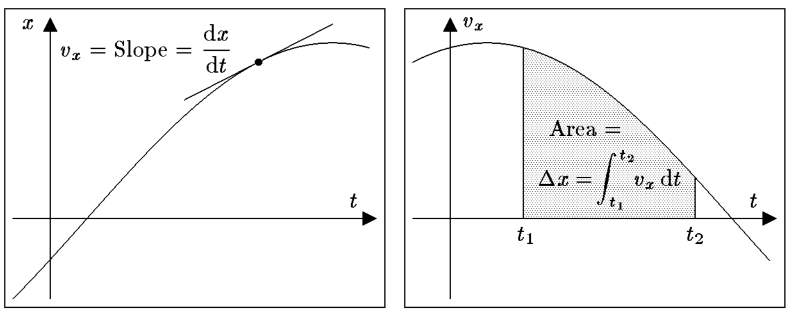
Since velocity is the ***derivative*** of position with respect to time, it follows that the position at a given time t_1 relative to the position at time $t = 0$ can be calculated by ***integrating*** the velocity:

$$\vec{\mathbf{r}}(t_1) = \vec{\mathbf{r}}_0 + \int_0^{t_1} \vec{\mathbf{v}}\, dt \ ,$$

where $\vec{\mathbf{r}}_0$ is the position at time $t = 0$. The velocity $\vec{\mathbf{v}}(t_1)$ can similarly be obtained by integrating the acceleration $\vec{\mathbf{a}}$:

$$\vec{\mathbf{v}}(t_1) = \vec{\mathbf{v}}_0 + \int_0^{t_1} \vec{\mathbf{a}}\,\mathrm{d}t \ .$$

The x-component of the velocity v_x is the derivative of x with respect to t, which can be displayed graphically on a plot of x versus t, as shown below on the left. For any time t, v_x is the slope of the line tangent to the curve at the specified value of t. The change Δx in the x-component of the displacement during the time interval between t_1 and t_2 is the area under the graph of v_x versus t, as shown on the graph below on the right. Analogous relations hold for the y- and z-components.



Problem 1A.6

The phrase ***average velocity*** requires definition, since the method of averaging must be specified. You will usually find a different answer if you average over distance than if you average over time. Most physics books, including this one, define average velocity to mean the average over ***time***:

Problem 1A.3

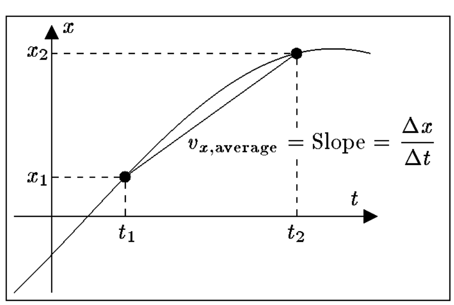
$$v_{x,\text{average}} \equiv \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} v_x(t)\,\mathrm{d}t \ .$$

Since the integral of the derivative $v_x(t) \equiv \mathrm{d}x/\mathrm{d}t$ is the original function $x(t)$, the average velocity can be written simply as

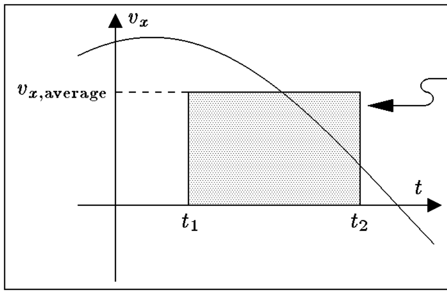
$$v_{x,\text{average}} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{\Delta x}{\Delta t} \ ,$$

Problem 1A.4

as illustrated on the following graph:



Since Δx is equal to both $v_{x,\text{average}}\,\Delta t$ and also $\int_{t_1}^{t_2} v_x\,\mathrm{d}t$, it follows that the area under the average velocity graph is equal to the area under the graph of v_x versus t:



Shaded region has the same area as the shaded region in the graph above.

The relations in this paragraph apply to the y- and z-components as well, so they can be written as vector equations:

$$\vec{\mathbf{v}}_{\text{average}} \equiv \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \vec{\mathbf{v}}(t)\,\mathrm{d}t = \frac{\Delta\vec{\mathbf{r}}}{\Delta t}\ .$$

Similarly, the ***average acceleration*** is defined to be the average over ***time***, so

$$a_{x,\text{average}} \equiv \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} a_x(t)\,\mathrm{d}t\ .$$

Integrating the derivative $a_x(t) = \mathrm{d}v_x/\mathrm{d}t$, one finds

$$a_{x,\text{average}} = \frac{v_x(t_2) - v_x(t_1)}{t_2 - t_1} = \frac{\Delta v_x}{\Delta t}\ .$$ Problem 1A.7

The full vector equation can then be written as

$$\vec{\mathbf{a}}_{\text{average}} \equiv \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \vec{\mathbf{a}}(t)\,\mathrm{d}t = \frac{\Delta\vec{\mathbf{v}}}{\Delta t}\ .$$

SUMMARY

* Natural phenomena can be described in terms of basic quantifiable concepts.
* Mathematical equations relating these basic concepts can be used to predict the outcome of experiments.
* Physical concepts introduced in this chapter: space, time, velocity, acceleration, speed.
* Near the Earth's surface a freely falling body accelerates downwards with uniform acceleration $\vec{\mathbf{g}}$ (magnitude g). To two significant figures the value of g is 9.8 m/s^2.
* Mathematical concepts introduced in this chapter: vector, scalar, derivative, integral.

  A vector $\vec{\mathbf{v}}$ is represented in component form as $\vec{\mathbf{v}} = [v_x, v_y, v_z]$.

  Vector addition: $\vec{\mathbf{a}} + \vec{\mathbf{b}} = [a_x + b_x, a_y + b_y, a_z + b_z]$.

  Multiplication by scalar s: $s\vec{\mathbf{v}} = [s\,v_x, s\,v_y, s\,v_z]$.

  Differentiation with respect to scalar s: $\dfrac{\mathrm{d}\vec{\mathbf{v}}}{\mathrm{d}s} = \left[\dfrac{\mathrm{d}v_x}{\mathrm{d}s}, \dfrac{\mathrm{d}v_y}{\mathrm{d}s}, \dfrac{\mathrm{d}v_z}{\mathrm{d}s}\right]$.

  Position in space $\vec{\mathbf{r}}$, velocity $\vec{\mathbf{v}}$, and acceleration $\vec{\mathbf{a}}$ are vectors; time t, distance r and speed v are scalars.
* Equations introduced in this chapter:

$$\vec{\mathbf{v}} = \frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t}; \qquad \vec{\mathbf{a}} = \frac{\mathrm{d}\vec{\mathbf{v}}}{\mathrm{d}t} = \frac{\mathrm{d}^2\vec{\mathbf{r}}}{\mathrm{d}t^2}; \qquad \vec{\mathbf{r}}(t_1) = \vec{\mathbf{r}}_0 + \int_0^{t_1} \vec{\mathbf{v}}\,\mathrm{d}t; \qquad \vec{\mathbf{v}}(t_1) = \vec{\mathbf{v}}_0 + \int_0^{t_1} \vec{\mathbf{a}}\,\mathrm{d}t .$$

  For ***constant*** acceleration $\vec{\mathbf{a}}$, if $\vec{\mathbf{r}} = \vec{\mathbf{r}}_0$ and $\vec{\mathbf{v}} = \vec{\mathbf{v}}_0$ at time $t = 0$, then

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}_0 + \vec{\mathbf{a}}t$$

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}_0 + \vec{\mathbf{v}}_0 t + \frac{1}{2}\vec{\mathbf{a}}t^2 .$$

  For one-dimensional motion with constant acceleration a:

$$v^2 = v_0^2 + 2a(x - x_0) .$$

  For circular motion at constant speed v:

$$a = \frac{v^2}{r} ,$$

  where r is the radius of the circle, and the acceleration is directed towards the center of the circle.

  If an object has position $\vec{\mathbf{r}}$ and velocity $\vec{\mathbf{v}}$, its position and velocity relative to an observer with position $\vec{\mathbf{r}}_0$ and velocity $\vec{\mathbf{v}}_0$ are given respectively by

$$\vec{\mathbf{r}}' = \vec{\mathbf{r}} - \vec{\mathbf{r}}_0 , \qquad \vec{\mathbf{v}}' = \vec{\mathbf{v}} - \vec{\mathbf{v}}_0 .$$

  Average velocity and acceleration are given by

$$\vec{\mathbf{v}}_{\text{average}} = \frac{\Delta\vec{\mathbf{r}}}{\Delta t} , \qquad \vec{\mathbf{a}}_{\text{average}} = \frac{\Delta\vec{\mathbf{v}}}{\Delta t} .$$

PROBLEMS AND QUESTIONS

By the end of this chapter you should be able to answer or solve the types of questions or problems stated below.

Note: throughout the book, in multiple-choice problems, the answers have been rounded off to 2 significant figures, unless otherwise stated.

At the end of the chapter there are answers to all the problems. In addition, for problems with an (H) or (S) after the number, there are respectively hints on how to solve the problems or completely worked-out solutions.

1A VELOCITY AND ACCELERATION IN ONE DIMENSION

1A.1 An athlete runs at a uniform speed of 9.5 m/s. How long does it take him to run a distance of 200 m?

(a) 19 s; (b) 21 s; (c) 22 s; (d) none of these

1A.2 An auto accelerates uniformly from rest to 100 km/h in 8.0 s. What is its acceleration?

(a) 12.5 m/s^2; (b) 12.5 km/h; (c) 3.5 m/s; (d) 3.5 m/s^2

1A.3 An athlete runs 50 m along a straight track at a constant speed of 10 m/s. She then slows to 8 m/s for another 50 m.

(a) How long does it take her to run each segment?

(b) Plot (i) her position as a function of time; (ii) her velocity as a function of time; and (iii) her velocity as a function of distance.

(c) Over the complete 100 meters, what is her average velocity, averaged over time? What is her average velocity, averaged over distance?

1A.4 (S) (a) You have arranged to meet a friend at his home, which is five miles from yours. You drive there, averaging about 20 miles per hour as both of you live in the city. How long does it take you to reach your friend's home? How long would it take if you lived in a rural district where it was possible to average 45 mph?

(b) If you were driving in Europe and saw a sign saying 'Paris 120 km', how long would it take you to reach there if you were traveling on a French autoroute at 75 miles per hour?

(c) A good sprinter takes about ten seconds for the 100 meter sprint. What is his average speed in miles per hour? If he could maintain this speed indefinitely, how long would it take him to run a marathon (26 miles 385 yards)?

1A.5 (H) An object moves along the x-axis with constant acceleration a. Its position and velocity at time $t = 0$ are $x = x_0$ and $v = v_0$ respectively; at some later time t it has position x and velocity v. Use the definitions of velocity and acceleration to prove that

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

and

$$v^2 = v_0^2 + 2a(x - x_0) \ .$$

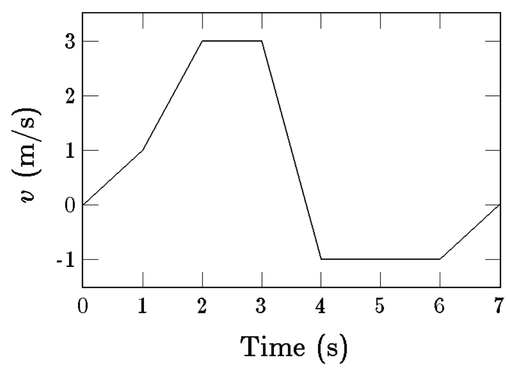
[It is very easy to prove this with calculus, but for constant acceleration you don't actually ***need*** calculus to derive these equations.]

1A.6 The graph shows the velocity of a particle (along the x-axis) as a function of time.

(a) When is the acceleration of the particle (i) positive, (ii) negative, (iii) zero?

(b) What is the particle's displacement after 3.5 s? After 7 s?

(c) Describe in words the motion of the particle.



1A.7 (S) Two cars race along a straight track for 1 km, starting from rest. The first accelerates at 4 m/s^2 for 10 s, then continues at constant velocity. The second accelerates at 5 m/s^2 for 5 s, then at 1.5 m/s^2 for 10 s, then at 0.5 m/s^2 for the rest of the race.

(a) How long does each car take to complete the race?

(b) Overall, what is the average acceleration and average velocity of each car?

(c) What is the average acceleration of each car after 15 s of the race?

(d) What distance has each car covered after 15 s of the race?

(e) Draw a graph of velocity and distance covered against time for each car.

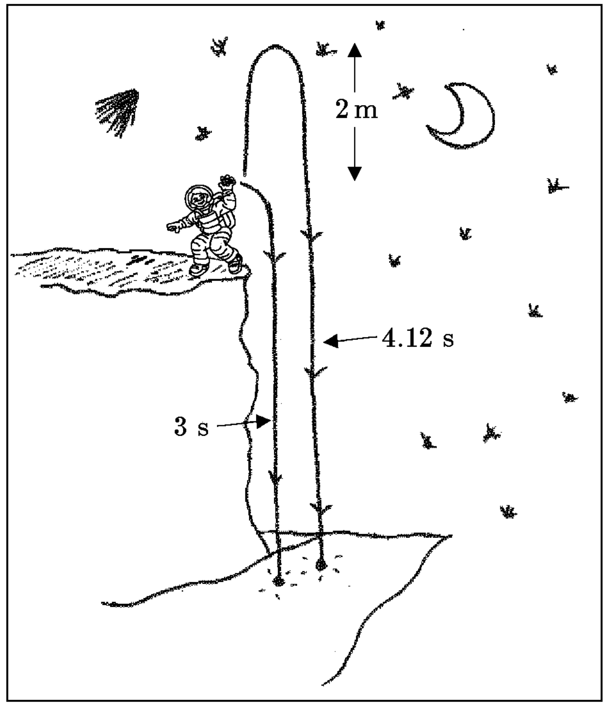
1A.8 (H) A bus is moving along a straight road 1 km long. Between stops it travels at the local speed limit, which is 40 km/h. Approaching a stop it decelerates at 1 m/s^2, stops for 30 s to let passengers on and off, and then accelerates at 1.5 m/s^2. There are two stops on this stretch of road, one at 200 m from the start and one at 650 m.

(a) How far before a stop does the bus start to decelerate?

(b) How long does it take the bus to complete this section of its route? What is its average velocity? Draw a graph of the bus's velocity against time.

(c) A cyclist doing 20 km/h entered the stretch of road at the same time as the bus. Draw a graph of position against time for both the cyclist and the bus. How often does the cyclist overtake the bus, and vice versa?

1A.9 (S) A crewman on the starship Enterprise is on shore leave on a distant planet. He drops a rock from the top of a cliff and observes that it takes 3.00 s to reach the bottom. He now throws another rock vertically upwards so that it reaches a height of 2.0 m before dropping down the cliff face. The second rock takes 4.12 s to reach the bottom of the cliff. The planet has a very thin atmosphere which offers negligible air resistance. How high is the cliff, and what is the value of g on this planet?



1B USING VECTORS

1B.1 You are standing 10 m south of a tree. A squirrel runs 6 m up the tree and then climbs out 4 m on an eastward-pointing branch. In a coordinate system where x is east, y north and z up, what are the coordinates of the squirrel ***relative to you***?

(a) [10, 6, 4] m; (b) [4, −10, 6] m; (c) [4, 10, 6] m; (d) [−10, 4, 6] m.

What is the distance between you and the squirrel, to the nearest meter?

(a) 20 m; (b) 14 m; (c) 12 m; (d) none of these

1B.2 (H) Albert, Betty, Carol, and Dave are playing frisbee in a square field whose sides happen to run due east and due north. Albert's position vector relative to one corner of the field is [10, 7, 0] m, where x is east and y north (z is up, but you can assume that the field is level).

(a) Betty is 14 m northeast of Albert, Carol is 10 m east of Betty, and Dave is 8 m south of Carol. What are the position vectors of Betty, Carol, and Dave? (Take the same corner of the field as origin for all position vectors.)

(b) How far is Albert from Carol?

(c) Dave's dog Ernie runs from Dave to Betty at 3 m/s. What is his velocity vector? How long does it take him to reach Betty? What is his position vector 4 s after leaving Dave?

(d) Make a scale drawing of the field, showing the positions of Albert, Betty, Carol, Dave, and Ernie 4 s after Ernie leaves Dave.

1C VELOCITY AND ACCELERATION AS VECTORS

1C.1 At a certain time, a particle has velocity [3, 1, −2] m/s. Its acceleration, which is constant, is [0, −0.6, 0] m/s^2. What is its speed after 10 s?

(a) 10 m/s; (b) 12 m/s; (c) 2 m/s; (d) none of these

What is the distance between its position at the start of the 10 s and at the end?

(a) 41 m; (b) 60 m; (c) 7 m; (d) 30 m

1C.2 (S) A cannonball emerges from a cannon with speed v, independent of the angle at which the barrel of the cannon is inclined. If the cannon is set on level ground, at what angle should the gunner set the cannon to maximize the range? What will the range be at this angle (in terms of v and g) and what height will the ball reach? Neglect any effects of air resistance.

1C.3 (H) A child is kicking a soccer ball in her backyard. If the ball leaves her foot with speed v_0 directed at an angle θ to the horizontal, derive expressions for the distance x that the ball travels and the height h that it reaches (assuming that it starts from $h = 0$, that the yard is level, and that air resistance is negligible).

(a) She kicks the ball with a speed of 8 m/s at an angle of 70° to the horizontal. How far from her does it hit the ground, and what maximum height does it reach? Take $g = 9.8$ m/s^2.

(b) She kicks the ball straight up and it reaches a height of 5 m. How far would it have gone horizontally if she had kicked it with the same speed, but at an angle of 45°? At what angle would she need to kick (again assuming the same speed) if she wants it to land a distance $x = 8$ m away? Draw the ball's trajectory for both possible answers. Can you do this problem if you do ***not*** know the value of g?

1C.4 (S) In a shooting contest, a clay pigeon is launched from the ground at a speed of 50 m/s at 60° to the horizontal, directed eastwards. The contestant is standing 100 m south of the clay's line of flight and 40 m east of the launch point. His gun has a muzzle velocity of 200 m/s and he hits the clay when it is directly ahead of him as he faces north. When did he fire, and at what elevation? (Neglect air resistance; take $g = 9.8$ m/s^2 and assume the shot is fired from a height of 1.6 m above ground level.)

1D CIRCULAR MOTION

1D.1 A car traveling at a steady 40 km/h negotiates a 60° left bend. The bend is an arc of a circle of radius 100 m. What is the magnitude of the acceleration of the car at any point in the bend?

(a) 1.2 m/s^2; (b) 0 m/s^2; (c) 1.6 m/s^2; (d) 1.4 m/s^2.

1D.2 (S) A motorcycle negotiates a 40° right-hand bend at 60 km/h. The bend consists of a 40° arc of a circle of radius 75 m. What is the centripetal acceleration of the bike at any point in the bend, and what is the total velocity change between entering the bend and leaving it?

1D.3 (a) A geosynchronous or geostationary satellite is so called because it takes 24 hours to complete one orbit. Such satellites orbit at a height of 35,800 km above the Earth's surface. What is the centripetal acceleration of geosynchronous satellites? [The radius of the Earth is 6,400 km.]

(b) What is the centripetal acceleration of a point on the Earth's equator, at sea level?

Compare your answers to both parts of the question with the value of g at sea level (9.8 m/s^2).

1E REFERENCE FRAMES

1E.1 (S) (a) A ferryboat crosses a river of width d. The speed of the boat (relative to the water) is v and the speed of the river current is V. Assuming that the landing point of the ferry is directly opposite its starting point, how long does it take for a round trip?

(b) In the Ferryman of the Year competition the ferryman is required to complete a course of the same distance $2d$ by rowing a distance d directly upstream and then back downstream to his starting point. (The distance is defined by posts on the river bank.) How long does it take him to complete the course, and is this longer or shorter than the round trip across the river?

(c) An invading army is approaching the river. The ferryman is anxious to get across to the other side as quickly as possible, without caring where he lands. How long does it take him, and where does he land?

1E.2 (S) The pilot of a light plane wishes to fly from Bristol to Edinburgh (625 km due north). Her cruising speed, measured relative to the air, is 150 km/h. There is a 20 km/h west wind blowing (where "west wind" refers to a wind blowing ***from*** the west).

(a) In what direction should she point her plane, and how long will her journey take? (Ignore the time spent in take-off and landing and assume that she flies at a constant altitude.)

(b) When she starts her return trip the wind has shifted to southwest and increased to 50 km/h. What heading should she take, and how long is the return journey?

1E.3 (H) You are in an airplane flying due west at 150 m/s. The plane has a glass window in the floor, through which you see a second airplane directly below you. It is flying northwest at 100 m/s. Both craft are maintaining a constant altitude.

What is the velocity (magnitude and direction) of the second plane relative to yours? Draw a sketch of the second airplane showing the direction of this relative velocity vector. Describe in words the apparent motion of the plane as you see it from your window.

COMPLETE SOLUTIONS TO PROBLEMS WITH AN (S)

1A.4 (a) *You have arranged to meet a friend at his home, which is five miles from yours. You drive there, averaging about 20 miles per hour as both of you live in the city. How long does it take you to reach your friend's home? How long would it take if you lived in a rural district where it was possible to average* 45 mph*?*

Conceptualize
In this problem we are given the distance that we need to travel, and the speed at which we are moving. We are asked to find the time that the trip will take.

What does the word "distance" mean in this context? To solve the problem, we have to assume that the distance of five miles quoted in the problem refers to the actual road distance that you travel. The road distance is usually longer than the value given by our formal definition of distance ("the magnitude of the displacement vector"), which would give the straight-line distance ("as the crow flies"). However, as long as one recognizes that the motion discussed in this problem is along a specified route, and that all distances are to be measured along that route, then the solution to this problem is essentially identical to the case of motion along a line. We can label each point along the route by a coordinate s, defined to be the road distance from the start. In this context the speed v is defined by $v = \mathrm{d}s/\mathrm{d}t$, and the average speed is just the total road distance divided by the time.

Formulate
In mathematical form our relationship is $t = s/v$, where s is the total road distance.

Solve
time in city = (5 miles)/(20 mi/h) = 0.25 h = 15 minutes;

time in rural district = (5 miles)/(45 mi/h) = 0.11 h = 6 minutes 40 seconds.

Scrutinize
The dimensions of the answer are correct: [length]/([length]/[time]) gives a time. One must also make sure that the ***units*** are correct—0.25 ***hours***, not seconds or years! In this case there was no problem, but see the next part.

(b) *If you were driving in Europe and saw a sign saying 'Paris* 120 km*', how long would it take you to reach there if you were traveling on a French autoroute at 75 miles per hour?*

Conceptualize
We are using the same concepts as in the previous problem, but this time we have the added complication that our units of ***distance*** are not consistent with our units of ***speed***. We will have to express the distance in miles, or alternatively the speed in km/h, before doing the calculation. Consulting reference books, we find that to two significant figures 1 mile = 1.6 km.

Formulate and Solve
Our conversion factor 1 mile = 1.6 km is equivalent to saying that 1 = (1 mile)/(1.6 km). Multiplying any quantity by 1 does not change it, so we can multiply 120 km by (1 mile)/(1.6 km) to obtain

$$120\,\mathrm{km} = 120\,\cancel{\mathrm{km}} \times \frac{1\,\mathrm{mile}}{1.6\,\cancel{\mathrm{km}}} = \frac{120}{1.6}\,\mathrm{miles} = 75\,\mathrm{miles}.$$

1A.4, continued:

We then apply $t = s/v$ to obtain

$$t = (75\,\text{miles})/(75\,\text{miles/hour}) = 1\,\text{hour}.$$

Alternatively, it is often convenient to leave the unit conversions for the final step, as numerical values are inserted. Again, the idea is to look for ways to insert expressions for 1 that cause the units to cancel and give the answer in the desired units. For this problem, one would write:

$$t = \frac{s}{v} = \frac{120\,\cancel{\text{km}}}{75\,\cancel{\text{mile}}\cdot\text{hour}^{-1}} \times \frac{1\,\cancel{\text{mile}}}{1.6\,\cancel{\text{km}}} = 1\,\text{hour}.$$

Even when no unit conversion is needed, cancellation of units like this is a good way to check for dimensional consistency.

Scrutinize
We can check the arithmetic by converting the speed to km/h instead of converting the distance to miles: multiplying by 1 = (1.6 km)/(1 mile) we have

$$75\text{ miles/hour} = 75\,\frac{\cancel{\text{miles}}}{\text{hour}} \times \frac{1.6\,\text{km}}{1\,\cancel{\text{mile}}} = 120\,\text{km/hour},$$

and it is clear that this gives $t = 1$ hour as before.

Learn
In numerical problems like this we always need to extend our checking of the ***dimensions*** of an equation to a check of the actual ***units***. Inches, meters and miles all have dimensions of length, but 1 inch is not the same as 1 mile!

The method of unit conversion used here may seem a little labored for this simple problem, but it is a general method which can be used successfully in much more complicated equations.

(c) *A good sprinter takes about ten seconds for the 100 meter sprint. What is his average speed in miles per hour? If he could maintain this speed indefinitely, how long would it take him to run a marathon (26 miles 385 yards)?*

Conceptualize
This problem uses the same concepts as the previous two, except that here we are given time and asked to calculate speed rather than vice versa. Once again we will have to take care to convert all our numerical values into consistent units.

Formulate
The only equation we need is $x = vt$.

Solve
The sprinter is traveling at 10 m/s. There are 3,600 s in 1 hour, so this is 36,000 meters per hour, or 36 km/h.

1A.4, continued:

Since 1 mile = 1.6 km, 36 km/h = 22.5 mph.

There are 1,760 yards in one mile, so 26 miles 385 yards is $26\frac{385}{1760} = 26.22$ miles. The time it would take to run the marathon is then (26.22 miles)/(22.5 mi/h) = 1.165 hours or 1 hour 10 minutes.

Scrutinize
The time calculated for the marathon is much shorter than we would expect in real life, but this is no surprise—we know that human beings cannot keep up sprinting speeds for longer than a few tens of seconds. The sprinter's speed seems reasonable: for comparison, a four-minute mile implies an average speed of 15 mph, and we would expect a sprinter to be running considerably faster than a miler.

Learn
Why don't we give our final answer more precisely, as 1 h 9 min 55 s? The reason is that it would be meaningless to do so. The original data in the question gave the sprinter's time for the 100 m as "about 10 s"—we could interpret this as "between 9.5 s and 10.5 s", or perhaps "between 9.8 s and 10.2 s", but surely not as "between 9.99 s and 10.01 s". If we take the time as 9.8 s, the average speed comes out to 23.0 mph, and the time for a marathon as 1 h 8 m 31 s. Thus quoting the time to the nearest second is unjustified—we simply do not have that accuracy in the data supplied to us. This is an important point in experimental science, where we often want to know if two measurements of a quantity are consistent or inconsistent. Much of the work of experimental physicists involves not so much determining a value as determining the ***precision*** with which that value is known.

1A.7 *Two cars race along a straight track for* 1 km*, starting from rest. The first accelerates at* $4\ \mathrm{m/s^2}$ *for 10* s*, then continues at constant velocity. The second accelerates at* $5\ \mathrm{m/s^2}$ *for* 5 s*, then at* $1.5\ \mathrm{m/s^2}$ *for* 10 s*, then at* $0.5\ \mathrm{m/s^2}$ *for the rest of the race.*

(a) *How long does each car take to complete the race?*

(b) *Overall, what is the average acceleration and average velocity of each car?*

(c) *What is the average acceleration of each car after* 15 s *of the race?*

(d) *What distance has each car covered after* 15 s *of the race?*

(e) *Draw a graph of velocity and distance covered against time for each car.*

Conceptualize
In this problem the acceleration of the cars is not constant. However, we can divide each car's trip into time segments during which the acceleration is constant—for the first car, $0 \le t \le 10$ s, with $a = 4\ \mathrm{m/s^2}$, and the rest of the race, with $a = 0$. The way to solve the problem is to take each individual time segment as a separate constant-acceleration problem. This is a case where it is likely to be easiest to work out the numerical results as we go along, since otherwise the equations are going to get very messy!

Formulate
We apply our standard formulas for constant acceleration a:

$$v = v_0 + at$$

1A.7, continued:

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

(note that this is a one-dimensional problem, so we can regard all quantities as numbers—only the x-components of vectors will be nonzero). We treat each time segment as a separate problem, resetting the clock at the end of each one. For the first segments, where we have the end time, solving the equations is simple: we have a and t, and the answer to the previous time segment will give us v_0 and x_0. The final time segment of each car's race is different, though: in this case we have x, because we know the distance over which the race is run, but not t. If the velocity is constant, we can use $t = x/v$, but if $a \neq 0$ we must solve a quadratic equation for t:

$$t = \frac{1}{a}\left(-v_0 \pm \sqrt{v_0^2 + 2a(x - x_0)}\right) .$$

In general, quadratic equations have two roots. How do we know which one we want? In this case there is no problem—one root is negative, and thus relates to a time before the start of this time segment, i.e. to a time for which this equation does not apply. If the acceleration were negative, however, we would have two positive roots. This corresponds to a situation in which the car passes the finish line, continues to decelerate and eventually goes into reverse, crossing the finish line again in the opposite direction! It is clear that in such a case we want the smaller value of t, trusting the time-keeper to stop the watch the ***first*** time the car hits the finish line.

Solve

We can display our information in tabular form as shown below. Starting from the top line of the table, we then use our equations $v = v_0 + at$ and $x = x_0 + v_0 t + \frac{1}{2}at^2$ to fill in most of the blanks, as in the second copy of the table.

Car	Time Segment	t_i (s)	$t_f - t_i$ (s)	a (m/s^2)	v_i (m/s)	x_f (m)
1	1	0	10	4	0	?
	2	10	?	0	?	1000
2	1	0	5	5	0	?
	2	5	10	1.5	?	?
	3	15	?	0.5	?	1000

1A.7, continued:

Car	Time Segment	t_i (s)	$t_f - t_i$ (s)	a (m/s^2)	v_i (m/s)	x_f (m)
1	1	0	10	4	0	200
	2	10	20	0	40	1000
2	1	0	5	5	0	62.5
	2	5	10	1.5	25	387.5
	3	15	14	0.5	40	1000

Finally we use the quadratic equation for t to fill in the last blank, shown shaded in the table. This was not necessary for car 1, because its acceleration is zero in stage 2, and so the time taken is simply the distance divided by the (constant) speed.

Now we are in a position to answer the questions posed in the problem.

(a) Calculating t_f from the last line for each car, we see that car 1 takes 30 s to complete the race and car 2 takes 29 s. The race is won by car 2.

(b) The average acceleration is the total change in velocity divided by the total time for the race, and the average velocity is the total displacement over the total time. To calculate these we need one further piece of information, the final speed of car 2. From $v = v_0 + at$ this comes out to 47 m/s. The average velocities and accelerations for the two cars are then:

$$\text{Car 1}: \quad v_{\text{average}} = \frac{1000 \text{ m}}{30 \text{ s}} = 33 \text{ m/s} ; \qquad a_{\text{average}} = \frac{40 \text{ m/s}}{30 \text{ s}} = 1.3 \text{ m/s}^2$$

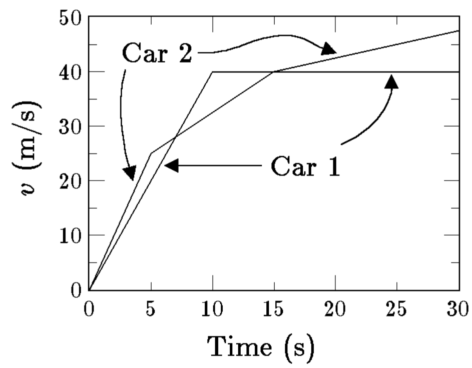
$$\text{Car 2}: \quad v_{\text{average}} = \frac{1000 \text{ m}}{29 \text{ s}} = 34 \text{ m/s} ; \qquad a_{\text{average}} = \frac{47 \text{ m/s}}{29 \text{ s}} = 1.6 \text{ m/s}^2$$

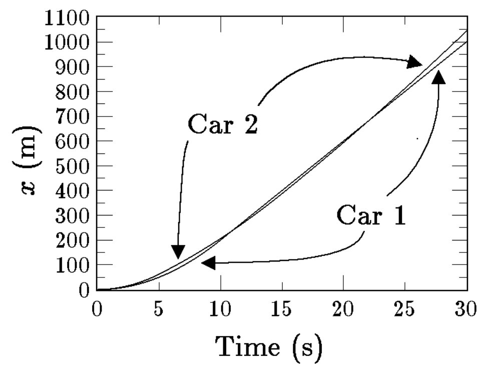
(c) After 15 s the speed of car 1 is 40 m/s, and that of car 2 is, as it happens, also 40 m/s: their average accelerations are therefore both the same, namely (40 m/s)/(15 s) = 2.7 m/s^2.

(d) The distance covered by car 2 after 15 s is in the table: rounded to 2 significant figures, it is 390 m. For car 1 we calculate the distance traveled in 5 s at a constant speed of 40 m/s, (40 m/s) $\times$ (5 s) = 200 m, and add this to the distance of 200 m covered in the first 10 s, giving a distance of 400 m after 15 s. At this point in the race, therefore, car 1 was in the lead.

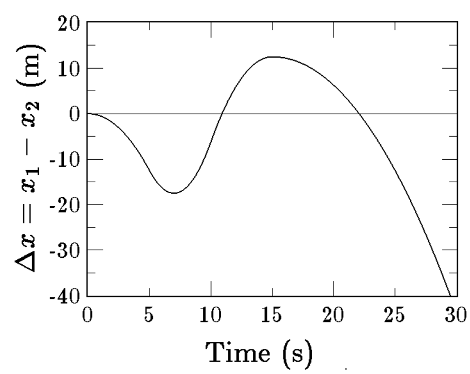
(e) The graphs of velocity and distance are shown on the next page.

1A.7, continued:





The graph of position is difficult to interpret, because the cars are close together. So let's also draw a graph of the ***difference*** in position $\Delta x = x_1 - x_2$. We can then see the progress of the race more clearly.

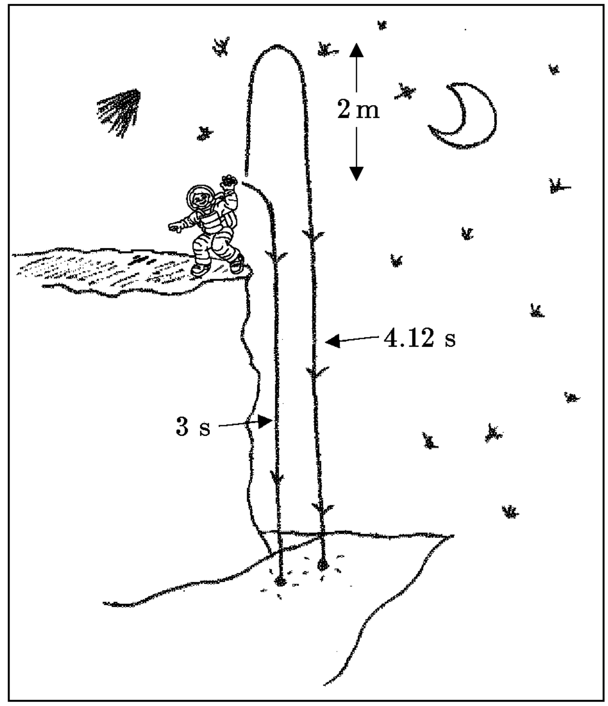


Scrutinize
The numbers appear to make sense—the car which wins the race has a higher average speed and a higher average acceleration, for example, and the average acceleration of each car lies between its minimum acceleration and its maximum acceleration (0 and 4 m/s^2 for car 1, 0.5 and 5 m/s^2 for car 2), as we expect of an average.

Learn
Notice that the formula $x = x_0 + v_0 t + \frac{1}{2}at^2$ definitely does not work for non-constant acceleration. We can see this clearly in the answers to parts (c) and (d), where the two cars have the same average acceleration, yet have covered different distances (and neither distance is that given by the formula, which comes out to 300 m).

This problem is one of the rare cases where working symbolically until the last moment does not pay. If you try it, you will find that the equations become steadily more complicated as you go on: the initial velocity for stage 2 is $v_2 = v_1 + a_1 t_1$, giving an initial velocity for stage 3 of $v_3 = v_1 + a_1 t_1 + a_2 t_2$, for example. Since a_1 is not related to a_2, nor t_1 to t_2, this is not increasing our understanding of the problem—it's just increasing the chances of getting confused. This is in fact the exception discussed in "Solving Problems": a problem consisting of multiple disconnected parts. If the parts were more closely related—if the acceleration halved after a fixed time interval, for instance, instead of having arbitrary values maintained for arbitrary times—it ***would*** pay to work algebraically.

1A.9 *A crewman on the starship Enterprise is on shore leave on a distant planet. He drops a rock from the top of a cliff and observes that it takes* 3.00 s *to reach the bottom. He now throws another rock vertically upwards so that it reaches a height of* 2 m *before dropping down the cliff face. The second rock takes* 4.12 s *to reach the bottom of the cliff. The planet has a very thin atmosphere which offers negligible air resistance. How high is the cliff, and what is the value of g on this planet?*



Conceptualize

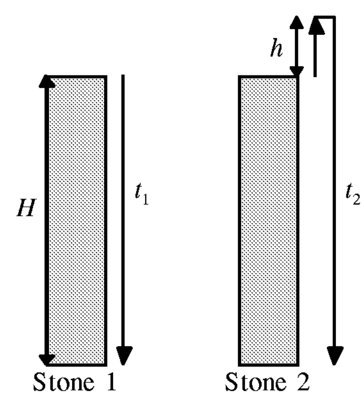
This problem involves projectile motion in one dimension. We have two unknowns, the height H of the cliff and the acceleration g of the falling stones, so in formulating the problem we need to construct at least two equations.

Formulate

The situation is shown in the diagrams (where the arrows representing the motion of the stones have been shifted sideways slightly for clarity). Taking a coordinate system such that z points vertically upwards and $z = 0$ at the bottom of the cliff, we have the following information:

for stone 1:
initial velocity $v_i = 0$;
initial position $z_i = H$ (unknown);
total time $= t_1$;

for stone 2:
initial velocity $v_i = v_0$ (unknown);
initial position $z_i = H$ (unknown);
total time $= t_2$.



We can construct the equations

$$z_f(1) = 0 = H - \tfrac{1}{2}gt_1^2 \ ;$$
$$z_f(2) = 0 = H + v_0 t_2 - \tfrac{1}{2}gt_2^2 \ .$$

This won't do at all: we have two equations, but three unknowns: H, g and v_0. We need at least one more equation.

Fortunately there is information about stone 2 that we have not used: we know its

1A.9, continued:

maximum height h above the cliff ($z = h + H$). At this point $v = 0$ (since immediately beforehand the stone is moving ***up***, and immediately after it's going ***down***). We can apply $v^2 = v_0^2 + 2a(z - z_0)$ to this point to get

$$0 = v_0^2 - 2gh.$$

This gives us a third equation, so we can now solve for three unknowns. Our new equation gives us v_0 in terms of g, and by eliminating H from the first two equations we can solve for g.

Solve

We first solve the third equation for v_0: $v_0 = \sqrt{2gh}$.

Our first two equations then become

$$H = \frac{1}{2}gt_1^2 ;$$

$$H = \frac{1}{2}gt_2^2 - t_2\sqrt{2gh}$$

and we can equate the right-hand sides of these to get

$$\frac{1}{2}g(t_2^2 - t_1^2) = t_2\sqrt{2gh} .$$

Squaring this and rearranging gives

$$\frac{1}{4}g^2(t_2^2 - t_1^2)^2 = 2ght_2^2$$

so

$$g = \frac{8ht_2^2}{(t_2^2 - t_1^2)^2} .$$

We can substitute this back into our original equation for stone 1 to discover that

$$H = \frac{1}{2}gt_1^2 = \frac{4ht_1^2t_2^2}{(t_2^2 - t_1^2)^2} .$$

Numerically, $g = 4.27\ \text{m/s}^2$ and $H = 19.2$ m.

Scrutinize

Our final expression for g has dimensions [length]×[time]^2/[time]^4, which gives [length]/[time]^2 as we expect for an acceleration. We can cross-check the consistency of our calculations by seeing if the results from ***both*** our original equations for H agree:

$$H = \frac{1}{2}gt_1^2 = \frac{1}{2}(4.27\ \text{m/s}^2)(3.00\ \text{s})^2 = 19.2\ \text{m};$$

$$H = \frac{1}{2}gt_2^2 - v_0t_2$$

$$= \frac{1}{2}(4.27\ \text{m/s}^2)(4.12\ \text{s})^2 - \left(\sqrt{2 \times 4.27\ \text{m/s}^2 \times 2\ \text{m}}\right)(4.12\ \text{s}) = 19.2\ \text{m}.$$

1A.9, continued:

This sort of check can find errors such as lost numerical factors which will not show up in considering the dimensions of an equation.

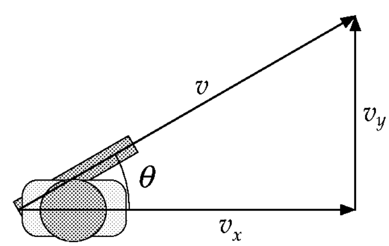
Learn

The most important step in this problem is the formulation. We had to develop a strategy for solving the problem which gave us enough equations to account for all our unknowns. The difficulty of this depends very much on the information that you have been given: this problem would be much easier to formulate if we had been given v_0 instead of h (or instead of t_2 or t_1, for that matter).

1C.2 *A cannonball emerges from a cannon with speed v, independent of the angle at which the barrel of the cannon is inclined. If the cannon is set on level ground, at what angle should the gunner set the cannon to maximize the range? What will the range be at this angle (in terms of v and g) and what height will the ball reach? Neglect any effects of air resistance.*

Conceptualize

This is a case of projectile motion in two dimensions. The initial velocity of the cannonball is shown in the diagram. Since we are told to neglect air resistance, the cannonball is freely falling and therefore has a constant downward acceleration g. Its vertical velocity, which is initially $v_y(0) = v\sin\theta$, will therefore decrease to zero and become negative (downwards), thus returning the cannonball to Earth. The horizontal distance it has traveled in the interim is its range. Our task is to find the value of θ which maximizes the range. Those of you who are already familiar with calculus will know that one standard method for doing this is to obtain an equation for the range x in terms of the angle θ and then differentiate. Minima and maxima of x correspond to zero values of $\mathrm{d}x/\mathrm{d}\theta$.



It is not always necessary to use calculus to identify the maximum (for example, the maximum value of $A\cos\theta$ is obviously A), but in any case the path to the solution clearly lies in finding an equation for x in terms of θ.

Formulate

We define a coordinate system with x horizontal along the cannon's direction of fire and y vertical, and the origin $x = y = 0$ at the location of the cannon. (We assume that the height of the cannon's barrel above the ground is negligible compared with the length of the ball's flight, i.e. the ball is launched from, and lands at, $y = 0$.) For uniform acceleration, the general equations for the velocity and position are

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}_0 + \vec{\mathbf{a}}t$$

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}_0 + \vec{\mathbf{v}}_0 t + \frac{1}{2}\vec{\mathbf{a}}t^2 \ .$$

1C.2, continued:

In this case $\vec{\mathbf{a}} = [0, -g, 0]$, and $\vec{\mathbf{v}}_0 = [v\cos\theta, v\sin\theta, 0]$. (If you are having difficulty distinguishing the sine from the cosine, a practical technique is to always draw your angles noticeably smaller than 45°, as was done here. Then the short side of the triangle is always proportional to the sine, and the long side is proportional to the cosine.) Writing the equations in components, one has

$$v_x = v\cos\theta$$
$$v_y = v\sin\theta - gt \ .$$

Similarly the position $\vec{\mathbf{r}} = [x, y, z]$ is given in components as

$$x = vt\cos\theta \qquad (1)$$

$$y = vt\sin\theta - \frac{1}{2}gt^2 = t\left(v\sin\theta - \frac{1}{2}gt\right) \ . \qquad (2)$$

The cannonball is at $y = 0$ at two times: $t = 0$, the launch, and

$$t = \frac{2v}{g}\sin\theta \ , \qquad (3)$$

which must be the landing. To solve the problem we will calculate the range using this value for t, and then find the maximum value of the resulting formula for x. The maximum height can be found by finding the time at which $v_y = 0$ for this value of θ, or alternatively we can recognize that the trajectory is symmetrical, and therefore the maximum height is reached halfway between launch and landing. The first method is safer, because it will still work if our cannon is not on level ground.

Solve

Substituting the value for t in Eq. (3) into the expression for x in Eq. (1), the range of the cannon is found to be

$$x = \frac{2v^2\sin\theta\cos\theta}{g}$$

for any angle θ. We want the angle that gives the largest possible x. To work this out without calculus we use the trigonometric identity

$$\sin 2\theta = 2\sin\theta\cos\theta \ .$$

Using this formula we can rewrite the equation for x as

$$x = \frac{v^2\sin 2\theta}{g} \ ,$$

from which it is clear that the maximum value occurs when $2\theta = 90°$, i.e. when $\theta = 45°$. The value of x at this maximum is

$$x_{\max} = \frac{v^2}{g} \ .$$

1C.2, continued:

The time for which $v_y = 0$ is $t = \frac{v}{g}\sin\theta$. This gives

$$y = \frac{v^2}{g}\sin^2\theta - \frac{1}{2}g\frac{v^2\sin^2\theta}{g^2} = \frac{v^2\sin^2\theta}{2g} ,$$

which for $\theta = 45°$ yields

$$y_{\max} = \frac{v^2}{4g} .$$

Scrutinize

The dimensions of v^2/g are $([\text{length}]^2/[\text{time}]^2)/([\text{length}]/[\text{time}]^2)$, i.e. length, which is correct. Note that dimensional analysis cannot give us the factor of 4 in the expression for y, nor can it confirm that our sines and cosines are in the right places.

The calculus approach to finding the maximum would be

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}\theta} &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{2v^2\sin\theta\cos\theta}{g}\right)\\ &= \frac{2v^2}{g}(\cos^2\theta - \sin^2\theta)\\ &= 0 \text{ when } \cos\theta = \pm\sin\theta .\end{aligned}$$

This gives $\theta = 45°$ or $135°$ (the other two quadrants, $-45°$ and $-135°$, are clearly unphysical). The second solution just corresponds to the cannon facing in the opposite direction.

Learn

This solution seems at first to contradict common sense—one might argue that to maximize the horizontal distance we should maximize the horizontal component of velocity, which is clearly not what we have done. The reason for this is that we also have to consider flight time: a ball with zero vertical velocity launched from zero height instantly plows into the ground. Flight time is maximized by maximizing the vertical component of velocity—but a ball launched vertically upwards comes back down on our heads. The trade-off between longer flight times and larger horizontal velocities is what gives us our 45° angle.

Golfers and field sports enthusiasts may feel that this calculation does not accord with their experience of reality. This is quite true: air resistance is rarely negligible in real situations, and other factors such as spin may come into play as well. We will consider some of these complications in later chapters.

1C.4 *In a shooting contest, a clay pigeon is launched from the ground at a speed of* 50 m/s *at 60° to the horizontal, directed eastwards. The contestant is standing* 100 m *south of the clay's line of flight and* 40 m *east of the launch point. His gun has a muzzle velocity of* 200 m/s *and he hits the clay when it is directly ahead of him as he faces north. When did he fire, and at what elevation? (Neglect air resistance; take* $g = 9.8$ m/s^2 *and assume the shot is fired from a height of* 1.6 m *above ground level.)*

Conceptualize

Here we have two projectile problems combined, making a three-dimensional problem (an individual projectile's motion is two-dimensional). Fortunately we can in fact deal with each projectile separately, making two two-dimensional problems. Our strategy will be:

- First find the time at which the clay pigeon is directly ahead of the contestant, and calculate the clay's height at that time.
- This gives us the position at which the shot hit the clay. We then use this information to determine the flight time and initial direction of the shot.

Formulate

The general equations for uniform acceleration are

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{v}}_0 + \vec{\mathbf{a}}t$$
$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}_0 + \vec{\mathbf{v}}_0 t + \frac{1}{2}\vec{\mathbf{a}}t^2 \; .$$

For this problem, let us call east the x direction, north y and up z, so $\vec{\mathbf{a}} = [0, 0, -g]$, for both the clay and the shot. The velocity vector of the clay is $\vec{\mathbf{v}} \equiv [v_x, 0, v_z]$, and that of the shot is $\vec{\mathbf{w}} \equiv [0, w_y, w_z]$. If the clay is fired at $t = 0$, the component equations for its velocity and position at time t are

$$v_x = v\cos\theta \qquad v_y = 0 \qquad v_z = v\sin\theta - gt \tag{1}$$
$$x_c = vt\cos\theta \qquad y_c = 0 \qquad z_c = vt\sin\theta - \frac{1}{2}gt^2 \; , \tag{2}$$

where we know $v = 50$ m/s and $\theta = 60°$. We are measuring all position vectors relative to the launch point of the clay.

The contestant fired his shot sometime later, at $t = t_0$, at an angle α to the horizontal. Its component equations are:

$$w_x = 0 \qquad w_y = w\cos\alpha \qquad w_z = w\sin\alpha - g(t - t_0) \tag{3}$$
$$x_s = x_0 \qquad y_s = y_0 + w(t - t_0)\cos\alpha \qquad z_s = z_0 + w(t - t_0)\sin\alpha - \frac{1}{2}g(t - t_0)^2 \; , \tag{4}$$

where $y_0 = -100$ m, $z_0 = 1.6$ m is the height from which the shot was fired, and $x_0 = 40$ m, since we are told that the contestant is 40 m east of the launch point. We also know

1C.4, continued:

$w = 200$ m/s. Note that $w_x = 0$, since we are told that the shot hits the clay directly in front of the contestant, and w_x is a sideways component of the shot's velocity.

We know that the shot hits the clay (i.e. they are in the same place at the same time!). With this information we can solve the clay's equations for t and z_c. This gives us z_s for the shot at impact, and we already know $y_s = 0$ since the clay has no y-component of velocity. We then have the information needed to solve the shot's equations for t_0 and α.

Solve

The impact occurs at $x_c = x_s = x_0$, so from the 1st of Eqs. (2) one has

$$t = \frac{x_0}{v\cos\theta} = \frac{40\text{ m}}{50\text{ m/s} \times \cos 60^\circ} = 1.6\text{ s} .$$

Therefore at the point of impact $z_c = z_s = vt\sin\theta - \frac{1}{2}gt^2 = 56.7$ m.

To solve the shot's equations we first note that at impact $y_s = y_c = 0$, so from the 2nd of Eqs. (4) one has

$$t - t_0 = -\frac{y_0}{w\cos\alpha} .$$

We can use this to eliminate $t - t_0$ from the 3rd of Eqs. (4):

$$\begin{aligned} z_s - z_0 &= -y_0\tan\alpha - \frac{gy_0^2}{2w^2}\sec^2\alpha \\ &= -y_0\tan\alpha - \frac{gy_0^2}{2w^2}(1+\tan^2\alpha) \end{aligned}$$

using a standard trigonometric identity (which you can easily prove if you remember that $\cos^2\alpha + \sin^2\alpha = 1$ for any angle α). This is a quadratic equation for $\tan\alpha$. Its solution is

$$\tan\alpha = \frac{w^2}{gy_0^2}\left[-y_0 \pm \sqrt{y_0^2 - \frac{2gy_0^2}{w^2}\left(\frac{gy_0^2}{2w^2} + z_s - z_0\right)}\right] .$$

Numerically this gives $\tan\alpha = 0.567$ or 81.1, corresponding to $\alpha = 29.6^\circ$ or 89.3° respectively. The first of these is clearly the one we want; the second is possible in principle, but not in practice (the flight time would be extremely long, requiring the contestant to shoot well before the clay was launched!).

Finally we use our value of α to find

$$t - t_0 = (100\text{m})/(200\text{m/s} \times \cos 29.6^\circ) = 0.58\text{ s}.$$

Since $t = 1.6$ s, $t_0 = 1.0$ s to 2 significant figures. The contestant fired 1.0 s after the clay was launched, at 30° to the horizontal.

1C.4, continued:

Scrutinize

The units of gy^2/w^2 are $(\text{m/s}^2)(\text{m}^2)/(\text{m/s})^2$, i.e. meters, and using this it is straightforward to check that $\tan\alpha$ is dimensionless, as it should be.

Are the values sensible? We can check the solution of the shot's equations by recognizing that the effect of gravity is quite small (if the flight time is 0.6 s, the change in velocity is only about 6 m/s, which is small compared to the initial value of 200 m/s). Setting $g = 0$ in the shot's equations gives $z_s - z_0 = -y_0 \tan\alpha$, so $\tan\alpha = 0.551$ and $\alpha = 28.9°$. This is reassuringly close to our exact value, the difference being about 3% (it is ***not*** a coincidence that this is equal to the percentage change in the velocity, 6/200 = 3%).

Learn

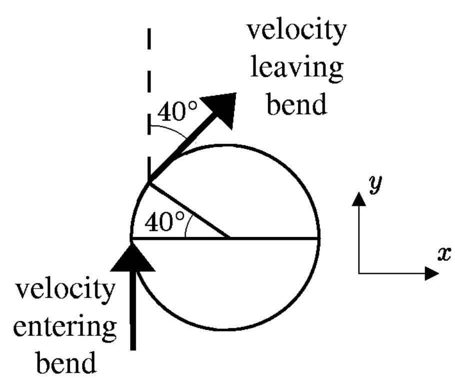
"Back-of-the-envelope" approximate calculations like this are very useful, especially if trying to decide whether something is likely to be possible or not (e.g. can a massive black hole supply enough power to account for the observed brightness of quasars?). If the approximate calculation suggests that it is possible, then we can go on to the more difficult exact calculation. If not, we have not expended large amounts of time and effort on a proposal that will not work.

Note that when the contestant fired, the x coordinate of the clay was only (50 m/s) $\times$ (1 s) $\times(\cos 60°) = 25$ m. To hit the clay, the contestant had to aim ***ahead*** of the clay's position at the time he fired. Did he in fact aim at the actual point of impact, [40, 0, 56.7] m? If not, why not?

1D.2 *A motorcycle negotiates a 40° right-hand bend at* 60 km/h. *The bend consists of a 40° arc of a circle of radius* 75 m. *What is the centripetal acceleration of the bike at any point in the bend, and what is the total velocity change between entering the bend and leaving it?*

Conceptualize

We can treat the motorcycle negotiating the bend as a point mass in circular motion. The velocity change is not zero, because its direction has changed (although its magnitude has not). The acceleration will be directed towards the center of the circle defined by extrapolating the arc of the bend, and its magnitude can be found by using the formula for circular motion.



Formulate

The acceleration of a particle traveling with speed v in a circle of radius r has magnitude v^2/r and is directed towards the center of the circle.

If we take the initial direction of the motorcycle to be the y-direction, its velocity entering the bend is $\vec{\mathbf{v}}_i = [0, v, 0]$, where $v = 60$ km/h, and its velocity leaving the bend is $\vec{\mathbf{v}}_f = [v\sin\theta, v\cos\theta, 0]$, where $\theta = 40°$ (as can be seen from the diagram with the aid of a little geometry).

1D.2, continued:

Solve

To get the acceleration in sensible units we need to convert 60 km/h into m/s: 60 km/h = (60,000 m/h)/(3,600 s/h) = 16.67 m/s. The magnitude of the acceleration is $v^2/r =$ $(16.67 \text{ m/s})^2/(75 \text{ m}) = 3.7 \text{ m/s}^2$; as we have said, it is directed towards the center of curvature of the bend.

The change in velocity is

$$\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i = [v\sin\theta, v(\cos\theta - 1), 0] = [39, -14, 0] \text{ km/h}.$$

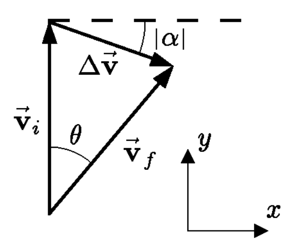
Its magnitude is 41 km/h. If we write its vector in the form

$$[w\cos\alpha, w\sin\alpha, 0] \ ,$$

where $w = 41$ km/h, we have $\cos\alpha = 0.940$ and $\sin\alpha = -0.342$, giving $\alpha = -20°$.

Scrutinize

We can make a geometrical check on the direction of our velocity change vector by redrawing the "before" and "after" velocities with their tails touching, as shown. The two vectors form an isosceles triangle with apex angle 40°, so each of the other two angles is 70° and $|\alpha| = 20°$.



Learn

Notice that the centripetal acceleration required to negotiate the bend increases rapidly with the speed of the vehicle. This is why it is necessary to slow down when taking a sharp curve. We will see in the next chapter how banking the curve can make it easier to negotiate (without changing the acceleration required).

1E.1 (a) *A ferryboat crosses a river of width d. The speed of the boat (relative to the water) is v and the speed of the river current is V. Assuming that the landing point of the ferry is directly opposite its starting point, how long does it take for a round trip?*

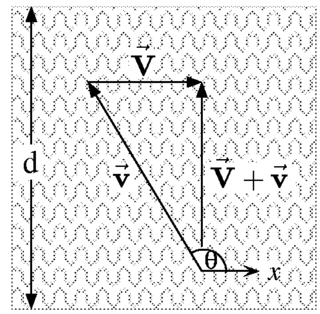
Conceptualize

There are two reference frames important to this problem, the ***bank*** frame which is stationary relative to an observer on the river bank, and the ***river*** frame which is stationary relative to an observer drifting with the current. The ferry's speed v is specified in the river frame, which has speed V downstream relative to the bank frame.

1E.1, continued:

Formulate

We will define a coordinate system in which the x-axis points downstream for both frames. If the ferry is moving at an angle θ to the downstream direction, its velocity in component form, in the river frame of reference, is



$$\vec{\mathbf{v}} = [v\cos\theta, v\sin\theta, 0] \ . \tag{1}$$

In the bank frame we must add on the velocity $\vec{\mathbf{V}} = [V, 0, 0]$ of the current: thus in this frame the boat's velocity is

$$\vec{\mathbf{v}}' = [V + v\cos\theta, v\sin\theta, 0] \ . \tag{2}$$

To solve the problem we note that the ferry's starting and finishing points are specified in the bank frame. We must use the second form of the ferry's velocity vector. The only unknown is θ.

Solve

Relative to its starting point, the ferry's position vector in the bank frame is just t times its velocity vector, assuming constant velocity. We want to reach a point on the bank opposite the starting point, i.e. with $x = 0$, so we must have

$$V + v\cos\theta = 0 \ ,$$

$$\text{i.e. } \cos\theta = -\frac{V}{v} \ .$$

An observer on the bank will then see the ferry traveling straight across the river with speed $v\sin\theta = v\sqrt{1 - \frac{V^2}{v^2}} = \sqrt{v^2 - V^2}$. A one-way trip will thus take a time $t = d/\sqrt{v^2 - V^2}$, where d is the width of the river, and the round trip will be twice this.

Scrutinize and Learn

The ratio $-V/v$ is dimensionless, as it should be, and the negative sign implies that $\theta > 90°$— the ferryman has to steer upstream, against the current. This is clearly correct.

Notice that for $v < V$ the equation for t involves the square root of a negative number. This is not a problem: if you think about the situation, you will see that in this case the boat cannot possibly make a landing directly opposite its starting point, because even if the ferryman steers directly upstream he is still being washed downstream at a speed $V - v$. In many cases physically impossible situations are signaled in the mathematical representation of the problem by mathematically illegal operations such as this. Another example might be a value of cosine or sine outside the range -1 to 1, and indeed in this example $v < V$ implies that $\cos\theta < -1$.

(b) *In the Ferryman of the Year competition the ferryman is required to complete a course of the same distance 2d by rowing a distance d directly upstream and then back downstream*

1E.1, continued:

to his starting point. (The distance is defined by posts on the river bank.) How long does it take him to complete the course, and is this longer or shorter than the round trip across the river?

Conceptualize

The situation is much the same as before, with the distance again being defined in the bank frame, and the same techniques can be used to solve this problem.

Formulate

The boat's velocity vectors in the two frames are still given by equations (1) and (2), with $\theta = 180°$ for the upstream leg of the course and $0°$ for the downstream leg. The velocity of the boat relative to the bank is therefore $[V - v, 0, 0]$ for the first leg (displacement $[-d, 0, 0]$) and $[V + v, 0, 0]$ for the second leg (displacement $[+d, 0, 0]$).

Solve

The total time for the course is

$$t = \frac{-d}{V - v} + \frac{d}{V + v} = \frac{2dv}{v^2 - V^2} .$$

The difference between this time and the return journey across the river is

$$\frac{2dv}{v^2 - V^2} - \frac{2d}{\sqrt{v^2 - V^2}} = \frac{2dv}{v^2 - V^2}\left(1 - \sqrt{1 - \frac{V^2}{v^2}}\right) .$$

This is always positive for $v > V$, so the up- and downstream course takes longer than rowing across the current.

Scrutinize

A good check of this answer is to consider the case where $V = 0$ (a lake rather than a river). River frame and bank frame are then identical, and so are the two round trips. The journey time is the distance divided by the speed, $2d/v$. It is easy to see that both our expressions for t do indeed reduce to this when $V = 0$, and the time difference becomes 0.

(c) *An invading army is approaching the river. The ferryman is anxious to get across to the other side as quickly as possible, without caring where he lands. How long does it take him, and where does he land?*

Conceptualize

The ferryman wants to minimize the journey time across the river. The distance across the river is d in the y-direction, so to minimize the journey time he should steer so as to maximize the y-component of his velocity.

Formulate and Solve

Equations (1) and (2) still apply, so to maximize the y-component he needs $\sin\theta = 1$, regardless of whether we work in the bank frame or the river frame. His velocity is $[0, v, 0]$ ***in the river frame of reference***, and he reaches the other side in a time $t = d/v$.

1E.1, continued:

In the bank frame his velocity is $[V, v, 0]$, so he lands a distance Vd/v downstream of his starting point.

Scrutinize

The total distance covered in the bank frame of reference is

$$d\sqrt{1+\frac{V^2}{v^2}}\ .$$

This is slightly surprising at first sight—surely the minimum journey time should correspond to the minimum distance traveled? However, the answer seems sensible if we remember that the maximum speed is specified in the ***river*** frame, not the bank frame. The distance traveled in the river frame is indeed d, the minimum required to cross the river.

Note that in this case the expression under the square root is never negative. As long as we don't care how far downstream we end up, it is always possible to get across the river.

Learn

It is never ***necessary*** to change reference frames to solve a problem. In some cases, however, it is certainly easier to visualize the situation by choosing a particular reference frame, and sometimes it can greatly simplify the mathematics.

The actual physical phenomena we are describing are of course independent of the choice of reference frame—the results of an experiment are unaffected by the motion of the observer. However, as we see in the next chapter, one must be careful if one frame of reference is accelerating relative to another frame.

1E.2 *The pilot of a light plane wishes to fly from Bristol to Edinburgh (*625 km *due north). Her cruising speed, measured relative to the air, is* 150 km/h. *There is a* 20 km/h *west wind blowing (where "west wind" refers to a wind blowing* ***from*** *the west).*

(a) *In what direction should she point her plane, and how long will her journey take? (Ignore the time spent in take-off and landing and assume that she flies at a constant altitude.)*

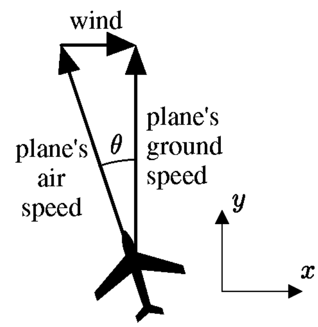
Conceptualize

The subtlety of this problem is that the speed of the plane is specified ***relative to the air around it*** (air speed). The velocity of the plane relative to the air will be called its air velocity. The velocity of the plane relative to the ***ground*** (ground velocity)is made up of the vector sum of the plane's air velocity and the wind velocity. We want this resultant ground velocity to point due north. We can therefore draw a vector diagram of the velocity as shown below. To solve the problem we simply choose a suitable coordinate system and construct this vector sum.

1E.2, continued:

Formulate

We choose a coordinate system in which x points east and y points north. If we call the plane's air velocity $\vec{\mathbf{v}}$, the wind velocity $\vec{\mathbf{w}}$, and the plane's ground velocity $\vec{\mathbf{V}}$, then the velocity relation can be written



$$\vec{\mathbf{V}} = \vec{\mathbf{v}} + \vec{\mathbf{w}} \ .$$

In component form this becomes

$$[V_x, V_y, 0] = [v_x, v_y, 0] + [w_x, 0, 0] \ .$$

We know the wind velocity ($w_x = 20$ km/h), the magnitude of the plane's air velocity ($v = \sqrt{v_x^2 + v_y^2} = 150$ km/h), and the fact that we want $V_x = 0$. Hence we have two unknowns, V_y and the angle θ between the plane's air velocity and north. Since each component of a vector equation is an equation in its own right, we have two equations (the z component gives $0 + 0 = 0$, which doesn't count!). Writing $v_x = -v \sin\theta$ and $v_y = v\cos\theta$, we have two equations in two unknowns, and can solve for θ and V. The time required for the journey is simply the distance divided by the speed, as usual.

Solve

Our two equations are

$$0 = -v \sin\theta + w_x \ , \qquad (1)$$

$$V_y = v\cos\theta \ . \qquad (2)$$

Eq. (1) gives

$$\sin\theta = w_x / v = 0.133 \ ,$$

from which $\theta = 7.7°$, and substituting this into Eq. (2) yields

$$V_y = (150 \text{ km/h}) \cos 7.7° = 149 \text{ km/h}.$$

The journey will take (625 km)/(149 km/h) = 4.2 h, or 4 h 12 min.

Scrutinize

From the diagram, we can see that the vectors form a right-angled triangle with the plane's airspeed as hypotenuse. So we can check our result with the Pythagorean theorem: the ground speed of the plane is $\sqrt{150^2 - 20^2} = 149$ km/h, in agreement with the component method.

(b) *When she starts her return trip the wind has shifted to southwest and increased to* 50 km/h. *What heading should she take, and how long is the return journey?*

Conceptualize

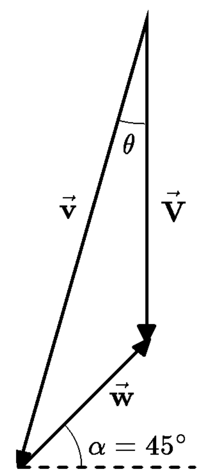
The setup is the same as in part (a), and we solve it in the same way. The only difference is that our two equations will be slightly more complicated, because the wind's vector now has two nonzero components.

1E.2, continued:

Formulate
The wind vector is now $\vec{\mathbf{w}} = [w\cos\alpha, w\sin\alpha, 0]$, where $w = 50$ km/h and $\alpha = 45°$ as shown in the diagram, and the plane's air velocity is $\vec{\mathbf{v}} = [-v\sin\theta, -v\cos\theta, 0]$, where $v = 150$ km/h and θ is defined in the diagram. Since we want to fly south, $V_x = 0$, and we can define $V \equiv -V_y$, with $V > 0$. The unknowns are V and θ.

The velocity relation $\vec{\mathbf{V}} = \vec{\mathbf{v}} + \vec{\mathbf{w}}$ can be written in components as

$$0 = -v\sin\theta + w\cos\alpha \ ,$$
$$-V = -v\cos\theta + w\sin\alpha \ .$$



Solve
The first equation gives

$$\sin\theta = \frac{w}{v}\cos\alpha = 0.236 \ ,$$

which yields $\theta = 13.6°$, and hence

$$-V = -v\cos\theta + w\sin\alpha = -110 \text{ km/h} \ .$$

The plane's air velocity must be directed 13.6° west of south, and its ground speed is 110 km/h. The journey will take (625 km)/(110 km/h) = 5.7 h, or 5 h 40 min.

Scrutinize
The ground speed is less on the return journey, because the plane is now flying into a headwind. Our sketch of the vector diagram is too rough to provide an exact check, but certainly indicates that our answers are in the right ballpark. Note that angles are dimensionless, as are trigonometric functions: all our equations for sines or cosines are in terms of the ratio of two speeds.

Learn
It would also be possible to check this result geometrically, as we did in part (a), but as the triangle is not right-angled we would have to use more complicated trigonometry, such as the cosine rule. Such a "check" would not be very useful, as the chances of making an error in the trigonometry are probably higher than in the original calculation! The power of the component approach is that it avoids the need for such tedious geometrical exercises and lets us set up the problem in a standard way (in contrast, for the geometric methods each vector addition is a different triangle). However, it is always worth drawing the vector triangle, roughly to scale and with vectors pointing in about the right directions, to provide a visual check on the results.

HINTS FOR PROBLEMS WITH AN (H)

The number of the hint refers to the number of the problem

1A.5 To do this without calculus, sketch the graph of v against t. Convince yourself that the distance traveled between times 0 and t is the area under this graph. What rectangular area (i.e. what constant velocity) would give the same value for the distance traveled?

1A.8 (a) How long does it take the bus to decelerate to zero velocity?

(b) What is the total time taken to negotiate a bus stop, from the start of deceleration to the end of acceleration? (Remember to include the time the bus is stationary!) What distance is covered during this time?

You may also find it helpful to review the solution to 1A.7.

1B.2 (a) If you are confused, it will help to start with a scale drawing. Put in what you already know and add additional information as you go along.

(b) What is the position vector of Carol relative to Albert?

(c) What is the position vector of Betty relative to Dave? What is its magnitude?

What is the ***direction*** of Ernie's velocity vector? What is its ***magnitude***?

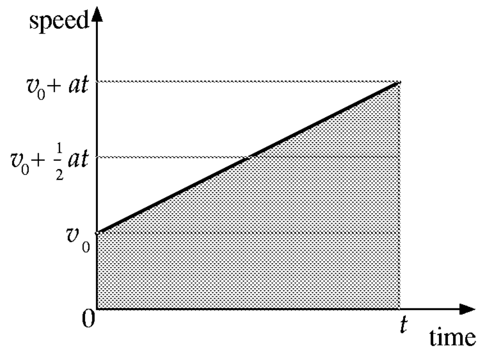
1C.3 Think of the horizontal and vertical motions separately. What is the acceleration in each direction? What is the vertical velocity at maximum height? How long does it take the ball to reach maximum height?

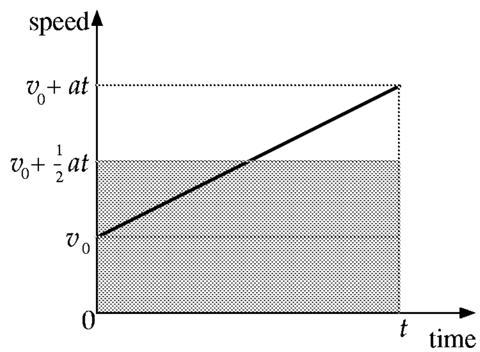
1E.3 (a) Draw the velocity vectors of the two airplanes in two reference frames: (i) the frame in which the air is stationary; (ii) the frame in which your plane is stationary.

(b) Which frame is best suited to this problem?

ANSWERS TO HINTS

1A.5 To get the same area for a constant speed, we need a speed of $v_0 + \frac{1}{2}at$.





1A.8 (a) 11.1 s.

(b) 48.5 s; 103 m.

1B.2 (b) [20, 10, 0] m.

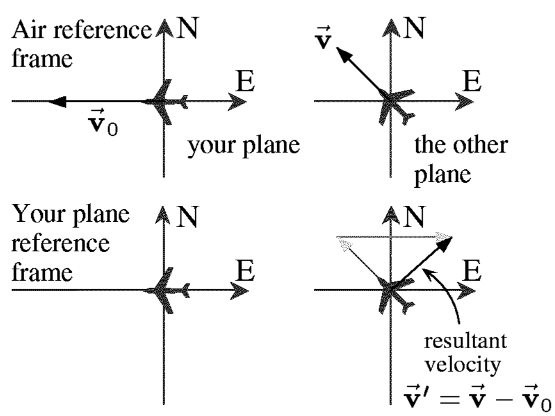
(c) [−10, 8, 0] m; 12.8 m.

Same direction as [−10, 8, 0];

3 m/s.

1C.3 0 in horizontal direction, $-g$ vertically; zero; $\frac{v_0}{g}\sin\theta$ (initial vertical velocity is $v_0 \sin\theta$).

1E.3 (a)



(b) The second frame.

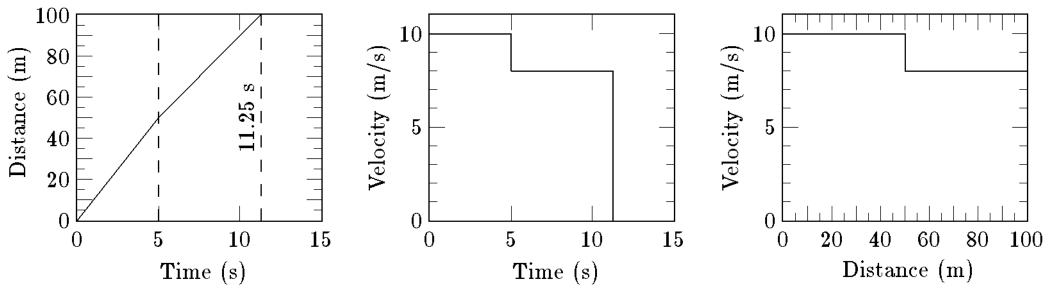
ANSWERS TO ALL PROBLEMS

1A.1 b.

1A.2 d.

1A.3 (a) 5 s; 6.25 s.

(b)



(c) 8.89 m/s; 9.00 m/s.

1A.4 15 min.; 6 min. 40 sec; 1 hour; 1 hour 10 min.

1A.5 Calculus:

As $a = \dfrac{dv}{dt}$, $v(t) = v_0 + \int_0^t a\,dt'$, so $v(t) = v_0 + at$, where v_0 is the velocity at time $t = 0$. Similarly,

$$v = \frac{dx}{dt} \Longrightarrow x(t) = x_0 + \int_0^t (v_0 + at')\,dt', \qquad \text{so } x(t) = x_0 + v_0 t + \tfrac{1}{2}at^2,$$

where x_0 is the position at $t = 0$.

(Non-calculus: see hints.)

For the second equation, use the expression for the velocity to solve for t:

$$t = \frac{(v - v_0)}{a} \; .$$

Then substitute this into the equation for x.

1A.6 (a) Acceleration is positive from 0 to 2 s and from 6 to 7 s, negative from 3 to 4 s, and zero from 2 to 3 s and from 4 to 6 s.

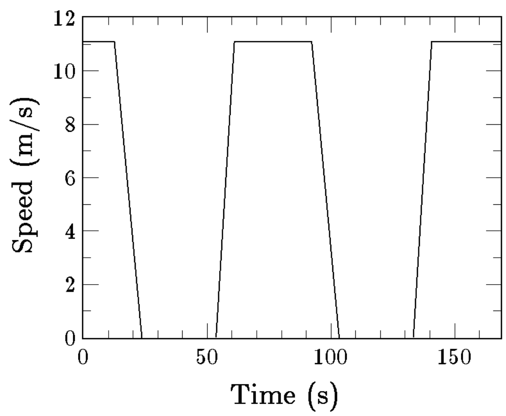
(b) After 3.5 s $x = 6.5$ m; after 7 s $x = 4$ m (taking $x = 0$ at $t = 0$).

(c) The particle starts from rest and accelerates uniformly for 1 s, reaching a speed of 1 m/s. It then accelerates at a higher rate for 1 s, reaching a speed of 3 m/s which it maintains for a further 1 s. It then decelerates rapidly, coming to a halt 3.75 s after $t = 0$ and reversing direction, so that 4 s after $t = 0$ it is moving back towards its starting point at 1 m/s. It maintains this velocity for a further 2 s before decelerating uniformly to come to rest 7 s after $t = 0$.

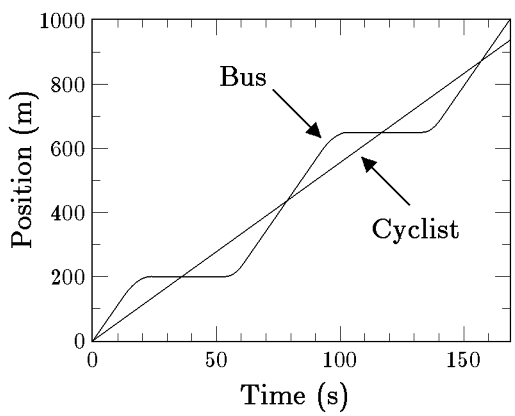
1A.7 See complete solution.

1A.8 (a) 62 m.

(b) 170 s; 5.9 m/s.



(c) Twice each.



1A.9 See complete solution.

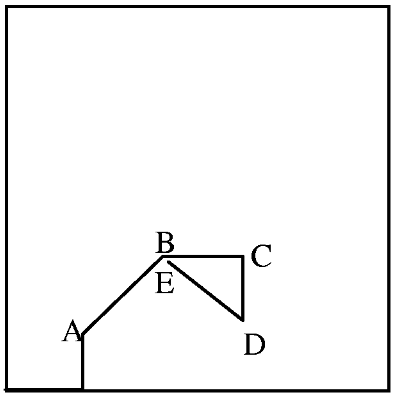
1B.1 c; c.

1B.2 (a) Betty: [20, 17, 0] m; Carol: [30,17, 0] m; Dave: [30, 9, 0] m.

(b) 22 m.

(c) [−2.3, 1.9, 0] m/s; 4.3 s; [21, 16, 0] m, to 2 significant figures. (The y-component of Ernie's position vector is almost exactly 16.5 m: depending on how and when you round off, you may get 17 m instead of 16.)

(d)

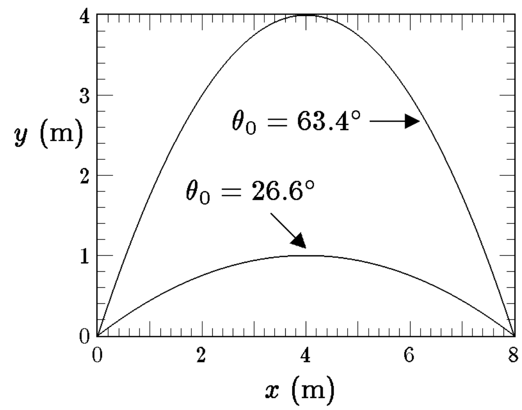


1C.1 d; a.

1C.2 See complete solution.

1C.3 (a) 4.2 m; 2.9 m.

(b) 10 m; 26.6° or 63.4°. See graph on right. Yes.



1C.4 See complete solution.

1D.1 a.

1D.2 See complete solution.

1D.3 0.22 m/s^2; 0.034 m/s^2.

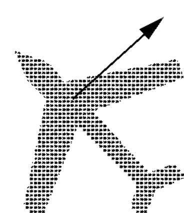
Both are very much less than g. In the second case this means that we can neglect the Earth's rotation in doing problems: the centripetal acceleration implied by the rotation is negligible compared to the acceleration of a freely falling body. The first case tells us something about the variation of g with height, which we will study further in the next chapter.

1E.1 See complete solution.

1E.2 See complete solution.

1E.3 106 m/s, at 138° clockwise to the direction of your plane (i.e. 138° northwards from west).

The airplane is apparently moving sideways to its right (see diagram).



CHAPTER 1

SUPPLEMENTARY NOTES

PHYSICS AS AN EXPERIMENTAL SCIENCE

What is physics, and why are we studying it?

Physics is our attempt to ***understand*** and ***predict*** natural phenomena. Studying physics is valuable both philosophically—most of the world's civilizations have a long history of seeking to understand the world around us—and practically, in dealing with such everyday questions as the design of bridges, energy generation and conservation, atmospheric circulation, etc.

Natural phenomena appear complex and infinitely varied. If I drop a glass on the floor, many things happen: there is a loud noise, the glass breaks into several pieces, the floor covering may be marked or damaged. Empirically we know that some things about this incident can be predicted—the glass ***will*** fall when I let go of it, and it ***will*** (under normal circumstances) break when it hits the ground. Other aspects appear to be unpredictable, at least with the information we have—I don't know how many pieces the glass will break into or where the fracture lines will be. Our predictions in this are based on our past experience with similar objects: if I came from a culture which used only wooden or basketwork utensils, I would not expect the glass to break.

The scientific study of natural phenomena is based on this empirical approach.

- We make ***observations*** (if I let go of a glass, it falls to the floor and breaks).
- If possible, we make ***controlled observations*** or ***experiments*** (I take several glasses, as near to identical as I can find, and drop them from different heights, or onto different types of floor covering).

  In an ideal controlled experiment, we make all the conditions of the experiment identical except one: for example, we take identical glasses and drop them onto identical floor coverings, but change the height from which they are dropped. If we in fact want to study the effect of varying several of the experimental conditions (for example height, nature of floor covering, and type of glass), this means that we have to repeat the experiment a large number of times, which is tedious and time-consuming. Nevertheless it is still the best method (why?).
- From these observations, we try to define basic features of the phenomenon which can be represented by mathematical symbols (the height at which I hold the glass, for example). Relations between these basic ***variables*** derived from our experimental results can then be expressed as mathematical equations (I would find that the speed at which the glass is moving when it hits the floor is determined by the height from which I dropped it, and I could deduce an equation relating these two variables).
- The equations we deduce can be used to ***predict*** what will happen in a new series of experiments (if I have measured the velocity of the glass for heights between one meter and two meters from the floor, I can predict what it will be if I drop the glass from a height of three meters, and then test this prediction by actually doing so).

In some sciences, particularly astronomy (but also, for example, palaeontology and geology), it is not possible to do controlled experiments in this way, because the necessary conditions cannot be duplicated in the lab or because the scale of the phenomena (in space or in time) is too large. In such cases one must use the theory to predict the results of observations yet to be made. For example, if we have a theory that birds are descended from dinosaurs, we may predict that well-preserved fossils of small dinosaurs will have feathers, or structures clearly ancestral to feathers. This is less satisfactory, because the crucial observations may be very difficult to make (fossils well enough preserved to show feathers are extremely rare), but the principle is the same.

- If the prediction succeeds, we can try other tests (does our prediction of the velocity of the falling glass work for heights of 300, rather than 3, meters? Does it depend on the type of glass?). If it fails, we must return to the original experimental measurements, with the new information gained from our second set, and make another attempt to deduce the correct relation between our variables.

This is the ***scientific method***. Its most fundamental feature is that ***it works***—it is indeed possible to deduce mathematical relations between observed quantities which allow us to predict the behavior of these quantities in different conditions. We can use our understanding of gravity as derived from observations of the planets and experiments on Earth to direct the Voyager spacecraft on their grand tour of the outer planets, and we can be confident that this will work even though no planet has an orbit remotely similar to the trajectory we want for our probe. This is a remarkable finding; it is not at all obvious philosophically that the universe is required to behave in this predictable manner.

Another important feature of the scientific method is that its findings are intimately related to the results of observation and experiment. A scientific theory must always be abandoned or modified if its predictions turn out to be in disagreement with a secure experimental result, no matter how many previous predictions have been successful. Newton's theory of gravity was extremely successful for some 250 years, but it was nonetheless necessary to replace it with General Relativity when it failed to predict the results of observations on the orbit of Mercury and the bending of starlight in the gravitational field of the Sun. New experimental results may force us to abandon a theory completely (the Ptolemaic system in which the Sun and the other planets revolve around the Earth is simply wrong) or just to modify it (the laws of gravity as derived from General Relativity are indistinguishable from those of Newton except under very extreme conditions—Newton's laws were used to guide the Voyager spacecraft and we will use them in this book).

ABSOLUTE TIME AND EUCLIDEAN SPACE

To describe the results of our observations of natural phenomena we need some quantitative concepts. The most basic of these is some way of identifying the particular event we have observed. The only unambiguous way of doing this is to state ***where*** and ***when*** it happened, i.e. its position in space and time, as accurately as possible. (Limitations on accuracy may be imposed by the precision of our measuring equipment or by the quantum mechanical properties of the phenomenon we are observing.)

Everyday observation indicates that ***the flow of time is absolute***, that is, it does not depend on the location or motion of the observer. If I leave my home at 6 a.m., drive to the airport, fly to another city and return home to meet a friend at 9 p.m., we will arrive at the rendezvous together, provided that both our wristwatches keep accurate time. My measurement of the difference between 6 a.m. and 9 p.m. has not been affected by the traveling I have done in the interim.

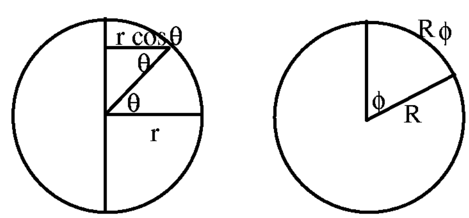
This property of time is fundamental to the way we measure and predict the motion of objects, but it is ***only approximately true***. If I were carrying a state-of-the-art atomic clock capable of measuring with a precision of 10^{-12} s, I would in fact observe that my clock ran more slowly while I was on board the airplane. However, the difference is so tiny that when we are dealing with normal, everyday situations it is impossible to detect. The results that we get by assuming that time is absolute are thus approximate, but practically speaking indistinguishable from what we would have obtained using more sophisticated theories.

Measurement of ***position*** requires us to state the location of the object relative to some agreed reference point. To specify the position unambiguously we need three numbers, or ***coordinates*** (this is what we mean by saying that space is three-dimensional). For example, to get from the bus stop to my apartment I might need to walk 100 meters north, turn left at an intersection and walk 50 meters west, then climb two floors (about six meters). The position of my apartment relative to the bus stop could be expressed in these coordinates as [100, 50, 6] m. This could be specified in different ways—if a spy standing at the bus stop were trying to bug my apartment he would probably think in terms of polar coordinates: a range of 112 meters, bearing 26.5 degrees west of north, elevation 3 degrees—but we always need three coordinates.

In calculating the distance between the bus stop and my apartment I used the Pythagorean theorem: the length of the hypotenuse h of a right-angled triangle of sides x and y is given by $h^2 = x^2 + y^2$. If this is true space is said to be ***Euclidean***, i.e. its geometry is that described in the treatises by the famous Ancient Greek mathematician Euclid. If we are considering distances on the Earth's surface the Pythagorean theorem is ***not*** true for large distances, and large triangles do not have angles that sum to 180° (for example, consider the triangle joining the North Pole, the point on the equator with longitude 0°, and the point on the equator with longitude 90°E. ***Each*** of the angles of this triangle is a right angle!). This is because the Earth is (approximately) spherical, not flat. For short distances this effect is negligibly small, but for large distances it is very important. Let's look at how this affects an attempt to describe distances on the Earth's surface in terms of vectors. Consider moving from Boston (longitude 71°W, latitude 42°N) to London (longitude 0°, latitude 51°N). If we first travel east to longitude 0°, latitude 42°N, we have gone a distance $(r\cos\theta)\phi$, where r is the radius of the Earth (6400 km), θ is the latitude, and ϕ is the difference in longitudes expressed in radians. This comes to 5900 km.

To reach London we must now travel north a distance $r\Delta\theta$ km, where $\Delta\theta$ is the difference in latitudes expressed in radians. This comes to 1000 km. In Euclidean space we would therefore argue that the (two-dimensional) position vector of London relative to Boston is [5900, 1000] km, and the distance from London to Boston should be the magnitude of this vector, or 6000 km.

Now let's do it by first moving to the point with longitude 71°W and latitude 51°N. This is a distance of 1000 km, as before. However, to get to London we now need to travel east only 5000 km, giving a 'vector' of [5000, 1000] km and a 'distance' of 5100 km. ***It matters which way round we do things***. This is completely unlike genuine vector algebra and shows we are not dealing with a flat two-dimensional surface. It is also the reason that flat maps of large portions of the Earth's surface are distorted: it is impossible to represent this non-Euclidean two-dimensional space on the



Euclidean two-dimensional space of a flat map without distorting the relationships between points.

Try this calculation for longitude and latitude differences of 10°, 1°, 0.1°. What happens to the discrepancy as you go to smaller angles? Can you explain why? [Notice that the non-Euclidean properties of distances on the Earth occur because we confine ourselves to moving on the Earth's surface. If we traveled from Boston to London through the Earth's interior rather than staying on the surface, we would have no problems, as the three-dimensional space in the vicinity of the Earth is very accurately Euclidean.]

If we leave the Earth's surface and consider interplanetary or interstellar space, is it Euclidean? For the purposes of this book, ***yes***, space is Euclidean. The distance between the Earth and Voyager 2, which recently left the solar system, can be calculated in the same way as the distance between the bus stop and my apartment, if we know the relevant coordinates.* However, as in the case of time, the Newtonian description is not the complete truth, but a very good approximation which breaks down when we consider extreme conditions. Our best theory of gravity, General Relativity, holds that gravitational effects are due to local distortions of space and time caused by the mass of the gravitating bodies. In the Solar System, therefore, space is not ***perfectly*** Euclidean, because it is distorted by the masses of the Sun and planets. The difference between this picture and the Newtonian theory of gravity which we will study in this book is detectable only very close to a very massive object, and only by making very precise measurements. In the solar system, the orbit of Mercury, the innermost planet, is perturbed by 43 seconds of arc per century, and light rays passing very close to the Sun are bent by 1.75 seconds of arc. These were two early tests of General Relativity (the latter requires a total solar eclipse [at least if we use visible light], so that we can ***see*** stars so close to the Sun). Although the effects of general relativity are small, they are nonetheless needed for very high precision projects. The clocks used in the satellites of the Global Positioning System have to be corrected to account for relativistic effects. Relativistic time dilation associated with by the motion of the satellites causes the clocks to run slowly by 7.11 microseconds per day, but gravitational effects cause then to run faster by 45.7 microseconds per day, leading to a net error of 38.59 microseconds per day.

If we ignore these local distortions caused by concentrations of mass, could space 'as a whole' be Euclidean? This turns out to be related to a deep question in modern cosmology, which is not yet definitively resolved. Nonetheless, very significant progress has been made in just the past few years, and it now appears that the space of the universe is extraordinarily close to being Euclidean.

Cosmologists usually assume that if the local distortions are ignored, the universe can be described as being ***homogeneous*** (i.e., it looks the same at all locations) and ***isotropic*** (i.e., it looks the same in all directions). Given these assumptions, the geometry of the universe is described by one number: its curvature. If the curvature is positive, then the universe is called ***closed***. In this case the space wraps back on itself in a manner very similar to the surface of a sphere. A closed universe has a finite volume, but no boundaries—if a spaceship traveled very far in what appears to be a straight line, it would eventually return to its starting point. In a closed universe, the sum of the angles in a triangle is more than 180°, and the ratio of the circumference of a circle to its diameter is less than π. If the curvature is negative then the universe is called ***open***. In such a universe the sum of the angles in a triangle is less than 180°, and the ratio of the circumference

* In the summer of 2002, Voyager 2 is still returning data, 25 years after its launch! See http://web.mit.edu/space/www/voyager/voyager.html.

of a circle to its diameter is more than π. The ideal mathematical version of an open universe is infinitely large, but we can only speculate about the nature of the real universe at distances larger than what we can observe. The third possibility is that the curvature is zero, in which case the axioms of Euclidean geometry are valid and the universe would therefore be called ***flat***. An ideal Euclidean space is also infinitely large.

According to general relativity, the geometry of the universe is determined by the relation between its (average) mass density and its expansion rate. The expansion of the universe is described by ***Hubble's law***, discovered in 1929 by Edwin Hubble, which states that, on average, any two galaxies separated by distance r are moving apart from each other with a relative speed $v = Hr$, where H is called the ***Hubble constant*** (or sometimes the Hubble parameter, in recognition of the fact that it changes with time over the life of the universe). The present value of the Hubble constant is not known precisely, but was measured in 2001 by the Hubble Space Telescope Key Project to be 72 ± 8 km·sec^{-1}·Mpc^{-1}, where 1 Mpc $= 3.26 \times 10^6$ light-year $= 3.09 \times 10^{22}$ m. The mass density that gives a precisely flat universe is called the ***critical density***, and is given by

$$\rho_c = \frac{3H^2}{8\pi G} ,$$

where G is Newton's gravitational constant (see Chapter 2). Using the Hubble Key Project value for H, the critical density is given by $\rho_c = (9.7 \pm 2.2) \times 10^{-30}$ g/cm$^3 \approx 10^{-29}$ g/cm^3. Note that this is a phenomenally small density, far lower than the density of the best vacuum that can be produced with current technology on Earth. Using ρ to denote the actual average mass density of the universe, cosmologists use the symbol Ω (upper-case Greek Omega) to denote the ratio ρ/ρ_c.

The simplest versions of inflationary theories of cosmology predict that $\Omega = 1$, so the curvature should be zero and the universe should be flat. Until 1998 most of the evidence pointed to $\Omega \approx 0.3$. ***Baryonic matter***, matter made of protons, neutrons, and electrons like the atoms of which we are composed, is known to make up only about 5% of the critical density. The rest of the 0.3 is attributed to ***dark matter***, matter which is not seen, but which is believed to exist because we see its gravitational effect on visible matter. The composition of the dark matter remains a mystery, but it is believed to be composed of something different from protons, neutrons, or electrons. Starting in 1998, however, astronomers have been accumulating evidence for yet another contribution to the cosmic inventory, now often called the ***dark energy***. The evidence for dark energy began with the observation that the expansion of the universe is apparently not being slowed by the force of gravity, but instead the relative velocity between galaxies has been speeding up over the last 5 billion years or so. If this observation is correct, it means that the universe is being influenced by an exotic form of gravity, which acts repulsively instead of attractively. Such a gravitational repulsion is consistent with general relativity, but only if the universe is permeated with a peculiar kind of material that would need to have a negative pressure. General relativity allows us to calculate what the mass density of this material would have to be in order to cause the observed acceleration, and it turns out to be just the right value to contribute 0.7 to Ω, bringing it up to one.

In addition, astronomers studying the cosmic microwave background radiation have also uncovered strong evidence for $\Omega = 1$. This radiation is interpreted as the afterglow of the heat of the big bang, and is found to have the same intensity in all directions (after correcting for the motion of the Earth) to an accuracy of one part in 100,000. Nonetheless, there are subtle ripples in the intensity of the radiation at the level of one part in 100,000, and these ripples can now be measured so accurately that their study has essentially become a new subfield of astronomy. The ripples reflect the oscillations of gases in the early universe, and their motions are believed to be so well understood that the observation of these ripples can be used to measure Ω (and a number of other cosmological parameters as well). For example, a 2002 study by the Cosmic Background Imager team combined the measurements of the cosmic background radiation with measurements of H and some information about large scale structure, concluding that $\Omega = 1.03 \pm 0.04$. When they included the data about the cosmic acceleration as well, they found $\Omega = 1.00 \pm 0.03$.

How Euclidean, then, is the geometry of the universe? Suppose we assume, as the data suggests, that Ω is equal to one to within 5%. In that case, for a circle the size of the solar system, the cosmological curvature causes the circumference to differ from π times the diameter by only about 1 part in 10^{29}, approximately the diameter of a single proton! (If the Sun were at the center of the circle, its gravitational field would cause a much larger difference, but still only about 2 parts per billion.) The possible deviations from Euclidean geometry become more significant at larger distances, but they are still very small. For a circle of radius 10 billion light-years, roughly the size of the visible universe, Euclid's relation between the circumference and diameter would still be accurate to one half of one percent.