ESSENTIALS OF INTRODUCTORY CLASSICAL MECHANICS

Sixth Edition

CHAPTER 2

MASS, FORCE, AND NEWTON'S LAWS

OVERVIEW

In this chapter we introduce the new basic concepts of ***mass*** and ***force***. If we consider an accelerating particle, the force on it is the external influence which is causing it to accelerate, and its mass determines the magnitude of the acceleration produced by a given force. A body which is not being acted on by any net external force has zero acceleration (but ***not*** necessarily zero velocity). The mathematical relationship between mass, force, and acceleration, $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, is Newton's second law. Although experimentally measured forces may be produced in many ways leading to widely varying apparent properties, there appear to be only four basically different ***fundamental forces*** in nature, and even these are very likely related to each other.

As Newton's laws depend upon our assumptions that space is Euclidean and time is absolute, they cease to be good descriptions of nature when these approximations are no longer valid, i.e. when we are considering relative motions close to the speed of light. We will assume for the rest of this book that we are dealing with objects moving at speeds much less than the speed of light, which is, of course, the case for almost all practical applications. We also assume that the systems are large enough so that we do not have to invoke the principles of quantum mechanics.

When you have completed this chapter you should:

- ✔ recognize the principle of inertia: a particle undisturbed by external influences will either be at rest or will maintain a constant velocity;
- ✔ understand the experimental definition of mass;
- ✔ be able to distinguish between mass and weight;
- ✔ understand the experimental definition of force, including the fact that force is a vector quantity;
- ✔ know the mathematical relationship between force, mass and acceleration, Newton's second law, and be able to use it to solve problems;
- ✔ understand the concept of inertial reference frames;
- ✔ qualitatively understand the relationship between the forces measured in the laboratory and the four underlying fundamental forces of nature.

ESSENTIALS

Observation shows that a body set in motion with a constant velocity will slow down and stop if left alone. However, it is clear from experiment that this is the result of an interaction between the body and its environment (e.g. friction with the ground, air resistance). In the absence of such effects, the body would continue to move: for example, the orbital speed of a satellite does not decrease appreciably with time if it is high enough so that atmospheric drag can be ignored. We conclude that

a body left undisturbed maintains a constant velocity. Problem 2A.1

This is ***Newton's first law***, also called the ***law of inertia***. (Note that the constant velocity could be zero.)

Acceleration is produced when the body is subjected to an external influence. The same external influence (e.g. a compressed spring) will produce ***different*** accelerations in different objects, but the ***ratio*** of accelerations of the two bodies is the same regardless of the nature of the external influence (except in the case of gravity—see below). Hence the factor which produces the difference is a property of the object: we call it the object's ***mass*** and define the masses of bodies 1 and 2 such that Problems 2A.3 and 2A.4

$$\frac{m_1}{m_2} = \frac{a_2}{a_1} \; .$$

Mass is a measure of a body's inertia—its resistance to acceleration. The SI unit of mass is the kilogram and is defined relative to a standard reference mass (a platinum-iridium alloy cylinder in Paris). Problem 2A.2

It follows from the above definition that the product $m_i a_i$ is constant for a given "external influence". It can be regarded as a quantitative measure of the experimental factors affecting the motion of the body. This quantitative concept is called the ***force*** exerted on the body. The direction of the acceleration gives the direction of the force, so force is a vector. In vector form we have Problem 2C.1

$$\boxed{\vec{\mathbf{F}} = m\vec{\mathbf{a}} \; .}$$

This is ***Newton's second law***. The unit of force is the ***newton***. Unlike the other units we have met so far, the newton can be expressed in terms of more basic units: 1 N = 1 kg·m/s^2. In words, one newton is the force necessary to accelerate a 1 kg mass at 1 m/s^2.

Newton's third law, which completes Newton's laws of motion, will be discussed in Chapter 5.

Since in practice the velocity of an object must be defined relative to some reference frame, the meaning of the word "velocity" in

Newton's first law requires clarification. Stated precisely, we are assuming that there exists a reference frame in which the law of inertia holds. That is, we assume that there exists a reference frame with respect to which any undisturbed body maintains a constant velocity. Such a reference frame is called ***inertial***. The inertial reference frame is not unique, however: any frame of reference that is moving at a constant velocity relative to an inertial reference frame is also an inertial reference frame. The measured value of an acceleration or force is unchanged when viewed by a second observer moving at a constant velocity relative to the first observer, so an acceleration or force has the same value in all inertial reference frames. Newton's laws of motion hold in all inertial reference frames, and sometimes a problem can be simplified by working it in a frame of reference that is different from the one in which it was posed.

Often two or more forces will be acting on the same body simultaneously. The ***net force*** (also called the ***total force***), which produces the observed acceleration, is the vector sum of all the forces acting on the body. For example, a light fixture hanging from the ceiling has zero acceleration relative to the room, but there are two forces acting: the gravitational force due to the mass of the fixture, and an upward force exerted by the wire by which it is suspended. Taking the ground to be an inertial reference frame (an approximation, as the Earth rotates, but we normally neglect the effects of this), these two forces are equal in magnitude and opposite in direction, so there is no net force and no acceleration. Problems 2C

Experimentally observed forces seem to arise from many sources (gravity, compression of a spring, contact with a hard surface, friction, etc.). Physicists believe, however, that all the forces of nature can be explained in terms of four ***fundamental forces***. Supplementary Notes.

The most familiar fundamental force in our everyday lives is gravity. We are accustomed to feeling the gravitational attraction of the Earth, but in fact any two objects exert a gravitational attraction on each other. The force of gravity on an object of mass m caused by an object of mass M is given by Problem 2B.1.

$$\boxed{\vec{\mathbf{F}} = -\frac{GMm}{r^2}\,\hat{\mathbf{r}}\ ,}$$

where G is a constant, r is the distance between the objects, and $\hat{\mathbf{r}}$ is a unit vector pointing from M to m. The force on M caused by m is given by the same formula, and therefore has the same magnitude. In this case, however, $\hat{\mathbf{r}}$ points from m to M, so the force on M is in the opposite direction from the force on m—each object is attracted towards the other. We usually consider cases in which $M \gg m$ (e.g.,

M is the Earth, and m is a block of cement): in these cases the acceleration of M can be ignored, as it is too small to be measured.

The constant G, called the ***gravitational constant*** or sometimes ***Newton's constant***, has the value 6.67×10^{-11} N·m^2/kg^2.

As a fundamental law, the equation for the gravitational force above applies when M and m are point masses. Larger objects are viewed as being composed of point masses, each of which experiences a gravitational force given by this formula. It can be shown, however, that the formula holds for any two spherically symmetric objects (e.g., solid spheres, spherical shells), where r is the distance between their centers. While the proof of this statement is beyond the scope of this book, we will use the result.

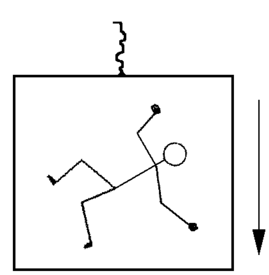
The fact that the 'masses' entering this formula (the ***gravitational mass***) are the same as the 'masses' found from $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ (the ***inertial mass***) is called the Principle of Equivalence, and is the starting point for the development of General Relativity. The fact that the gravitational mass is proportional to the inertial mass, which is not logically necessary, has been confirmed experimentally to an accuracy of one part in 10^{11}.

Closely related to gravity is the concept of ***weight***. As long as one deals with inertial frames of reference only, the weight of an object is simply the magnitude of the gravitational force acting on it. We learned in the last chapter that a freely falling object near the Earth's surface has an acceleration $\vec{\mathbf{g}}$. Therefore, if we work in the approximation that the Earth's surface can be taken as an inertial frame, the gravitational force on such an object is $m\vec{\mathbf{g}}$, and the weight is mg, where $g = |\vec{\mathbf{g}}|$. Since the value of g on the moon is less than it is on Earth, your weight would be lower if you were on the moon, but your mass would be the same as it is on Earth. Since the mass of an object is independent of its location, it tends to be a more useful physical concept than weight. Note that mass and weight have different dimensions and are measured in different units: mass is measured in kilograms, whereas weight, being the magnitude of a force, is measured in newtons.

Problem 2B.7

The definition of weight becomes more complicated if one considers non-inertial frames of reference. In this book we will rarely mention non-inertial frames, but the concept of weight in a non-inertial frame is important enough to make an exception. The surface of the Earth, for example, is not truly an inertial frame, since the Earth is rotating about its axis and revolving around the Sun. One might also want to talk about the weight of an astronaut in a space capsule, which is a highly non-inertial frame of reference. For such non-inertial frames, the vector $\vec{\mathbf{g}}$ is defined to be the acceleration

Problem 2B.4



of a freely falling object relative to the frame. The weight, according to the official SI (Système International) definition, is given by $W \equiv m|\vec{\mathbf{g}}|$. For example, suppose all cables attached to an elevator are cut, so the elevator falls freely downward. The occupants would fall at the same rate, so they could float inside the elevator with no contact with the walls. They would feel weightless (until they hit the ground). In the non-inertial frame of the elevator these occupants would be freely falling with zero acceleration, so their weight would be zero. (Note that the weight of an object is affected by the acceleration of the frame of reference (or, equivalently, the acceleration of the observer), but it is not affected by the acceleration of the object: to a stationary observer, a book on a desk is not accelerating, but it has the same weight as a similar book falling off the desk!)

A force similar in form to gravity is the ***electrostatic*** force between two electrically charged particles, which has the form

Problems 2B.5 and 2B.6

$$\boxed{\vec{\mathbf{F}} = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}\hat{\boldsymbol{r}}\ ,}$$

where Q and q are the charges and $1/(4\pi\epsilon_0)$ is a constant. In this case the charge of the particle is different from its mass, and there is no 'equivalence principle'. Charges are measured in a unit called the coulomb, abbreviated as C, with

$$\frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\ .$$

It seems peculiar to denote the constant by $1/4\pi\epsilon_0$, rather than by a single symbol, but you will see when you study electromagnetism that other equations are simplified by this choice.

The electrostatic force is an aspect of the fundamental force of electromagnetism. The electromagnetic force encompasses the effects of both electric forces—such as the force that holds electrons in orbit about the atomic nucleus—and magnetic forces. Electric and magnetic fields can interact with each other to form electromagnetic waves, which include microwaves, radio waves, visible light, and X-rays.

Gravity and electromagnetism are two of the four fundamental forces. The other two are both short-range, acting over distances comparable to the diameter of an atomic nucleus. The ***weak force*** is responsible for the radioactivity of some types of atomic nuclei. The ***strong force*** is responsible for the structure of protons and neutrons, each of which are believed to be composed of three particles called ***quarks***, bound together by the strong force. The strong force also holds protons and neutrons together inside the atomic nucleus. Gravity is by far the weakest of the four forces, and the strong force is the strongest. It is hard to believe that gravity is the weakest of the forces, since it exerts such a strong influence on our everyday world. Gravity gives the illusion of being strong, however, because it is long-range and always attractive, so we feel the combined attraction of all 10^{52} particles that make up the Earth; in contrast, although the electromagnetic force is much stronger, the Earth contains almost exactly equal numbers of positively and negatively charged particles, so the total electrostatic force at long range is practically zero.

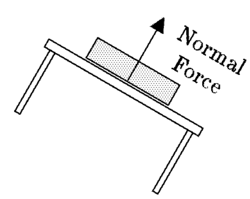
It is now believed that the weak and electromagnetic forces are different aspects of a single force, called the ***electroweak force***, and it is possible that all four fundamental forces can be explained in terms of a single, unified force.

Supplementary Notes

Many commonly encountered forces are not fundamental, but rather are the large-scale observable effects of the electromagnetic force acting on a microscopic scale between atoms and molecules. For example, contact forces between surfaces are caused by temporary electromagnetic bonds being formed between neighboring atoms on the two surfaces. We describe here several of the most commonly encountered ***macroscopic*** forces: the ***normal force*** between surfaces in contact, the ***tension*** in a string, and the force associated with the compression or stretching of a spring.

The ***normal force*** is that part of the contact force that one object exerts on another which is in the direction perpendicular to the surface between the two objects. (The part of the contact force tangential to the surface is called friction, and will be ignored in this book until Chapter 6.) When a book rests on a table, the force of gravity acts downward on the book. The book does not fall through the table, however, because the table exerts a normal force upward, of equal strength. (Of course the book could be so heavy that the table breaks apart, but for now we will assume that all our tables, inclined planes, road surfaces, roller coaster tracks, etc., are completely rigid and indestructible.) The situation is slightly more complicated if the table is tilted, and the book is sliding along its surface. Again we know that the book will not fall through the table, and that the book will not fly upward. The general rule is that any two objects that touch each other exert normal forces on each other. The force is by definition normal (i.e., perpendicular) to the surface joining them, directed so as to push the objects apart. Its magnitude is just large enough to prevent the objects from penetrating each other. The force can vanish if no force is needed to prevent penetration, but it can never pull the objects together.

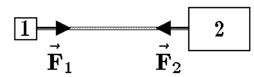
Problems 2C.1 and 2C.5



Another frequently encountered force is the ***tension*** in a string, rope, or wire. Just as we are assuming for simplicity that table surfaces cannot bend or break, we will also assume for now that

Problems 2C.3 and 2C.5

all ropes can be approximated as massless and inextensible (i.e, they cannot be stretched). When such a rope is pulled taut, it exerts forces on the objects at both ends, in each case pulling the object towards the rope. The forces at the two ends have the same magnitude, called the tension of the rope. The magnitude of the tension is whatever is necessary to prevent the rope from stretching. The tension of a rope can be positive or zero, but never negative.



$|\vec{\mathbf{F}}_1| = |\vec{\mathbf{F}}_2| = T$

T = tension

If a body is at rest with no net force acting on it, it will clearly remain in the same position: it is then said to be in ***equilibrium***. [Note that we are presently assuming that the body we are dealing with is a point mass, with negligible size, and we can therefore assume that all the forces acting on it act at the same point. Later in the book we will see what happens when this is not the case.]

Equilibrium will be further discussed in Chapter 4.

An equilibrium position is ***stable*** if a particle slightly displaced from the position of equilibrium is subject to a force which tends to restore equilibrium. An example of this is a mass suspended from a string: in equilibrium the mass hangs directly below the suspension point of the string, and if it is displaced slightly from this position in any direction it will move back towards it when released—i.e. there is a restoring force. In this and many other cases, the magnitude of the restoring force is proportional to the displacement from equilibrium, so it can be written (in one dimension) as

Problems 2D

$$F_x = -kx \qquad (k \text{ a suitably dimensioned constant}).$$

Using $F_x = ma_x$, one finds the ***differential equation***

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx\ .$$

It is useful to rewrite this formula in the standard form

$$\boxed{\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x\ ,}$$

where in this case $\omega^2 = k/m$. This equation recurs frequently in physics, and the motion it describes is called ***simple harmonic motion***. Note that x and ω might have different meanings for different problems, but the mathematical solution is always the same. It is easy to check that the differential equation is satisfied if

$$\boxed{x(t) = A\sin\omega t\ ,}$$

where A is any constant. It can be shown that any function which satisfies the differential equation can be written this way, provided that one chooses to start one's clock so that $t = 0$ when $x = 0$. The quantity ω is called the ***angular frequency***; its units are radians per second, so that ωt is an angle in radians*. Since one cycle of the sine function is 2π radians,

$$\boxed{\omega = 2\pi f \ ,}$$

where f is the frequency measured in cycles per second. 1 cycle per second is also called a ***hertz***, abbreviated Hz. The ***period*** of oscillation is the time for one cycle,

$$\boxed{T = \frac{1}{f} = \frac{2\pi}{\omega} \ .}$$

An example of simple harmonic motion is a mass attached to a stretched or compressed spring. To a good approximation, springs are found to exert a restoring force proportional to the amount of stretching or compression:

Problems 2B.2 and 2B.3

$$\boxed{F_x = -kx \ .}$$

In this case x is the difference between the spring's natural length and its length when compressed or stretched, and the constant k is called the ***spring constant***. The units of k are N/m. (This is called 'Hooke's law', although in this case the word 'law' is ill-chosen—Hooke's law is a simple experimental relation valid for a restricted class of objects, not a widely applicable fact of nature like Newton's laws.)

* The radian is the SI unit of angle. It is defined as the angle between two radii of a circle that cut off on the circumference an arc equal in length to the radius: i.e., as the circumference of a circle of radius r is $2\pi r$, there are 2π radians in a complete circle (360°). Angles are dimensionless, so the dimensions of ω are 1/[time].

SUMMARY

* A body subject to no disturbance from outside will either be and remain at rest or maintain a state of uniform unaccelerated motion (Newton's first law).
* If external forces are applied to a body, it will accelerate with an acceleration equal to the total applied force, divided by the mass of the accelerated body (Newton's second law). Two observers in uniform relative motion will observe the same acceleration and the same force, provided that their relative speed is small compared to that of light.
* The gravitational force between two objectsis attractive, with a magnitude proportional to the product of their masses and inversely proportional to the square of the distance between them. The fact that the mass as defined in this way is proportional to the mass defined by the ratio of accelerations produced by a given force is known as the Principle of Equivalence, a fundamental property of nature and one of the cornerstones of the theory of General Relativity.
* The magnitude of the electrostatic or ***Coulomb*** force between two objects is proportional to the product of their charges and inversely proportional to the square of the distance between them: it is therefore analogous in form to the gravitational force. The force is repulsive if the two charges have the same sign, and otherwise it is attractive.
* Physical concepts introduced in this chapter: mass, force, inertial reference frame, electrical charge.
* Mathematical concepts introduced in this chapter: differential equation.
* Equations introduced in this chapter:

$\vec{\mathbf{F}} = m\vec{\mathbf{a}}$	(Newton's second law);
$\vec{\mathbf{F}} = -\dfrac{GMm}{r^2}\hat{\mathbf{r}}$	(the gravitational force between two particles);
$\vec{\mathbf{F}} = \dfrac{1}{4\pi\epsilon_0}\dfrac{Qq}{r^2}\hat{\mathbf{r}}$	(the electrostatic force between two particles);
$\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x$	(for a particle near a point of stable equilibrium; equation leads to simple harmonic motion);
$x = A\sin\omega t$	(a solution to the above equation; any solution can be written this way if we choose $t = 0$ when $x = 0$);
$\omega = 2\pi f$	(relation between angular frequency and frequency);
$T = \dfrac{1}{f} = \dfrac{2\pi}{\omega}$	(period of an oscillator).

* The ***weight*** of an object measured in an inertial reference frame is the magnitude of the gravitational force on the object. In a non-inertial frame the weight of an object is mg, where m is its mass and g is the magnitude of the acceleration that it would have if allowed to fall freely in that reference frame. (An astronaut thus has zero weight in the non-inertial reference frame of her orbiting spacecraft, although the magnitude of the gravitational force acting on her is not greatly decreased from its value at the surface

of the Earth.) The Earth's surface is not strictly an inertial reference frame, although we usually treat it as such when doing calculations.

* A position at which there is no net force on a body is called a position of ***equilibrium***. If a body is displaced from a position of equilibrium, in many cases the force acting on the body is (to a good approximation) proportional to the distance from the equilibrium point and directed towards it. In this case the body will undergo ***simple harmonic motion***, which means qualitatively that it will ***oscillate*** back and forth about the point of equilibrium.
* Any two objects that touch each other can exert ***normal forces*** on each other, perpendicular to the surface joining them and directed so as to push the objects apart. The magnitude is just large enough to prevent the objects from penetrating each other. The force can vanish if no force is needed to prevent penetration, but it can never pull the objects together.
* Strings, ropes, and wires can be approximated as being massless and inextensible. Such a rope pulls the objects at each end toward the rope with a force whose magnitude is equal to the ***tension*** of the rope. The value of the tension is whatever is necessary to prevent the rope from stretching. The tension is never negative.

PROBLEMS AND QUESTIONS

By the end of this chapter you should be able to answer or solve the types of questions or problems stated below.

Note: throughout the book, in multiple-choice problems, the answers have been rounded off to 2 significant figures, unless otherwise stated.

At the end of the chapter there are answers to all the problems. In addition, for problems with an (H) or (S) after the number, there are respectively hints on how to solve the problems or completely worked-out solutions.

2A FUNDAMENTAL CONCEPTS (FORCE, MASS, AND NEWTON'S SECOND LAW)

2A.1 An airplane is flying due west (relative to the ground) at a constant speed of 600 km/h. The mass of the plane is 8500 kg, and the engines are supplying a constant forward thrust of 5000 N. To the nearest 10 N, what is the magnitude of the net force acting on the plane?

(a) 83300 N; (b) 5000 N; (c) 83450 N; (d) none of these

2A.2 In the context of the physics of this chapter, why is it easier to catch and hold a tennis ball than it is to catch and hold a lead ball of the same size, moving with the same velocity?

2A.3 A man pushing a cart with a mass of 180 kg can accelerate it from rest to 3 m/s in 4s. Approximately how long would you expect it to take him to accelerate the same cart to 3 m/s if a 120 kg mass is placed on the cart?

(a) 5.2; (b) 5.2 s; (c) 6.7 s; (d) none of these

2A.4 (H) A spring gun is used to accelerate small pucks horizontally across a frictionless surface. A puck with a mass of 100 g is found to accelerate at 3 m/s^2. When the experiment is repeated with two other pucks of unknown mass, one accelerates at 1.7 m/s^2 and the other at 4.1 m/s^2. Calculate their masses. If the three pucks were glued together so as to form one larger mass, what would its acceleration be in this experiment?

2B FUNDAMENTAL AND MACROSCOPIC FORCES

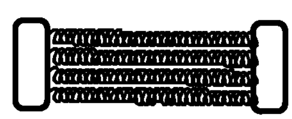
2B.1 An astronaut of mass 80 kg is a member of the crew of a space shuttle orbiting the Earth at an altitude of 220 km. What is the magnitude of the gravitational force on the astronaut? Assume that the radius of the Earth is 6400 km, and take g at the surface of the Earth as 9.8 m/s^2.

(a) 730 N; (b) 780 N; (c) 0 N; (d) none of these

2B.2 A spring balance is constructed using a spring with k = 150 N/m. How far will the spring extend when a mass of 1.8 kg is suspended from it?

(a) 12 mm; (b) 8.5 cm; (c) 8.5 mm; (d) 12 cm

2B.3 (H) A body-building accessory consists of two handgrips joined by four identical springs. If each individual spring obeys Hooke's law with constant k, what is the spring constant of the whole device? What if the four springs had different constants?



2B.4 (H) An astronaut aboard an orbiting spacecraft observes that he and other objects within the spacecraft are weightless. Is it true that no net force acts on them? If not, how is the presence of a force reconciled with the weightlessness of the contents of the spacecraft?

2B.5 (S) Protons have a mass of 1.67×10^{-27} kg and a charge of 1.60×10^{-19} coulomb. The protons in an atomic nucleus are separated by distances of around 10^{-15} m. Calculate the electrostatic force between neighboring protons. Estimate the acceleration with which protons would depart from the nucleus if this were the only force operating, and comment on your result.

2B.6 (H) Calculate the electrostatic force exerted on an electron by a proton at a distance of 10^{-10} m. Compare this with the gravitational force between the two. In the light of your comparison, discuss why gravity, and not electromagnetism, is the fundamental force most apparent to us on a macroscopic scale. (The mass of the proton is 1.67×10^{-27} kg, that of the electron is 9.11×10^{-31} kg, and their charges are $\pm 1.60 \times 10^{-19}$ coulomb respectively. The numerical values (in SI units) of the relevant constants are $1/4\pi\epsilon_0 = 8.99 \times 10^9 \mathrm{N \cdot m^2/C^2}$ for the electrostatic force and $G = 6.67 \times 10^{-11} \mathrm{N \cdot m^2/kg^2}$ for the gravitational force.)

2B.7 Explain, in 100 words or less, the difference between ***mass*** and ***weight***. Would an object have the same mass on the Moon as it does on the Earth? Would it have the same weight?

2C NEWTON'S SECOND LAW: FORCE, MASS, AND ACCELERATION

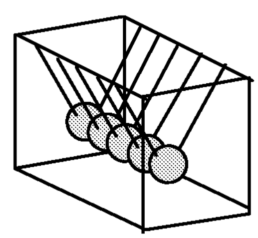
2C.1 A block of mass 1.5 kg slides down a frictionless slope inclined at 40° to the horizontal. What is the magnitude of its acceleration down the slope?

(a) 9.8 $\mathrm{m/s^2}$; (b) 6.3 $\mathrm{m/s^2}$; (c) 7.5 $\mathrm{m/s^2}$; (d) none of these

What is the magnitude of the force exerted on it by the slope?

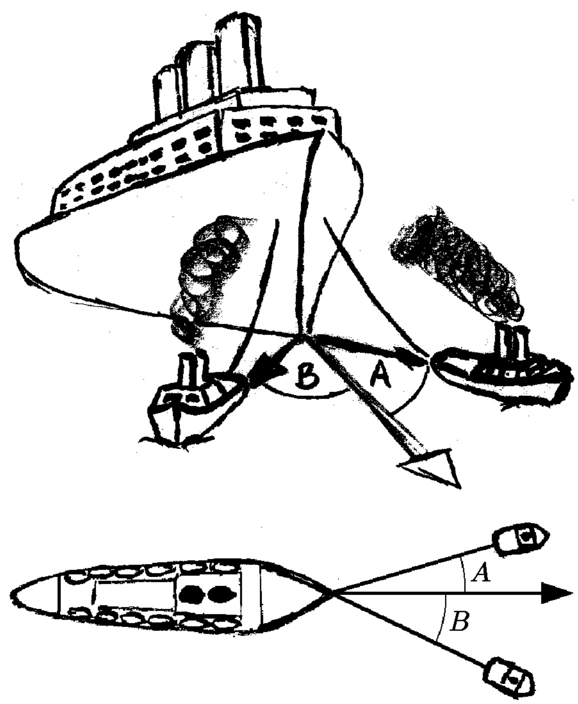
(a) 6.3 N; (b) 7.5 N; (c) 11.3 N; (d) 9.4 N

2C.2 (S) In the toy known as a Newton's cradle, steel balls are suspended by fine threads from a wooden frame. In a particular specimen, each ball has a mass of 50 g and the threads each make an angle of 20° to the vertical. Assuming that the ball is not moving, what forces are acting on it?



2C.3 (H) A tight-rope walker stands midway along a high wire of length ℓ. If her mass is m, what must the tension T in the wire be if it sags by an amount y? Would it be possible to arrange the wire so that it did not sag at all ($y = 0$)? If the length of the wire is 25 m and the acrobat's mass is 55 kg, calculate the tension in the wire if it sags by 5 cm.

2C.4 (H) Two tugboats are towing a liner out of harbor. Their lines are arranged as in the diagram. If tug A exerts a force of magnitude F_A, derive an expression (in terms of F_A and the angles A and B) for the magnitude F_B of the force that tug B must exert if the net acceleration of the liner is to be straight ahead. Calculate F_B if $F_A = 3.1 \times 10^5$ N, with angle $A = 15°$ and angle $B = 18°$.



2C.5 (S) A mother tows her daughter on a sled on level ice. The friction between the sled and the ice is negligible, and the tow rope makes an angle of 40° to the horizontal. The combined mass of the sled and the child is 25 kg. If the sled accelerates at 1 m/s^2, calculate the tension in the rope. What is the normal force exerted by the ice? (As will be justified in Chapter 5, the child and sled can be treated in this problem as if they comprised a single particle.)

2D VARIABLE FORCES

2D.1 (S) (a) A spring compressed by some amount is found to give a 1.3 kg mass an acceleration of 1.1 m/s^2. The experiment is repeated with a second object which accelerates at 2.4 m/s^2. Calculate the mass of the second object.

(b) The amount of compression was 1.0 cm. Calculate the force constant of the spring, assuming it behaves according to Hooke's law. By how much would you have had to compress the spring to give the second mass the same acceleration as the first?

(c) The first mass is now hung vertically from the same spring. By how much does the spring extend?

(d) If you hold the hanging mass slightly below its equilibrium position, extending the spring an additional amount Δx, what forces are now acting on the mass? What happens if you let go?

2D.2 (S) (a) A pendulum consists of a 250 g bob suspended by a string of negligible mass. If I hold the bob so that the string is taut and makes an angle θ with the vertical, and then release it, what forces are acting at the moment of release?

(b) If I agree to displace the bob by no more than 5° from the vertical, find a differential equation that describes (to a good approximation) the subsequent motion of the bob. Show that this equation is satisfied if the motion of the bob takes the form

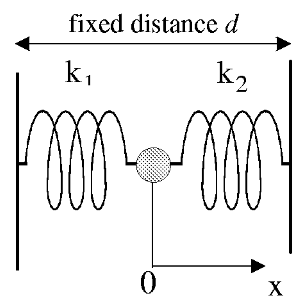
$$\theta = A \sin(\omega t + \phi)$$

2D.2, continued:

where A and ϕ are arbitrary constants, and derive an expression for ω. Describe in words the motion of the bob.

(c) I wish to use the pendulum to drive a clock. What length of string should I use to get a period of 1.00 s, and how fast will the bob then move at the bottom of its sweep if the maximum angle is $\pm 2°$?

2D.3 (H) A small object of mass m is held by two springs with spring constant k_1 and k_2 as shown. The mass rests on a smooth surface so that the effects of friction are negligible. Left undisturbed, the mass sits at position $x = 0$.



(a) If it is now displaced to position $x = -A$ and released, what forces are acting on it at the instant of release?

(b) Obtain a differential equation describing the motion of the mass after its release, and use this to derive an expression for its period of oscillation.

(c) If I turn the device on end, so that the springs hang vertical, what is the effect on the position of the stable configuration and on the motion of the mass when displaced? Assume the springs themselves have negligible mass.

COMPLETE SOLUTIONS TO PROBLEMS WITH AN (S)

2B.5 *Protons have a mass of* 1.67×10^{-27} kg *and a charge of* 1.60×10^{-19} *coulomb. The protons in an atomic nucleus are separated by distances of around* 10^{-15} m. *Calculate the electrostatic force between neighboring protons. Estimate the acceleration with which protons would depart from the nucleus if this were the only force operating, and comment on your result.*

Conceptualize

This is an application of $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$. We know the form of the electrostatic force, which depends on the charges of the interacting bodies and the distance between them. To calculate the acceleration we also need to know the mass of the accelerating body. All this information is provided in the question.

Formulate

The electrostatic force exerted by a proton on another is

$$\frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2},$$

where q is the charge on each proton and r is the separation of the two. The numerical value of the constant $1/4\pi\epsilon_0$ is $8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}$.

Solve

The force is $(8.99 \times 10^9 \mathrm{N \cdot m^2/C^2}) \times (1.60 \times 10^{-19}\mathrm{C})^2/(10^{-15}\mathrm{m})^2 = 230$ N. This would yield an acceleration of $(230\ \mathrm{N})/(1.67 \times 10^{-27}\ \mathrm{kg}) = 1.4 \times 10^{29}\ \mathrm{m/s^2}$!

Obviously, we must conclude (since nuclei containing 80 or so protons are perfectly stable) that the electrostatic force is not the only one acting. Of course one additional force is gravity, but you can check for yourself (see problem 2B.6) that this is far too weak to counteract the electrostatic repulsion.

Scrutinize

The dimensions of the electrostatic force are correct once we take into account the dimensions of the constant $1/4\pi\epsilon_0$. The numerical values do not appear reasonable, but this is the point of the question (signaled in the wording of the question by the phrase "comment on your result").

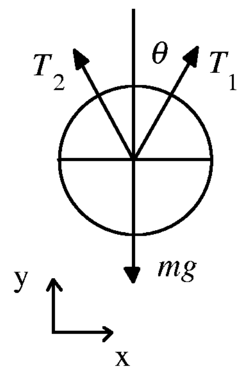
Learn

We are of course neglecting all sorts of quantum mechanical and relativity complications here, but the conclusion is certainly sound. Protons are bound in the nucleus by the effects of one of the two short-range fundamental forces: this one is called the 'strong force', for reasons which should now be apparent.

2C.2 *In the toy known as a Newton's cradle, steel balls are suspended by fine threads from a wooden frame. In a particular specimen, each ball has a mass of* 50 g *and the threads each make an angle of 20° to the vertical. Assuming that the ball is not moving, what forces are acting on it?*

Conceptualize

We start by drawing a force diagram for the ball. One force acting on the ball is obviously gravity, so there is a force $m\vec{g}$ directed downwards. The threads holding the ball must also be exerting a force and it is apparent from everyday experience that this force is directed along the line of the thread (think of towing a car, hanging a picture, etc.). This very common type of force is called the ***tension*** in the thread.



We also know that the ball is not accelerating. Therefore the ***net*** force on the ball is zero. We solve this problem by constructing and solving the component equations for the net force.

Formulate

We define a coordinate system with x and y axes as shown. The z-axis is directed out of the page; all z-components in this problem are zero. The component equations for the net force are:

$$\begin{aligned} F_x &= T_1 \sin\theta - T_2 \sin\theta &&= 0\,; \\ F_y &= T_1 \cos\theta + T_2 \cos\theta - mg &&= 0\,. \end{aligned}$$

Solve

The first equation tells us that both string tensions are equal in magnitude. Since the arrangement is symmetrical, this is what we would expect. Defining $T \equiv T_1 = T_2$, the second equation gives us

$$T = \frac{mg}{2\cos\theta}\ .$$

Substituting the numerical values, with $g = 9.8\ \mathrm{m/s^2}$, the forces acting on the ball are

$$(0.05\ \mathrm{kg}) \times (9.8\ \mathrm{m/s^2}) = 0.49\ \mathrm{N}\ \text{downwards;}$$

$$(0.49\ \mathrm{N})/(2 \times \cos 20°) = 0.26\ \mathrm{N}\ \text{at}\ 20°\ \text{left of vertically upwards;}$$

$$(0.49\ \mathrm{N})/(2 \times \cos 20°) = 0.26\ \mathrm{N}\ \text{at}\ 20°\ \text{right of vertically upwards.}$$

Scrutinize

Sine and cosine functions arise trigonometrically from the ratio of two lengths (two sides of a triangle) and are therefore dimensionless, so a force divided by a cosine is another force. As mentioned above, the system is clearly symmetrical about a line drawn vertically through the center of the ball, so the two string tensions should be equal in magnitude, as indeed they are.

Learn

Notice that θ can have any value from 0 up to a maximum of 90°, which corresponds to the strings having no sag at all. It can be seen from our result, however, that the

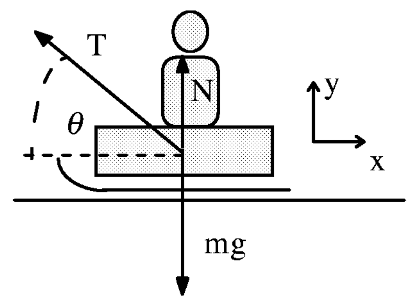
2C.2, continued:

tension approaches infinity as θ approaches 90°. Since the tension can never really be infinite, it is impossible for θ ever to be exactly 90°.

2C.5 *A mother tows her daughter on a sled on level ice. The friction between the sled and the ice is negligible, and the tow rope makes an angle of 40° to the horizontal. The combined mass of the sled and the child is* 25 kg. *If the sled accelerates at* 1 m/s^2, *calculate the tension in the rope. What is the normal force exerted by the ice? (As will be justified in Chapter 5, the child and sled can be treated in this problem as if they comprised a single particle.)*

Conceptualize

We treat the system of child and sled as a point particle, assuming that all the forces act at the same point. The forces acting on this "particle" are gravity, a normal force from the ice, and the tension in the tow-rope. The net force must be in the horizontal direction, since the sled is not rising into the air or sinking into the ice.



The force diagram is shown on the right. We know the acceleration of the sled, and can therefore calculate the components of the total force. Our strategy will be to construct and solve the equations giving this total force in terms of the three individual forces acting.

Formulate

The component equations for the net force are

$$\begin{aligned} F_x &= -T\cos\theta &&= ma_x\,; \\ F_y &= T\sin\theta + N - mg &&= ma_y\,. \end{aligned}$$

Our unknowns are T and N; we are given $a_x = -1$ m/s^2, $a_y = 0$, $m = 25$ kg, and $\theta = 40°$.

Solve

The x-component of the force on the sled is $ma_x = -25$ N. This is $-T\cos 40°$, so the tension in the rope must be 33 N.

The y-component of the force is zero, so $N = mg - T\sin 40° = 220$ N to two significant figures, taking $g = 9.8$ m/s^2.

Scrutinize

This is a straightforward problem presenting no difficulties. The units of our answer are obviously consistent and the numerical values seem reasonable.

2D.1 (a) *A spring compressed by some amount is found to give a* 1.3 kg *mass an acceleration of* 1.1 m/s^2. *The experiment is repeated with a second object which accelerates at* 2.4 m/s^2. *Calculate the mass of the second object.*

2D.1, continued:

Conceptualize
We defined the ratio of the masses of two bodies in terms of the ratio of their accelerations when subjected to the same force. This is exactly the situation we have here, so we can use that defining equation to calculate the mass ratio.

Formulate
The relevant equation is

$$\frac{m_2}{m_1} = \frac{a_1}{a_2} ,$$

where the subscript 1 refers to the first object and 2 to the second.

Solve
The second mass must be $(1.3 \text{ kg}) \times (1.1 \text{ m/s}^2)/(2.4 \text{ m/s}^2) = 0.60$ kg.

Scrutinize
The dimensions are correct, since the ratio of two accelerations is dimensionless. The smaller mass has the greater acceleration, which conforms to our expectations—pushing a wheelbarrow is easier than pushing a truck.

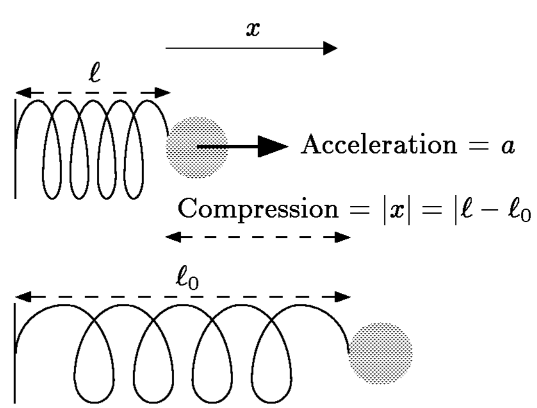
Learn
Note that we don't need to know the magnitude of the force, or its source (we have nowhere used the fact that a spring is involved). All we need is the information that the same experimental conditions (and thus the same net force) applied in both cases. The only exception is gravity, where the force is proportional to the mass, and thus applying the same experimental conditions to different masses does ***not*** produce the same net force.

(b) *The amount of compression was* 1.0 cm. *Calculate the force constant of the spring, assuming it behaves according to Hooke's law. By how much would you have had to compress the spring to give the second mass the same acceleration as the first?*

Conceptualize
We know the mass and acceleration of each of the two bodies, and can therefore use $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ to calculate the force. We can also obtain an expression for the force, in terms of the unknown spring constant, from Hooke's law. We should therefore be able to equate these two expressions to calculate the spring constant. Since this is a one-dimensional problem, we have only one component equation.

Formulate and Solve
Hooke's law says that $F_x = -kx$, where $x = \ell - \ell_0$ is the difference between the present length ℓ of the spring and its unstretched length ℓ_0 ($x < 0$ if the spring is compressed and $x > 0$ if it is stretched). Applying Newton's second law, $F = ma$, gives us

$$k = -\frac{ma}{x} .$$



With $x = -0.01$ m, $a = 1.1$ m/s^2, and $m = 1.3$ kg, we obtain $k = 140$ N/m to

2D.1, continued:

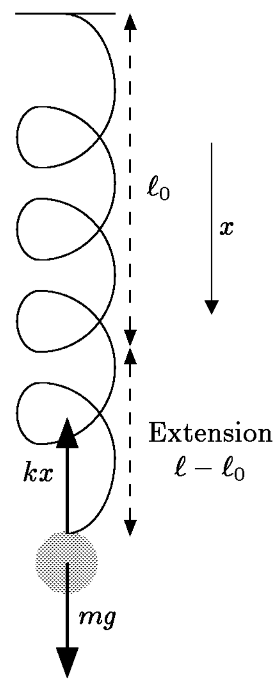
two significant figures. Note the sign of x compared to a: the mass accelerates in the positive direction ($F_x > 0$) if the spring is compressed ($x < 0$).

Using the same equation with $m = 0.60$ kg, $a = 1.1$ m/s^2, and $k = 140$ N/m gives $x = -0.0046$ m $= -4.6$ mm for the compression required to give the second mass the same acceleration as the first.

Scrutinize

Note that the constant k must have units N/m (dimensions [force]/[length], or in terms of more basic quantities [mass]/[time]^2), in order for $-kx$ to be a force. This agrees with the dimensions of ma/x.

In the second part a and k are the same for both masses: if $ma = -kx$ it follows that the x values must have the same ratio as the masses. We can use this to check our answer: $0.6/1.3 = 0.46$, so $x = -0.46$ cm.



(c) *The first mass is now hung vertically from the same spring. By how much does the spring extend?*

Conceptualize

If the mass is stationary, the total force on it must be zero. We can therefore calculate the extension of the spring by equating the magnitudes of the spring force and the weight of the mass.

Formulate and Solve

The net force acting on the mass in the x-direction (downwards) is

$$F_x = mg - kx \ .$$

When the mass is stationary the net force must vanish, so

$$x = \frac{mg}{k} \ , \qquad \text{where } x = \ell - \ell_0 \ .$$

Numerically $x = (1.3 \text{ kg}) \times (9.8 \text{ m/s}^2)/(140 \text{ N/m}) = 0.089$ m.

Scrutinize

This is the same equation we used in part (b), so for the same mass and k the ratio of compression lengths must be the same as the ratio g/a. So our spring extension should be $(9.8 \text{ m/s}^2)/(1.1 \text{ m/s}^2)$ times 1 cm, i.e. 8.9 cm, in agreement with the above calculation.

Learn

We could apply this equation in reverse, i.e. measure the extension of a spring of known k when a mass is suspended from it, and use this to determine the mass. This is the operating principle of a ***spring balance***. In contrast to a ***beam balance***, where we measure the mass m relative to a known mass M (essentially by balancing mg against Mg), the reading of a spring balance is sensitive to the value of g. Spring balances measure weight, so they should really be calibrated in newtons rather than kilograms.

2D.1, continued:

(d) *If you hold the hanging mass slightly below its equilibrium position, extending the spring an additional amount Δx, what forces are now acting on the mass? What happens if you let go?*

Conceptualize

The situation is the same as in part (c), but the magnitude of the spring force is increased by the larger extension. Therefore we expect that the total force on the mass from gravity and the spring is nonzero and is directed upwards. While you are holding the mass stationary, this upward force is balanced by a downward force exerted by you, and the total force on the mass is zero (as it must be, since the mass is not accelerating). If you let go, the mass will accelerate upwards. It will continue to accelerate upwards until it reaches the equilibrium position, where the force will go through zero and then become downwards. The mass will cross the equilibrium position with some nonzero speed, but then the downward force will cause it to slow down, stop, and start to move downwards. Once again it will be moving when it passes the equilibrium position, and the cycle will repeat. We therefore expect that the mass will ***oscillate*** above and below its position of equilibrium.

Formulate

The forces acting are mg downwards from gravity and kx upwards from the spring, plus the force you exert while holding the mass. We already know that $mg = kx_0$, where x_0 is the extension of the spring when the mass hangs in the equilibrium position (i.e. 8.9 cm, as we calculated in part (c)), so the spring force and gravity contribute a total of $k(x - x_0)$ upwards. While you are holding the mass, the total force is zero, so you must exert a force $k(x - x_0)$ downwards.

When you let go, the total force on the mass is $F_x = -k(x - x_0) = -kx'$, where $x' \equiv x - x_0$. The equation of motion of the mass is then

$$F_x = -kx' = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = m\frac{\mathrm{d}^2 x'}{\mathrm{d}t^2} \ ,$$

so

$$\frac{\mathrm{d}^2 x'}{\mathrm{d}t^2} = -\frac{k}{m}\, x' \ .$$

Solve

Our equation is of the form $\mathrm{d}^2 x'/\mathrm{d}t^2 = -\omega^2 x'$, where in this case $\omega^2 = k/m$. We saw in the *Essentials,* and will see again in Problem 2D.2, that a solution to this equation is

$$x' = A \sin \omega t \ ,$$

where A is a constant. If we have no preference for how the clock that defines t was

2D.1, continued:

started, then any solution to the differential equation can be written this way. Replacing x' by $x - x_0$ and ω by $\sqrt{k/m}$, the solution can be rewritten as

$$x = x_0 + A \sin\left(\sqrt{\frac{k}{m}}\, t\right) .$$

Thus the mass will oscillate sinusoidally about the equilibrium position x_0, with angular frequency $\omega = \sqrt{k/m}$.

Scrutinize

The form of the differential equation agrees with our expectation: larger extensions produce greater acceleration, and the acceleration is always directed towards the equilibrium point. The acceleration is increased if the spring constant k is increased, but decreased if the mass m is increased.

The solution is oscillatory, as we predicted when conceptualizing the problem. The period of the oscillations is $2\pi/\omega$, i.e. $2\pi\sqrt{m/k}$, which has the dimensions of time as we expect (recall that the dimensions of k are [mass]/[time]^2). The period increases if the mass increases, but decreases if we use a larger k (i.e., a "springier" spring).

Learn

An essential part of this solution was the changing of variables necessary to cast the differential equation into a simple form: we introduced $x' \equiv x - x_0$, and $\omega \equiv \sqrt{k/m}$. This is the standard technique for solving problems involving simple harmonic motion, and more generally one finds that differential equations can often be simplified by a judicious change of variables.

Above we wrote down a general solution to the differential equation: neither the constant A nor the starting value of t was specified. With more thought, however, we can determine the precise solution that applies to this problem. If we choose to use a time variable t that starts at $t = 0$ at the moment the mass is released, then we know that at $t = 0$ the value of x' was Δx and the value of $\mathrm{d}x'/\mathrm{d}t$ was 0. The differential equation can be solved by a sine or cosine function (or a sum of the two), but only the cosine has zero derivative when its argument vanishes. The initial value of the time derivative can therefore be satisfied by writing

$$x' = A \cos\left(\sqrt{\frac{k}{m}}\, t\right) .$$

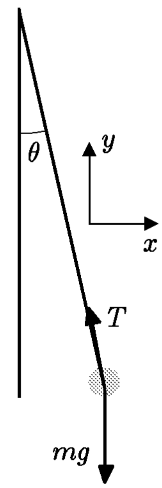
The initial value of x' can then be matched by choosing $A = \Delta x$, completely determining the solution. Thus, the amplitude of the oscillation is equal to the initial displacement.

Earlier we said that the most general solution can be written as $x' = A \sin \omega t$, so you might be wondering how this can be consistent with the cosine solution above. Remember, however, that $\cos \omega t = \sin(\omega t + \pi/2) = \sin \omega t'$, where $t' = t + (\pi/2\omega)$, so the two forms are related by a redefinition of the origin of time.

2D.2 (a) *A pendulum consists of a* 250 g *bob suspended by a string of negligible mass. If I hold the bob so that the string is taut and makes an angle θ with the vertical, and then release it, what forces are acting at the moment of release?*

Conceptualize

One force acting on the bob is obviously gravity. If this were the only one, the bob would accelerate straight downward on being released, which it clearly does not do. It is prevented from doing so by the string, which must therefore be exerting a tension force. We saw in Problem 2C.2 that tension forces act along the line of the string, so we can draw a force diagram for the bob as shown. It is apparent that there is a net force in the x-direction, which will accelerate the bob back towards its equilibrium position (hanging directly under the point of suspension); there must also be a net force along the line of the string, providing the centripetal acceleration to maintain the bob on its circular arc.



Formulate

We define a coordinate system with y vertical and x horizontal as shown. Then the component equations for the total force are

$$F_x = ma_x = -T\sin\theta;$$
$$F_y = ma_y = T\cos\theta - mg.$$

Solve

In this context, 'solving' the equations means using them to determine the subsequent motion of the bob. It would be very difficult to do this for the equations in the form we have them at present. Instead we consider, in part (b), the special case of small θ, where the equations are simplified considerably.

(b) *If I agree to displace the bob by no more than* 5° *from the vertical, find a differential equation that describes (to a good approximation) the subsequent motion of the bob. Show that this equation is satisfied if the motion of the bob takes the form*

$$\theta = A\sin(\omega t + \phi)\ ,$$

where A and ϕ are arbitrary constants, and derive an expression for ω. Describe in words the motion of the bob.

Conceptualize

The conceptual picture here is the same as before. Only the formulation will differ, because we can use small angle approximations.

Formulate

It can be shown that for small angles

$$\cos\theta \approx 1 - \tfrac{1}{2}\theta^2 \approx 1;$$
$$\sin\theta \approx \theta \text{ (in radians)}.$$

2D.2, continued:

Is 5° a small angle? Converting to radians (2π radians $= 360°$) gives 0.0873 radians, which is certainly a small number compared with 1 or π. In fact $\sin(0.0873\,\text{radians}) = 0.0872$, a difference of 0.13%. As a general rule, angles up to about 10° can be regarded as 'small'.

If $\cos\theta \approx 1$, the vertical motion of our bob is negligible, since its distance below the suspension point is $\ell\cos\theta \approx \ell$. Therefore, the vertical force must also be negligible, so we can write

$$F_y = T\cos\theta - mg \approx T - mg \approx 0.$$

This tells us that $T = mg$. From the diagram, $\sin\theta = x/\ell$, so the x-component of the force becomes

$$ma_x = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{mg}{\ell}x,$$

which is a differential equation for x. As $x = \ell\sin\theta \approx \ell\theta$, a differential equation for x is equivalent to a differential equation for θ, so solving this equation would give us an expression for θ.

Solve

Solving differential equations from scratch is beyond the mathematical level of this book, but in this case we have already been given a possible solution to try. We simply differentiate it twice and see what we get:

$$\begin{aligned} x &\approx \ell\theta = \ell A\sin(\omega t + \phi) \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= \omega\ell A\cos(\omega t + \phi) \\ \frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= -\omega^2\ell A\sin(\omega t + \phi) = -\omega^2 x\ . \end{aligned}$$

This has the required form, and is a solution if $\omega = \sqrt{g/\ell}$. A and ϕ are arbitrary, in the sense that any value of A or ϕ yields a valid solution, but they do have well-defined interpretations in terms of the physical motion of the bob: A gives the maximum deviation from equilibrium (the ***amplitude***) and ϕ gives the starting point.

The sine function has maximum and minimum values ± 1 and repeats every 2π radians. Therefore the mass will oscillate between the positions $\theta = -A$ and $\theta = +A$, where A is the original displacement from vertical, and it will complete one full cycle during each time interval of length $2\pi/\omega$. Since sine has its maximum value where cosine is zero, and vice versa, the bob will reach its maximum speed as it passes through equilibrium, at $\theta = 0$. Its maximum acceleration occurs at the same time as its maximum displacement, when it reverses direction at the end of each sweep.

Scrutinize

The dimensions of ω^2 are ([length]/[time]^2)/[length], so ωt is dimensionless, as it must be since it is the argument of a sine function. Note that our final equations have no dependence on the mass of the bob; this is typical of systems where the force acting is

2D.2, continued:

gravity, because we tend to end up with equations of the form $F = ma = Kmg$, where K is some dimensionless coefficient such as a sine or cosine, and the mass then cancels out. However, the bob must be massive enough to allow us to neglect the mass of the string by comparison, and also compact enough that we can treat it as a point mass.

Learn

Why does the mathematical solution to our differential equation leave A and ϕ undetermined? This is typical of differential equations; it arises because in solving the differential equation we are effectively doing an integration, and therefore introducing an unknown integration constant. The number of undetermined quantities we get comes from the number of integrations we have to do—two, in this case, because $\mathrm{d}^2\theta/\mathrm{d}t^2$ is a second derivative. Physically, we can understand the two undetermined constants by thinking about the role of the initial conditions. As in the simple case of uniform acceleration along a line, $\mathrm{d}^2x/\mathrm{d}t^2 = a_x$, the differential equation contains no information about how the motion was started. The differential equations of classical mechanics generally tell us the rate of change of the velocity, but we need to know the initial velocity before the velocity at any given time can be determined. Similarly, the velocity tells us the rate of change of the position, but we need to know the initial value of the position to determine its value at later times. Thus, the solution to the uniform acceleration equation is written as $x = x_0 + v_0t + \frac{1}{2}a_xt^2$, where x_0 and v_0 are the initial position and velocity. Similarly, for the pendulum we could determine the actual values of A and ϕ if we are told the initial position and velocity of the pendulum.

(c) *I wish to use the pendulum to drive a clock. What length of string should I use to get a period of* 1.00 s, *and how fast will the bob then move at the bottom of its sweep if the maximum angle is* $\pm 2°$*?*

Solve

(This is simply a numerical application of part (b), so we have already carried through the conceptualization and formulation.)

The period of oscillation T is $2\pi/\omega$, i.e.

$$T^2 = 4\pi^2\ell/g \text{ , or } \ell = gT^2/4\pi^2 \ .$$

For $T = 1$ s, the length is $(9.8\ \mathrm{m/s^2}) \times (1\ \mathrm{s})^2/4\pi^2 = 0.25$ m.

The speed of the bob is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \ell\frac{\mathrm{d}\theta}{\mathrm{d}t} = \ell\omega A\cos(\omega t + \phi) \ .$$

At the bottom of the swing the speed is maximized, so the cosine is 1 and the speed is $\ell\omega A = A\sqrt{g\ell}$, where A is the swing amplitude ***in radians*** ($2° = 0.035$ radians). This comes to 0.054 m/s.

Scrutinize

Since the dimensions of ω are 1/[time], it follows that $1/\omega$ represents a time. The factor of 2π is dimensionless and cannot be detected by dimensional analysis; in this problem the same is true of A (thus $\ell\omega A$ and $\ell\omega$ both have dimensions of [length]/[time]). There is

2D.2, continued:

no easy way to detect errors involving dimensionless factors, though checks of particular numerical values can help.

The numerical value of the length seems plausible when we consider the size of real pendulum clocks (which admittedly might have pendulum periods of 0.5 s or 2 s rather than 1 s).

Learn

From the point of view of a clock designer, the important feature of a pendulum is that the swing period is independent of the swing amplitude, provided that the amplitude is small. The typical clock pendulum is not simply a heavy bob on a light string, so the analysis is slightly different, but the motion still turns out to be simple harmonic.

What happens if the swing amplitude is not small? Clearly the motion is still oscillatory, because the force on the bob is always directed towards the equilibrium position. Analyzing the exact equation mathematically is beyond the scope of this book, but just by looking at the form of the force we can get a qualitative idea of what would happen. For angles which are not small, $\sin\theta < \theta$, so the acceleration will be smaller than you would calculate from the small angle approximation. Therefore the velocity will also be smaller, and so we conclude that the period of a pendulum increases when the amplitude is increased beyond the limits of the small angle approximation.

To summarize, we have made three approximations in this analysis compared to a real experiment: we have assumed that we can treat a pendulum as a point mass suspended on a massless string; we have assumed small swing amplitudes and therefore used the approximation that $\sin\theta \simeq \theta$; and we have neglected frictional effects in assuming that the only forces acting are the string tension and gravity. If we had not made these approximations, there would have been differences in the details of the motion—the "effective length" of a real pendulum is different from its physical measured length; the period of a pendulum depends on its amplitude if the amplitude is not small; friction causes the pendulum to "run down" instead of swinging with the same amplitude forever—but the essential physics would be unaltered. On the other hand, the equations we would have had to solve would have been very much more complicated, and quite beyond the sort of simple math we have been using here. This use of idealization to simplify complex problems without changing their essential features is one of the most important features of the scientific approach to problem solving.

HINTS FOR PROBLEMS WITH AN (H)

The number of the hint refers to the number of the problem

2A.4 What is the meaning of the word "mass"? How could you tell that one mass was three times another mass?

2B.3 If the device is extended by an amount Δx, what is the force exerted on the handgrip by one spring? What is the total force exerted on the handgrip? If you replaced the four springs by one single spring, what would its spring constant be?

2B.4 What is the motion of the spacecraft and its contents? Is this motion unaccelerated? What is the acceleration of the astronaut in the (non-inertial) rest frame of the spacecraft? What is the sensation of weight?

2B.6 What is the total charge Q of a body consisting of a large number of protons and an exactly equal number of electrons? What is the net electrostatic force on such a body due to a distant charged object?

2C.3 Draw a force diagram for the tight-rope walker. What is the net horizontal component of force? The net vertical component? If the wire were absolutely straight, could the net vertical component of the force be exactly zero?

A review of the solution of 2C.2 might be useful if you are having difficulty with 2C.3 or 2C.4.

2C.4 What is the liner's acceleration perpendicular to its direction of motion? What must the net force be in that direction? What's the net force in that direction written in terms of F_{A} and F_{B}?

2D.3 (a) A tricky aspect of this problem is that we are not told the unstretched lengths of the springs. So consider three possibilities:

(i) Suppose the springs are initially unstretched at $x = 0$. If the object is moved to the left a distance A, is the spring on the left compressed or stretched? What force $F_{x,1}$ will it exert on the object? Will the spring on the right be compressed or stretched, and what force $F_{x,2}$ will it exert?

(ii) Now suppose both springs are initially compressed; for simplicity, assume that they are compressed by a distance greater than A. What is the net force on the object at $x = 0$? When the object is moved to $x = -A$, is the spring on the left compressed more or less? By what amount $\Delta F_{x,1}$ does the force it exerts on the object change? Answer these same questions for the spring on the right.

(iii) If the springs are initially stretched, by a distance greater than A, what are the answers to the questions in the previous part?

(iv) Draw a force diagram for the mass at position $x = -A$.

(b) What is the net force on the object at an arbitrary position x?

(c) In the vertical orientation, where is the mass in equilibrium? If we call this position $y = 0$, what vertical forces act on the mass at some arbitrary position y? Define the positive y-direction to be downward.

ANSWERS TO HINTS

2A.4 Ratio of two masses is defined through $m_1/m_2 = a_2/a_1$ for the same applied force. The absolute value is defined by reference to an arbitrary standard mass.

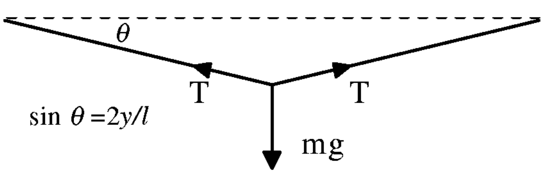
2B.3 $k\Delta x$; $4k\Delta x$; $4k$.

2B.4 Orbiting the Earth in uniform circular motion with the same velocity and therefore the same acceleration; no. Zero. You feel pushed against the floor—if the floor were not there, you would accelerate downwards.

2B.6 Zero; almost zero—it would be exactly zero if the protons and electrons in the body were at exactly equal distances from the charged object.

2C.3 Zero ($T\cos\theta - T\cos\theta$);

$2T\sin\theta - mg$ ($= 0$, as she is not moving).



No, because θ would be 0.

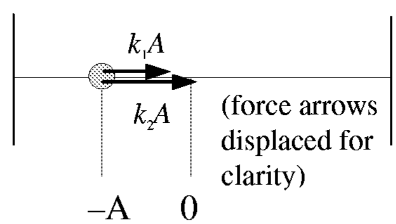
2C.4 Zero; zero; $F_{\mathrm{A}}\sin A - F_{\mathrm{B}}\sin B$.

2D.3 (a) (i) Spring on left compressed, $F_{x,1} = k_1A$. Other spring stretched, $F_{x,2} = k_2A$ (same direction).

(ii) Net force at $x = 0$ is zero. Spring on left compressed more, $\Delta F_{x,1} = k_1A$. Other spring compressed less, $\Delta F_{x,2} = k_2A$ (same direction again).

(iii) Net force at $x = 0$ is again zero. Spring on left stretched less, $\Delta F_{x,1} = k_1A$. Other spring stretched more, $\Delta F_{x,2} = k_2A$. In all cases, $\Delta F_{x,1} = k_1A$ and $\Delta F_{x,2} = k_2A$.

(iv)



(b) $F_x = -(k_1 + k_2)x$.

(c) When vertical, equilibrium is lower than original equilibrium position $x = 0$ by a displacement d, where $mg = (k_1 + k_2)d$. At arbitrary position y,

$$\begin{aligned} F_y &= mg - (k_1 + k_2)(d + y) \\ &= -(k_1 + k_2)y \ . \end{aligned}$$

ANSWERS TO ALL PROBLEMS

2A.1 d (The net force is zero.)

2A.2 Since the lead ball is more massive, more force is required to decelerate it to zero velocity, if one tries to stop it in the same distance.

2A.3 c

2A.4 180 g; 73 g; 0.86 m/s^2.

2B.1 a

2B.2 d

2B.3 4k; $\sum k_i$, where k_i is the constant of spring i ($i = 1$ through 4).

2B.4 No: the force of gravity is not much less than its surface value. However, the spacecraft and its contents are all undergoing circular motion around the Earth, with a centripetal acceleration equal to the acceleration due to gravity at this distance from Earth. The astronaut therefore has no acceleration ***relative to the spacecraft***, and thus has no tendency to 'fall' within the craft. He would experience the same feeling if unfortunate enough to be trapped inside an elevator with a broken cable. (Hence the more physically motivated term for this situation—'free fall'.)

2B.5 1.4×10^{29} m/s^2.

2B.6 Electrostatic force: 2.3×10^{-8} N;

gravitational force: 1.0×10^{-47} N;

both directed towards the proton.

Electric charge comes with both positive and negative sign, and matter on a large scale is generally neutral, so a large object exerts no net electrostatic force on a charged particle, if the particle is sufficiently far away that the different positions of individual charges within the large body are not significant. Mass, on the other hand, is always positive, so the gravitational force exerted by a large amount of matter is cumulative.

2B.7 There is, of course, no single "correct" answer to the first part of this problem. An acceptable answer would be:

"***Mass*** is an intrinsic property of a given object which will have the same value regardless of the object's environment and of the frame in which the measurement is made. ***Weight***, on the other hand, is the magnitude of the local force of gravity on the object as measured in the relevant reference frame, which may not be inertial. It therefore depends on the environment (the mass and distance of the body exerting the gravitational force) and the choice of reference frame."

An object would have the same mass on the Moon, but different weight.

2C.1 b; c.

2C.2 See complete solution.

2C.3 $T = \ell mg/4y$, for small θ; 6.7 kN; no.

2C.4 2.6×10^5 N.

2C.5 33 N; 220N perpendicular to the ice.

2D.1 (a) 0.60 kg.

(b) 140 N/m

(c) 8.9 cm.

(d) See complete solution.

2D.2 See complete solution.

2D.3 (a) In equilibrium, the forces from spring 1 and spring 2 acting on the object cancel each other. When the object is displaced, the force $F_{1,x}$ from the first spring increases by $k_1 A$; i.e. there is now an additional force $k_1 A$ acting toward positive x. The force $F_{2,x}$ from the second increases by $k_2 A$, which also corresponds to an additional force toward positive x. Hence the net force is $F_x = (k_1 + k_2)A$ towards positive x.

(b)

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -(k_1 + k_2)x \ .$$

The mass will oscillate around $x = 0$ in simple harmonic motion, i.e. with $x = -A\cos\omega t$, where ω is given by $\sqrt{(k_1 + k_2)/m}$. The limits of the oscillation are $\pm A$ and the period is $2\ \pi/\omega$.

(c) The stable position will be displaced downwards by an amount $d = mg/(k_1 + k_2)$; the motion will be unaffected, except for the shift in the equilibrium point.

CHAPTER 2

SUPPLEMENTARY NOTES

SIMPLE EXAMPLES VS THE REAL WORLD

The examples and problems we have considered to date have introduced the well-known artifacts of Physics-Problem Land: frictionless slopes, projectiles immune to air resistance, superdense objects with masses of 1 kg but negligible size, massless strings, and so on. These objects are not common in everyday experience—friction and air resistance ***matter*** in calculating, for example, the acceleration and top speed of a newly designed automobile. Yet we have claimed that physics, like all other sciences, is a structure of mathematical laws derived from and based upon experimental results. Are we, therefore, cheating by ignoring these important practical effects in formulating our problems? Can we really gain insight by considering such artificial examples? In some sense, of course, we are indeed cheating. If we were doing a calculation with a view to applying the results to a particular experimental situation, we would have to consider ***all*** the circumstances pertaining to the experiment and decide if any of them has an effect (e.g. friction in the suspension bearing of a pendulum, or calibration errors in the clock we are using to measure its period). We might conclude that there is no effect (it doesn't matter what color I paint the bob of the pendulum), that an effect is present in principle but too small to measure (we have already seen that the differences between the properties of space and time in Newtonian theory and General Relativity are completely negligible in everyday experience), that an effect is present and must be considered in the analysis (if we wanted to use the pendulum to calculate the mass of the Earth to three or more significant figures by measuring g, we would have to consider the effect of the Earth's rotation), or that the effect is present and so large that we must redesign the experiment to avoid it (we are unlikely to get an accurate measurement of g by dropping a feather, unless we conduct the experiment in a vacuum tube).

The sources and sizes of effects making significant contributions to the result of the experiment will depend on the details of the experimental setup. In a well-designed experiment, the phenomenon you want to study will be the dominant effect, and other contributions will be small. For example, the effect of friction on the motion of objects can be reduced to very near zero by conducting experiments on a linear air track, and air resistance can be minimized by using a small dense projectile moving at a comparatively low speed, or eliminated altogether by conducting the experiment in a vacuum.

Physics-Problem Land is an idealized version of these well-designed experiments in which the additional effects have been reduced to zero. In many cases the degree of idealization is quite small: it is not too difficult to construct a pendulum in which the mass of the string is very small compared to the mass of the bob, and where frictional effects are unimportant. In other examples the difference between the idealized version and the real thing is significant: the effect of air resistance on the flight of an arrow is certainly not negligible. However, if we included the effect of air resistance, we would not change the ***principles*** of the problem—we would just make it very much more difficult to solve. In practice, problems of this sort often do not have neat algebraic solutions; they are actually 'solved' by calculating the trajectory by computer, in a series of tiny steps (this is called 'numerical integration'). Doing this is a useful exercise in programming skills for a computer science student, and a vital part of the design process for an aeronautical engineer, but it does not provide further illumination of the underlying physics. The principles governing the motion of freely falling bodies were worked out by Galileo in a series of carefully controlled experiments with balls rolling down inclined planes which allowed him to distinguish between the fundamental physics (the action of gravity) and the environmental effects such as friction and air resistance. (Contrary to popular belief, he did ***not*** drop cannonballs off the Leaning Tower of Pisa: if he had, air resistance effects might have led him to the wrong conclusions!)

Do the idealized, soluble problems of Physics-Problem Land have any practical application? Certainly they do. A well-chosen idealization can allow us to extract the basic physics of a system without the need to set up complex and expensive computer simulations or engineering models. Theoretical physicists often set up "toy models"—highly idealized systems—to see if a new theory has any prospect of describing real phenomena, before proceeding with more realistic calculations. Likewise, an engineer may use approximate, idealized calculations to check whether the basic principles of a new design are sound, before proceeding to fill in the details for computer simulations and wind-tunnel test models. The art of choosing good idealizations, which retain the essential physical properties of the real system while simplifying its mathematical description, is a vital part of scientific work.

The apparent complexity of the behavior of objects in the real world stems largely from the fact that we are not dealing with single particles, but with assemblies of 10^{25} or so atoms and molecules. The forces acting between the individual atoms are quite simple and readily calculable, although because of the small sizes involved we have to add the ideas of quantum mechanics to the tools we are developing in this book. If it were possible to keep track of the motion of each individual atom, we would not have to worry about concepts such as friction and air resistance, and we could do everything in terms of the fundamental forces, which all behave very simply. However, it is clearly not possible to do this—we can neither make the observations (what is the 45 millionth iron atom in my desk doing at this precise moment?) nor do the calculations (imagine drawing 10^{25} individual force diagrams!). The complicated behavior of macroscopic forces like friction is not a fundamental property of nature, but a consequence of our insistence on working with macroscopic objects instead of basic building blocks. We will see later in the book how physical phenomena not obviously related to Newton's laws, such as heat and the pressure of a gas, are also consequences of the unobserved motion of atoms and molecules.

FUNDAMENTAL FORCES

Physicists distinguish between ***macroscopic*** forces, such as the force produced by a compressed spring, and ***fundamental*** forces such as gravity. This distinction is of very little value in terms of calculating the acceleration of an object—we draw the same force diagrams and use the same equations regardless of the source and nature of the force. Conceptually, however, it is a vital step in understanding the laws governing the structure of the universe, rather as the construction of the Periodic Table was a vital step towards the modern understanding of chemistry. If we had to try to understand every experimentally observed type of force individually (e.g. spring forces, friction, normal forces, muscular forces, etc., etc.), there would be little hope of learning more about the underlying structure of natural phenomena. (Newton was in this position, and hence the only force for which he could formulate a general law was gravity, which happened to be a fundamental force accessible to him; Newton's first and second laws, on the other hand, relate measured forces to measured accelerations, but tell us nothing about what the force in a given situation will actually be.) Once we know that measured forces are derived from only a few truly different fundamental forces, we are in a much better position to develop a real understanding. In fact, as this understanding progresses, we are learning that three of the four forces we now recognize (all but gravity) act in very similar ways at the elementary particle level, so that there is a possibility of including all of them in a single theoretical framework (a so-called ***grand unified theory***). Many theoretical physicists are even exploring the possibility that all four forces can be described by a single interaction in the context of what is called ***superstring*** theory. Either of these unifications would be a further advance along the road which started in the nineteenth century when electricity and magnetism, previously believed to be different things, were recognized as different aspects of the electromagnetic force, and continued in the 1960's and 70's with the uncovering of the close relationship of electromagnetism and the weak nuclear force.