ESSENTIALS OF INTRODUCTORY CLASSICAL MECHANICS

Sixth Edition

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CHAPTER 3

THE MOTION OF A POINT PARTICLE

OVERVIEW

This chapter contains no new ideas. Instead, we take the material covered in Chapters 1 and 2 and use it to solve more complicated problems. Notice that in this chapter we will not divide the problems up into sections dealing with specific topics; instead you will have to decide for yourself which of the physical principles you have learned are relevant to a given problem. This is a very important aspect of the technique of problem solving—real-life problems, such as designing a bridge support or an artificial hip joint, do not come with neat subtitles directing you to the right equations!

When you have completed this chapter you should:

- ✔ be able to extract the essential features of a problem and express them in mathematical equations;
- ✔ be capable of manipulating the equations relevant to a problem to obtain an expression for the required quantity, either symbolically or numerically;
- ✔ be able to analyze a hypothetical situation in terms of the physics presented in previous chapters, and explain that analysis clearly in non-mathematical terms.

PROBLEMS AND QUESTIONS

By the end of this chapter you should be able to answer or solve the types of questions or problems stated below.

Note: throughout the book, in multiple-choice problems, the answers have been rounded off to 2 significant figures, unless otherwise stated.

At the end of the chapter there are answers to all the problems. In addition, for problems with an (H) or (S) after the number, there are respectively hints on how to solve the problems or completely worked-out solutions.

3.1 (S) (a) A particle moves along the x-axis. Its position at any time is given by

$$x = x_0 + bt + ct^2 \ ,$$

where $x_0 = 1$ m, $b = 2$ m/s and $c = -3$ m/s^2.

(i) What are the particle's velocity and acceleration at any time t?

(ii) Calculate its velocity and acceleration for each time it passes through the point $x = 0$.

(iii) Is its velocity ever zero? If so, what are its position and acceleration at that time?

(iv) Plot its position and velocity as functions of time for the time interval -10 s to $+10$ s.

(b) Now consider a particle moving in three dimensions with position vector

$$\vec{\mathbf{r}} = [a_1 + a_2 t, b_1 + b_2 t^3, c_1 + c_2 t^2] \ ,$$

where $a_1 = 1$ m, $a_2 = 6$ m/s, $b_1 = 4$ m, $b_2 = -1$ m/s^3, $c_1 = 2$ m and $c_2 = -1$ m/s^2. What are its velocity and acceleration vectors at time t? What is its speed and the magnitude of its acceleration at time $t = -3$ s?

3.2 An astronaut on the Moon throws a stone with velocity $\vec{\mathbf{v}}$. It is in flight for a time Δt, reaches a height h and travels a horizontal distance x. He now throws a second stone with velocity $2\vec{\mathbf{v}}$. Which of the following statements (A) through (L) are true? (List ***all*** that apply.)

(A) Its flight lasts four times as long.
(B) Its flight lasts twice as long.
(C) It reaches a height $2h$.
(D) It travels four times as far.
(E) It travels twice as far.
(F) It goes four times as high.
(G) It travels half as far.
(H) It goes half as high.
(I) It travels the same distance, but goes higher.
(J) It goes higher, but travels less far.
(K) It reaches the same height, but travels further.
(L) The maximum height is less, but the horizontal distance is greater.

He throws a third stone with the same speed v as the first. Its flight lasts twice as long as that of the first stone. Which of the statements (C) through (L) above ***must*** be true? (List ***all*** that apply.)

3.3 A toy train moves at constant speed on a level circular track of radius r, taking a time T to complete one circuit. Obtain expressions for the speed v of the train, the magnitude a of its centripetal acceleration, and the x- and y-components of its velocity at time t. What is the source of the force producing the acceleration? If $r = 1.2$ m and $T = 30$ s, calculate v and a, and plot the x- and y-components of the position, velocity and centripetal acceleration of the train over one complete circle, assuming that it is traveling counterclockwise and that its position at $t = 0$ is $x = r,\ y = 0$.

3.4 (S) A local agricultural show features a display of archery. An archer is practicing at the target as the celebrity who is to open the show is arriving by helicopter. If the archer directs his arrow at 45° with a speed of 60 m/s and the helicopter is descending vertically with a constant speed of 10 m/s, plot the trajectory of the arrow as seen by (a) an observer on the ground and (b) an observer in the helicopter. What are the velocity and acceleration vectors at time t in each frame of reference? Take $g = 10$ m/s^2, ignore air resistance, and choose coordinates so that the launch point of the arrow is at the origin in both frames of reference.

3.5 (H) A certain unorthodox archeologist is trapped on top of a truck traveling at 40 km/h. His friends, pursuing in a jeep, finally overtake the truck at a speed of 45 km/h. With a perfectly timed leap our hero throws himself from the top of the truck to land in the back of the jeep. The top of the truck is 3 m above the jeep and there is 2 m between the vehicles as they pass.

(a) Assuming our man launches himself horizontally and directly outward from the truck, at what speed does he need to jump to land safely in the jeep—say 0.5 m from the side of the jeep, i.e. 2.5 m from the side of the truck?

(b) He was standing 4 m from the front of the truck and lands 2 m from the front of the jeep. If he jumped straight outward from the truck, what were the relative positions of truck and jeep at the moment of takeoff?

(c) Plot his trajectory, in terms of height relative to initial vertical position against horizontal distance traveled, as seen (i) from the truck, (ii) from the jeep, (ii) by a baffled Bedouin standing by the side of the road.

Take $g = 10$ m/s^2 and neglect air resistance throughout.

3.6 (H) Household scales, although (when metric) always calibrated in kilograms, actually balance the ***weight*** of the object against some other force—either the force of gravity on a reference mass (a beam balance) or the compression or extension of a spring (a spring balance). If I accurately calibrate a beam balance and a spring balance in Boston and then take them to the top of Mount Everest, will they give the same reading when I use them to weigh a rock? Explain your reasoning.

3.7 The acceleration of freely falling bodies, g, and the radius of the Earth, R, are far easier to measure than the gravitational constant G. Derive an expression for the orbital period of a satellite in a circular orbit with radius r (that is, its height above the Earth's surface is $h = r - R$) which involves ***only*** g, R and r (i.e. does not require a measurement of G or of the mass of the Earth).

In the light of this result, discuss why it is that G is so difficult to determine experimentally (it is one of the least well known of the fundamental constants).

3.8 (S) A bobsled run includes a bend in the form of a circular arc of radius 25 m. A sled approaches the bend at a constant speed of 90 km/h. At what angle must the sled bank in order to take the curve successfully? Assume that this part of the course is level, and that there is no friction between the sled runners and the ice.

3.9 (H) In a conical pendulum, the bob describes a horizontal circle while the string (which is of negligible mass) makes a fixed angle with the vertical. By what fraction does (i) the speed of the bob and (ii) the period of the conical pendulum change if I change the angle the string makes with the vertical from $5°$ to $10°$? In the small angle approximation, how does the period of a pendulum working in this way compare with the period of the same pendulum swinging in the conventional fashion (i.e. back and forth with small amplitude in a line passing under the point of suspension)?

3.10 (S) Calculate the final velocity of a mass m (a) falling freely from rest through a distance h; (b) sliding a distance ℓ down a frictionless slope making an angle ϕ to the horizontal, where $\sin\phi = h/\ell$; (c) forming the bob of a simple pendulum of length ℓ which makes a maximum angle with the vertical of $\theta = \sqrt{2h/\ell}$ (in this case calculate the velocity at $\theta = 0$). Give an expression for the net force acting in each case. Assume $h \ll \ell$.

3.11 A child has a toy rocket of mass m attached to a string of negligible mass. She whirls the toy at speed v in a horizontal circle of radius ℓ, at height h above the (level) ground.

Draw a force diagram for the rocket and use it to find:

(a) the angle θ that the string makes with the horizontal;

(b) the tension T in the string.

Assume the motion of the child's hand is negligible compared to ℓ.

(c) At a certain point the string breaks. How far from the child's feet does the rocket hit the ground?

3.12 (H) It's raining, and you forgot your umbrella. You have 2 km to walk back to your apartment. There is no wind and the rain is coming down vertically. What are you going to do: walk slowly, on the grounds that then only your head and shoulders are exposed to the rain, or run, arguing that the less time you are out the less wet you will get? This problem is a mathematical analysis of your situation, albeit with several simplifying approximations.

Assume for simplicity that a person is basically cuboidal, with dimensions $h \times w \times d$ (height, width, depth), and walks at a speed v. The rain is coming down with constant speed V, and the amount of rain per cubic meter is ρ kg/m^3.

(a) If you walk erect, how much rain will fall on you if you walk a distance ℓ? Your answer will be a function of h, w, d, v, V, ρ and ℓ.

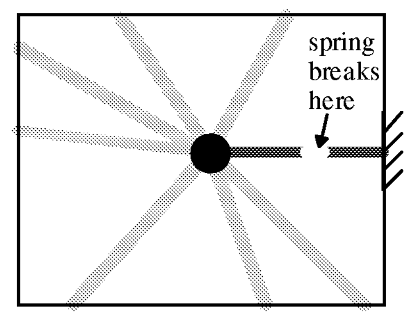
(b) How does your formula behave as v becomes very small or very large? What do you conclude about the best strategy for minimizing the amount of rain you catch?

(c) In practice, you are not compelled to walk erect. If the rain is coming down at 20 km/h, and the fastest speed at which you can run 2 km is 12 km/h, what is your best plan?

3.13 (S) The distance between the Earth and the Moon is 384 thousand kilometers, and the mass of the Moon is 7.4×10^{22} kg. Calculate the acceleration of the Earth due to the Moon's gravitational attraction. If a point mass were describing a circular orbit with this centripetal acceleration and a period of 27.3 days, what would be the radius of the orbit? Take $G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$.

3.14 The mass of the Sun is 2.0×10^{30} kg and the mass of the Earth is 6.0×10^{24} kg. The Moon (which has a mass of 7.4×10^{22} kg) orbits the Earth at a distance of 384 thousand kilometers, and the Earth and Moon together orbit the Sun at a distance of 150 million kilometers. Find the gravitational pull on the Moon from the Sun and from the Earth. Hence calculate the acceleration of the Moon (a) when it is directly between the Sun and the Earth; (b) when the Earth is directly between it and the Sun; (c) when Sun, Earth and Moon form a right-angled triangle. Which of the forces acting is more important? Why do we nonetheless see the Moon as simply orbiting the Earth?

3.15 A mass M is held stationary by many light springs as shown. If the spring on the right breaks, what is the acceleration of the mass immediately afterwards? Explain your reasoning clearly. Assume that the spring in question obeys Hooke's law with a force constant k, and that before breaking it was extended by an amount x; also assume that the total mass of all the springs is negligible compared to M.



COMPLETE SOLUTIONS TO PROBLEMS WITH AN (S)

3.1 (a) *A particle moves along the x-axis. Its position at any time is given by*

$$x = x_0 + bt + ct^2 \ ,$$

where $x_0 = 1$ m, $b = 2$ m/s *and* $c = -3$ m/s^2.

(i) *What are the particle's velocity and acceleration at any time t?*

(ii) *Calculate its velocity and acceleration for each time it passes through the point* $x = 0$.

(iii) *Is its velocity ever zero? If so, what are its position and acceleration at that time?*

(iv) *Plot its position and velocity as functions of time for the time interval* -10 s *to* $+10$ s.

Conceptualize

We are given the position x, and asked to find the velocity and the acceleration. This is a one-dimensional problem, so we have only one component equation—instead of $\vec{\mathbf{r}}$, $\vec{\mathbf{v}}$ and $\vec{\mathbf{a}}$ we have to consider only x, v_x and a_x. To solve it we use the definitions of velocity and acceleration from Chapter 1: velocity is the rate of change of position with time, and acceleration is the rate of change of velocity with time.

$\Sigma\int$

Formulate

The definitions of velocity and acceleration are:

$$\vec{\mathbf{v}} = \frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t} \ ; \qquad \vec{\mathbf{a}} = \frac{\mathrm{d}\vec{\mathbf{v}}}{\mathrm{d}t} = \frac{\mathrm{d}^2\vec{\mathbf{r}}}{\mathrm{d}t^2} \ .$$

In this one-dimensional problem these reduce to

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} \ ; \qquad a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} \ .$$

Solve

To answer (i), we differentiate the expression for $x(t)$ to find

$$v_x = b + 2ct \ ;$$

$$a_x = 2c = -6 \ \mathrm{m/s}^2 \ .$$

Note that the acceleration is constant—no factors of t remain.

For part (ii) we first solve the quadratic equation $x(t) = 0$ to find the times t at which $x = 0$:

$$t = \frac{1}{2c}\left(-b \pm \sqrt{b^2 - 4cx_0}\right) \ .$$

3.1, continued:

Numerically this yields $t = -\frac{1}{3}$ s and $t = 1$ s. The corresponding values of the velocity $v_x = b + 2ct$ are

$$\begin{aligned}(2\ \mathrm{m/s}) + \left[2 \times \left(-3\ \mathrm{m/s^2}\right) \times \left(-\tfrac{1}{3}\ \mathrm{s}\right)\right] &= 4\ \mathrm{m/s} \quad \text{for} \quad t = -\tfrac{1}{3}\ \mathrm{s}\ , \quad \text{and} \\ (2\ \mathrm{m/s}) + \left[2 \times \left(-3\ \mathrm{m/s^2}\right) \times (1\ \mathrm{s})\right] &= \ -4\ \mathrm{m/s} \quad \text{for} \quad t = 1\ \mathrm{s}\ .\end{aligned}$$

The acceleration, being constant, is of course $-6\ \mathrm{m/s^2}$ in both cases.

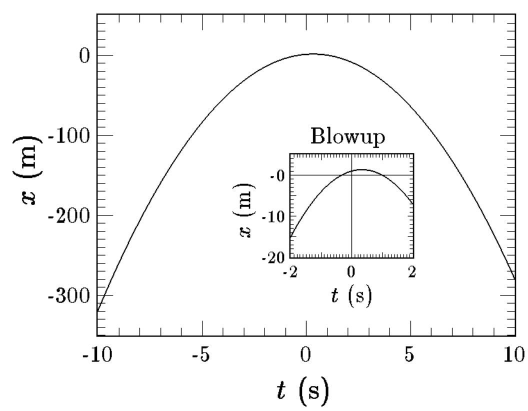
For part (iii) we solve the velocity equation for $v = 0$, which gives

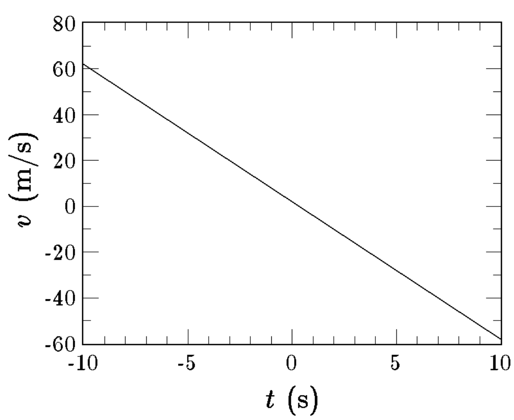
$$t = -\frac{b}{2c} = \frac{1}{3}\ \mathrm{s}.$$

The position at this time is

$$\begin{aligned} x &= (1\ \mathrm{m}) + \left[(2\ \mathrm{m/s}) \times \left(\tfrac{1}{3}\,\mathrm{s}\right)\right] \\ &\qquad + \left[\left(-3\ \mathrm{m/s^2}\right) \times \left(\tfrac{1}{3}\,\mathrm{s}\right)^2\right] \\ &= \tfrac{4}{3}\,\mathrm{m}\ , \end{aligned}$$

and the acceleration is still $-6\ \mathrm{m/s^2}$.





The graphs of position and velocity are shown at right. Note that the position graph is quadratic (proportional to t^2) whereas the velocity graph is linear (proportional to t). What would the acceleration graph look like?

Scrutinize
The units of $b + ct$ are m/s, and c is in $\mathrm{m/s^2}$, so the dimensions of our equations for velocity and acceleration are correct. Note that, dimensionally, differentiating with respect to a variable is like dividing by it: the dimensions of $\mathrm{d}x/\mathrm{d}t$ are the same as those of x/t.

We can check our answer to part (ii) by substituting our values of t back into the original

3.1, continued:

equation:

$$
\begin{aligned}
x(t = -\tfrac{1}{3}\,\mathrm{s}) &= (1\,\mathrm{m}) + \left[(2\,\mathrm{m/s}) \times \left(-\tfrac{1}{3}\,\mathrm{s}\right)\right] + \left[\left(-3\,\mathrm{m/s^2}\right) \times \left(-\tfrac{1}{3}\,\mathrm{s}\right)^2\right] \\
&= \left(1 - \tfrac{2}{3} - \left(3 \times \tfrac{1}{9}\right)\right)\ \mathrm{m} = 0; \\
x(t = 1\,\mathrm{s}) &= (1\,\mathrm{m}) + \left[(2\,\mathrm{m/s}) \times (1\,\mathrm{s})\right] + \left[\left(-3\,\mathrm{m/s^2}\right) \times (1\,\mathrm{s})^2\right] \\
&= (1 + 2 - 3)\,\mathrm{m} = 0.
\end{aligned}
$$

Learn

If the derivative of a function is zero at some point, the value of the function there is a local maximum, a local minimum, or occasionally a saddle point. In this case the position graph shows clearly that the zero of velocity corresponds to a global maximum value of x—the particle never reaches $x > \frac{4}{3}$ m.

(b) *Now consider a particle moving in three dimensions with position vector*

$$\vec{\mathbf{r}} = \left[a_1 + a_2 t, b_1 + b_2 t^3, c_1 + c_2 t^2\right] ,$$

where $a_1 = 1$ m, $a_2 = 6$ m/s, $b_1 = 4$ m, $b_2 = -1$ m/s^3, $c_1 = 2$ m *and* $c_2 = -1$ m/s^2. *What are its velocity and acceleration vectors at time t? What is its speed and the magnitude of its acceleration at time $t = -3$ s?*

Conceptualize

This is the same type of problem, but in three dimensions. We can solve it in the same way, treating each component as a separate one-dimensional problem.

Formulate

The equations for the x-component are

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t}\ ; \qquad a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\ ,$$

and likewise for the y- and z-components.

Solve

The velocity and acceleration vectors are

$$
\begin{aligned}
\vec{\mathbf{v}} &= \left[a_2, 3b_2 t^2, 2c_2 t\right]\ ; \\
\vec{\mathbf{a}} &= \left[0, 6b_2 t, 2c_2\right]\ .
\end{aligned}
$$

The acceleration this time is not constant. At $t = -3$ s, the velocity and acceleration are

$$
\begin{aligned}
\vec{\mathbf{v}} &= [6, -27, 6]\ \mathrm{m/s}\ ; \\
\vec{\mathbf{a}} &= [0, 18, -2]\ \mathrm{m/s^2}\ .
\end{aligned}
$$

3.1, continued:

We want the magnitudes of these vectors, namely

$$v = \sqrt{6^2 + (-27)^2 + 6^2} = \sqrt{801} = 28 \text{ m/s (to 2 sig. fig.) and}$$
$$a = \sqrt{0^2 + 18^2 + (-2)^2} = \sqrt{328} = 18 \text{ m/s}^2 \ .$$

Scrutinize
Checking units for the expression for $\vec{\mathbf{v}}$, we see that they are correct: the units of $b_2 t^2$ are $(\text{m/s}^3)(\text{s}^2)$, or m/s, and likewise $c_2 t$ has units of $(\text{m/s}^2)(\text{s})$. The differentiation to find the expression for $\vec{\mathbf{a}}(t)$ eliminates one factor of t, resulting in units of m/s^2 for both nonzero components. Our equations are therefore dimensionally correct. Note that in this case the velocity is never zero overall, although both v_y and v_z are zero at $t = 0$ (a saddle point in y, a global maximum in z).

Learn
Working with symbols instead of substituting in the numbers allows us to maintain a check on the consistency of our units through the calculation. It also results in a more general solution: if we were told that b_2 was $+1$ m/s^3 instead of -1 m/s^3, we would not have to redo the whole calculation. If you do choose to put in the numbers earlier, you ***must*** remember that they are not ***really*** numbers (in the sense that the 3 in $3b_2 t$ is really a number) but represent quantities with dimensions and units.

3.4 *A local agricultural show features a display of archery. An archer is practicing at the target as the celebrity who is to open the show is arriving by helicopter. If the archer directs his arrow at 45° with a speed of* 60 m/s *and the helicopter is descending vertically with a constant speed of* 10 m/s, *plot the trajectory of the arrow as seen by (a) an observer on the ground and (b) an observer in the helicopter. What are the velocity and acceleration vectors at time t in each frame of reference? Take* $g = 10$ m/s^2, *ignore air resistance, and choose coordinates so that the launch point of the arrow is at the origin in both frames of reference.*

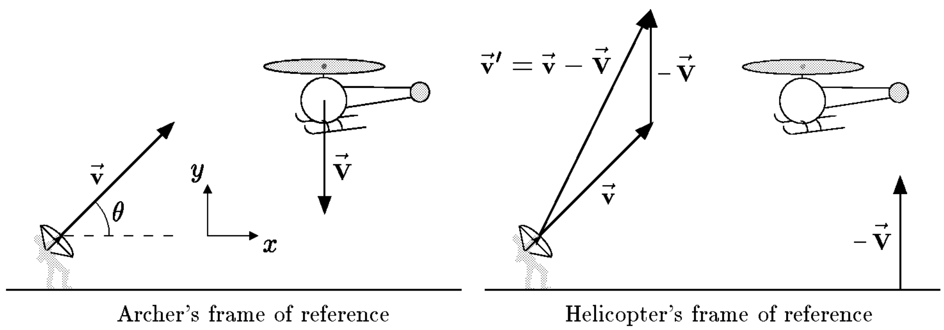
Conceptualize
This problem involves projectile motion and the concept of reference frames. We have two natural reference frames: the archer frame, which is fixed relative to the ground, and the helicopter frame, which is fixed relative to the helicopter. (The third possible frame, fixed relative to the arrow, is a non-inertial frame, with a nonzero acceleration relative to the inertial frames. We will postpone discussion of non-inertial frames until Chapter 6.) The archer frame is the simplest to work in, since the velocities specified in this problem are all given in this frame.

Formulate
We define a coordinate system in which y points vertically upwards and x is the horizontal direction of the arrow. In the archer's frame of reference, the component equations of the arrow's velocity $\vec{\mathbf{v}}(t)$ are

$$v_x(t) = v_0 \cos\theta \ ;$$
$$v_y(t) = v_0 \sin\theta - gt \ ,$$

3.4, continued:

where $t = 0$ at the moment the arrow is launched, $v_0 = 60$ m/s is the initial speed of the arrow, and $\theta = 45°$ is the angle its initial velocity makes with the horizontal.



In the helicopter's frame of reference, the arrow's velocity is

$$\vec{v}'(t) = \vec{v}(t) - \vec{V} \ ,$$

where $\vec{V} = [0, -10, 0]$ m/s is the velocity of the helicopter. (Note that $-\vec{V}$, the velocity of the ground relative to the helicopter, is a vector pointing upward of magnitude 10 m/s.) In components,

$$v'_x(t) = v_0 \cos\theta \ ;$$

$$v'_y(t) = V + v_0 \sin\theta - gt \ ,$$

where $V = |\vec{V}| = 10$ m/s.

To plot the trajectory we will need the equations for x and y in both frames: in the archer frame, the trajectory of the arrow is

$$x(t) = v_0 t \cos\theta \ ;$$

$$y(t) = v_0 t \sin\theta - \frac{1}{2} g t^2 \ .$$

In the helicopter frame we know the velocity $\vec{v}'(t)$ of the arrow, so we can determine the position either by integrating the velocity, or by using its initial value and the fact that the acceleration is uniform with value $[0, -g, 0]$. With either method, we need to specify the initial position as $x'_0 = 0$, $y'_0 = 0$, since the problem tells us to choose the coordinate system with the launch point of the arrow at the origin. The flight of the arrow in the helicopter frame is therefore described by

$$x'(t) = x(t) = v_0 t \cos\theta \ ;$$

$$y'(t) = (V + v_0 \sin\theta) t - \frac{1}{2} g t^2 \ .$$

Solve

The trajectory that we are asked to plot is a graph of y against x, but so far we have calculated each as a function of time. However, the equation for $x(t)$ (or equivalently $x'(t)$) can easily be solved to give $t = x/(v_0 \cos\theta) = x'/(v_0 \cos\theta)$, which can then be

3.4, continued:

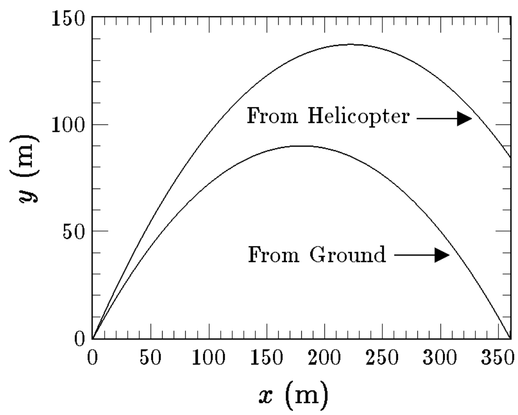
used to eliminate t from the equations for $y(t)$ and $y'(t)$. One finds that

$$y = x \tan\theta - \frac{gx^2}{2v_0^2 \cos^2\theta}$$

and

$$y' = x' \tan\theta + \frac{Vx'}{v_0 \cos\theta} - \frac{gx'^2}{2v_0^2\cos^2\theta} \ .$$

These equations are graphed at the right.



We have already written algebraic expressions for the velocity of the arrow in each frame, so all that remains is to insert numbers. The components of the arrow's velocity in the archer frame are

$$v_x = (60 \text{ m/s}) \cos 45^\circ = 42 \text{ m/s} \ ;$$

$$v_y = (60 \text{ m/s}) \sin 45^\circ - (10 \text{ m/s}^2)t = [42 - (10 \text{ s}^{-1})t] \text{ m/s} \ .$$

In the helicopter frame $v'_x = v_x = 42$ m/s, and

$$v'_y = 10 \text{ m/s} + (60 \text{ m/s}) \sin 45^\circ - (10 \text{ m/s}^2)t = [52 - (10 \text{ s}^{-1})t] \text{ m/s} \ .$$

In the archer frame, the acceleration is clearly $\vec{\mathbf{g}} = [0, -g, 0]$, and this is unchanged by adding a constant (i.e., unaccelerated) velocity to transform into the helicopter frame. Summarizing:

$$\vec{\mathbf{v}} = [42, 42 - (10 \text{ s}^{-1})t, 0] \text{ m/s} \ , \qquad \vec{\mathbf{a}} = [0, -10, 0] \text{ m/s}^2$$

in the archer frame, and

$$\vec{\mathbf{v}}' = [42, 52 - (10 \text{ s}^{-1})t, 0] \text{ m/s} \ , \qquad \vec{\mathbf{a}}' = [0, -10, 0] \text{ m/s}^2$$

in the helicopter frame.

Learn

Notice that the ground is moving upwards at 10 m/s as seen from the helicopter, so the arrow in the helicopter frame does not return to the same y-coordinate value. Also, it reaches its maximum y-coordinate value later in the helicopter frame. However, the acceleration in both frames is the same: an observer in the helicopter using the flight of the arrow to measure g would get the same result as an observer on the ground.

In the *Essentials* of Chapter 1 we learned that if an object has position $\vec{\mathbf{r}}$ and velocity $\vec{\mathbf{v}}$ in some reference frame, then its position and velocity in the frame of an observer O are given by

$$\vec{\mathbf{r}}' = \vec{\mathbf{r}} - \vec{\mathbf{r}}_0$$

$$\vec{\mathbf{v}}' = \vec{\mathbf{v}} - \vec{\mathbf{v}}_0 \ ,$$

3.4, continued:

where $\vec{\mathbf{r}}_0$ and $\vec{\mathbf{v}}_0$ are the position and velocity of the observer O in the original frame. In the solution above we used the second of these equations, but you might wonder why we did not also use the first. Our choice was just a matter of convenience: we were given the velocity of the helicopter, but not its position. Furthermore, the first equation leads to a coordinate system with the observer O at the origin, while we were asked to describe the arrow in a reference frame with the velocity of the helicopter, but with the launch point at the origin.

An unusual feature of this problem is the peculiar mix of variables with numerical constants in the final answer for the velocities. One tricky point is the handling of units. The answer for v_y, for example, was correctly written as $[42 - (10\ \mathrm{s}^{-1})t]$ m/s. It would ***not*** be correct, however, to write it as $(42 - 10t)$ m/s, even though this expression has the right units at the end. The problem with the latter expression is that 42 is a pure number, while $10t$ is a time, with units of seconds, hours, fortnights, or something else. There is no well-defined way to add a time to a pure number.

3.8 *A bobsled run includes a bend in the form of a circular arc of radius* 25 m. *A sled approaches the bend at a constant speed of* 90 km/h. *At what angle must the sled bank in order to take the curve successfully? Assume that this part of the course is level, and that there is no friction between the sled runners and the ice.*

Conceptualize

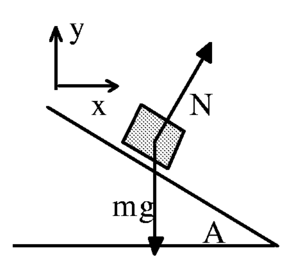
The bobsled needs to move in a horizontal circle. It must therefore have an acceleration v^2/r directed towards the center of the circle, i.e. horizontally. The forces acting are gravity, which is vertical and therefore cannot possibly produce a horizontal acceleration, and the normal force from the surface of the ice. If the sled is to negotiate the bend successfully, the normal force must have a vertical component which cancels the weight of the sled—there is no vertical acceleration—and a horizontal component producing the required centripetal acceleration. The ratio of these two components gives $\tan A$, the angle the normal force makes with the vertical, and since the normal force is always directed perpendicular to the surface this is necessarily the angle that the surface of the ice makes with the horizontal, which is what we are asked to calculate.

Formulate

Defining a coordinate system with y up and x horizontally along the direction of motion, the components of the total force are

$$F_x = N \sin A \ ;$$

$$F_y = N \cos A - mg \ .$$



The orientation of $\vec{\mathbf{N}}$ is defined by the fact that (as its name implies) it is perpendicular to the surface of the slope. The horizontal component of $\vec{\mathbf{N}}$ is the only horizontal force acting, so it must produce the necessary centripetal acceleration:

$$N \sin A = \frac{mv^2}{r} \ ,$$

3.8, continued:

while, since the sled has no vertical acceleration, $F_y = 0$, implying

$$N \cos A = mg \ .$$

Solve
Dividing the first equation by the second gives

$$\tan A = \frac{v^2}{gr} \ , \quad \text{or } A = 69^\circ \ .$$

Scrutinize
The dimensions of gr are $([\text{length}]/[\text{time}]^2) \times [\text{length}] = ([\text{length}]/[\text{time}])^2$, so v^2/gr is dimensionless. If the velocity goes to zero, there is no centripetal acceleration, so no banking is required, and indeed in this case $\tan A = 0$; conversely, a very large velocity will produce a very large $\tan A$, and hence require a nearly vertical bank (as in circus "Wall of Death" motorcycle rides).

Learn
The same principle explains why sharp bends on roads are banked, and why if you take the bend too fast you end up running off the outer edge of the bend. You have ***not*** been 'thrown outwards'—rather, you are still being acted on by a force $N \sin A = mv^2/r$, but because your v is greater than the road engineers expected it to be, r is larger (N and A are fixed by the mass of your car and the angle of bank), and you do not turn sharply enough to make the bend. In the case of automobiles frictional forces also act (unless the road is very icy), which is why you don't have to take the curve at ***exactly*** its design speed.

3.10 *Calculate the final velocity of a mass m (a) falling freely from rest through a distance h; (b) sliding a distance ℓ down a frictionless slope making an angle ϕ to the horizontal, where $\sin \phi = h/\ell$; (c) forming the bob of a simple pendulum of length ℓ which makes a maximum angle with the vertical of $\theta = \sqrt{2h/\ell}$ (in this case calculate the velocity at $\theta = 0$). Give an expression for the net force acting in each case. Assume $h \ll \ell$.*

Conceptualize
By now we have developed a standard approach to dealing with problems involving forces, accelerations and velocities:

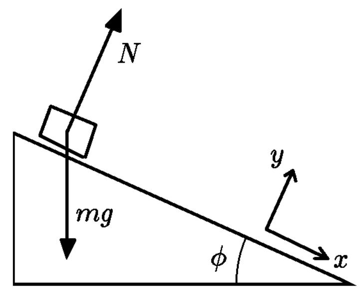
- draw a force diagram;
- choose a suitable coordinate system;
- write down the component equations for $\vec{\mathbf{F}}$, $\vec{\mathbf{v}}$ and (if necessary) $\vec{\mathbf{r}}$;
- solve for the requested quantities.

The choice of coordinate system depends on the individual problem. In the case of part (c), we will also want to assume that the angle θ is small—we saw in Problem 2D.2 that without this assumption the pendulum's equations of motion are too difficult to solve.

3.10, continued:

Formulate and Solve (a)
This is a one-dimensional constant-acceleration problem. Taking down to be positive, we have $v = gt$ and $h = \frac{1}{2}gt^2$. If we substitute $t = v/g$ in the equation for h we can solve for v to obtain $v = \sqrt{2gh}$, directed downwards. The net force is obviously $m\vec{g}$, also directed downwards.

Formulate (b)
The force diagram is as shown. Choosing a coordinate system with the x direction directed down the slope, the components of the forces acting are



$$F_x = mg\sin\phi$$
$$F_y = N - mg\cos\phi$$

There is no acceleration, and thus no net force, in the y direction. In the x direction,

$$v = gt\sin\phi$$
$$\ell = \frac{1}{2}gt^2\sin\phi \ .$$

Solve (b)
We can solve these equations for v as we did for case (a) to get

$$v = \sqrt{2g\ell\sin\phi} \ .$$

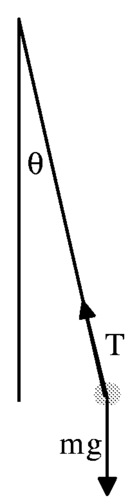
Putting in $\sin\phi = h/\ell$ gives

$$v = \sqrt{2gh}$$

in the x-direction, i.e. down the slope.

The net force, as we have already determined, is $mg\sin\phi$ directed down the slope.

Formulate (c)
As in Problem 2D.2(b), we assume θ is small and choose a coordinate system with y along the string (i.e., for small θ, vertical) and x perpendicular to the string.



$$F_y = T - mg\cos\theta \approx T - mg$$
$$F_x = -mg\sin\theta \approx -mg\theta \ .$$

The y-component provides the centripetal force which maintains the bob's circular motion (i.e. it produces an acceleration v^2/r). The x-component gives us the equation of motion for the bob

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\theta \ .$$

3.10, continued:

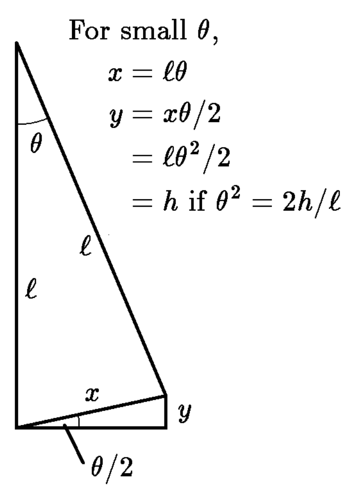
Solve (c)
We saw in Chapter 2 that the solution of this equation is $\theta = A\cos\omega t$, where in this case $A = \sqrt{2h/\ell}$ and $\omega = \sqrt{g/\ell}$, and the choice of cosine rather than sine defines $t = 0$ at maximum amplitude. The tangential speed of the bob is given by

$$v = \ell\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\sqrt{2gh}\sin\omega t \ ,$$

so at $\theta = 0$ we once again have $v = \sqrt{2gh}$, directed horizontally (with or without a minus sign, depending on the direction in which the bob is swinging when it passes through $\theta = 0$). The net forces are given above.

Scrutinize
The dimensions of $\sqrt{2gh}$ are $\sqrt{[\text{length}]/[\text{time}]^2 \times [\text{length}]} = [\text{length}]/[\text{time}]$, which is appropriate for v. We can also see that $v \to 0$ if $g \to 0$ or $h \to 0$, as we would expect—in zero gravity the objects would not fall, and falling through zero height is equivalent to not falling at all.

Learn
All three cases were set up so that the mass starts from rest and falls through a vertical distance h (see diagram at right if you are unconvinced about this for part (c)). Notice that in each case the mass ends up with exactly the same speed that it would have had if it had simply fallen freely from rest through a height h. The ***velocities*** are not the same, because the directions are different.



This result is not a coincidence. In any experimental situation we could contrive where a mass starts from rest and moves under the influence of gravity through a vertical distance h, it will end up with the same speed $\sqrt{2gh}$. To understand why this is so, we need to introduce the new physical concept of ***energy***, which is the subject of the next chapter.

3.13 *The distance between the Earth and the Moon is 384 thousand kilometers, and the mass of the Moon is* 7.4×10^{22} kg. *Calculate the acceleration of the Earth due to the Moon's gravitational attraction. If a point mass were describing a circular orbit with this centripetal acceleration and a period of 27.3 days, what would be the radius of the orbit? Take* $G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$.

Conceptualize
The diameters of the Earth and the Moon are small compared to the distance between them, so we can treat them as point particles. The force acting is gravity, but because we are considering distance large compared to the sizes of the gravitating objects we must use Newton's inverse square law, not simply $m\vec{g}$! Recognizing this supports our

13.13, continued:

decision to treat Earth and Moon as point masses, because, as mentioned in Chapter 2, the gravitational force exerted by any spherically symmetric body is the same as that of a point particle of the same mass.

Formulate

Newton's laws for the Earth-Moon system give

$$\frac{GM_{\text{earth}}M_{\text{moon}}}{r^2} = M_{\text{earth}}\,a \ .$$

A particle in uniform circular motion has acceleration v^2/r, where v is its speed and r the radius of the circle. To express v in terms of the period P, we note that

$$v = \frac{\text{circumference of circle}}{\text{period}} = \frac{2\pi r}{P} \ .$$

Solve

The Earth's acceleration is

$$a = \frac{(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)\times(7.4\times10^{22}\ \text{kg})}{(384\times10^6\ \text{m})^2} = 3.3\times10^{-5}\ \text{m/s}^2 \ .$$

A circular orbit has

$$a = \frac{v^2}{r} = \frac{4\pi^2 r}{P^2} \ ,$$

so for this acceleration and a period of 27.3 days

$$r = \frac{(3.3\times10^{-5}\ \text{m/s}^2)\times(27.3\times24\times60\times60\ \text{s})^2}{4\pi^2} = 4.7\times10^6\ \text{m} \ .$$

Scrutinize

$1\ \text{N} = 1\ \text{kg}\cdot\text{m/s}^2$, so the units of G can be expressed as $\text{m}^3/\text{kg}\cdot\text{s}^2$. In this form it is clear that the units of GM/r^2 are m/s^2, as expected. In this problem it is certainly necessary to check ***units***, rather than simply ***dimensions***, since forgetting to convert 27.3 days into seconds would not affect the dimensions of your answer, but would definitely change its numerical value!

For a given period, the radius of the orbit is proportional to the centripetal acceleration, which in turn is proportional to the mass exerting the gravitational force. The Moon's mass is about 1% of the Earth's (more exactly, 1/81), so we expect the radius we calculate to be about 1% of the Earth-Moon distance. This is in good agreement with the value that we get.

Learn

The fact that 4700 km is so small compared to the distance between the Earth and the Moon (in fact it is smaller than the radius of the Earth) explains why we can treat the Earth-Moon system as if only the Moon moved. Astronomers have studied many ***binary stars***, in which two stars form a gravitationally bound system. In this case the masses of the two stars may be quite similar, and the motion of both partners has to be taken into account.

HINTS FOR PROBLEMS WITH AN (H)

The number of the hint refers to the number of the problem

3.5 Visualize the man's motion in the truck's frame of reference. What is his horizontal motion in this frame? His vertical motion? How long does he take to cover the vertical distance from truck roof to jeep? In the truck's frame of reference, how far does the jeep move in this time? In the Bedouin's frame, in which the road is stationary, how far does the truck move?

If you're still stuck, try reviewing the solution to problem 3.4.

3.6 What is the general form of the gravitational force? What does this tell you about the value of g on Everest? What other forces are involved in weighing the rock?

3.9 What is the acceleration required to keep an object moving in a horizontal circle? Draw a force diagram for the bob. What net force acts to produce this acceleration?

You may also find it helpful to review the solution to problem 2D.2.

3.12 (a) If you move with speed v, at what angle to the vertical is the rain in your reference frame? How much rain will cross a plane area A in 1 s if the velocity of the rain makes an angle θ with the normal (i.e. perpendicular) to the surface? A diagram will probably help.

(c) How can you minimize the effective area you present to the rain?

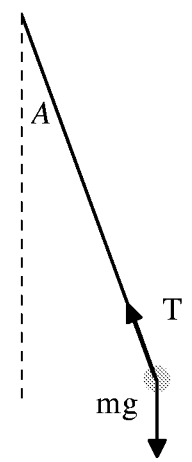
ANSWERS TO HINTS

3.5 None; $v = -gt$, where t is the time since he jumped.

0.77 s; 1.1 m; 8.6 m.

3.6 $-GMm/r^2$; on Everest g will be less, because r is increased. For beam balance, no other force; for spring balance, $-kx$.

3.9 v^2/r, directed horizontally inwards towards the axis of rotation.



$T \sin A$.

3.12 (a) At an angle θ such that $\tan\theta = v/V$; $\rho A v \cos\theta$, i.e. ρA times the component of the rain's velocity normal to the plane area.

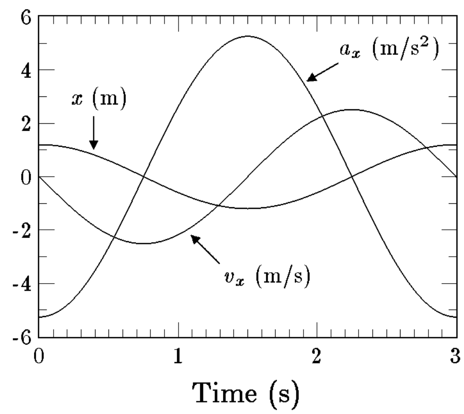
(c) By leaning forward at the angle θ to the vertical (so the rain strikes only your head and shoulders).

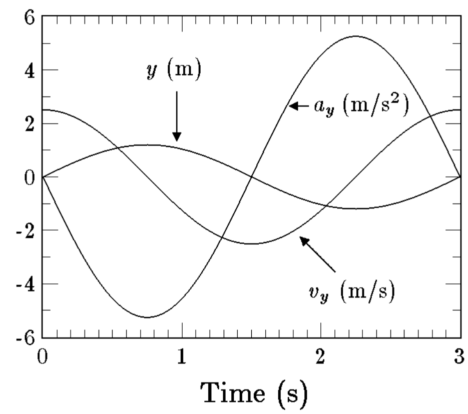
ANSWERS TO ALL PROBLEMS

3.1 See complete solution.

3.2 B,D,F; F.

3.3 $v = 0.25$ m/s, $a = 0.053$ m/s^2. The normal force from the rails.



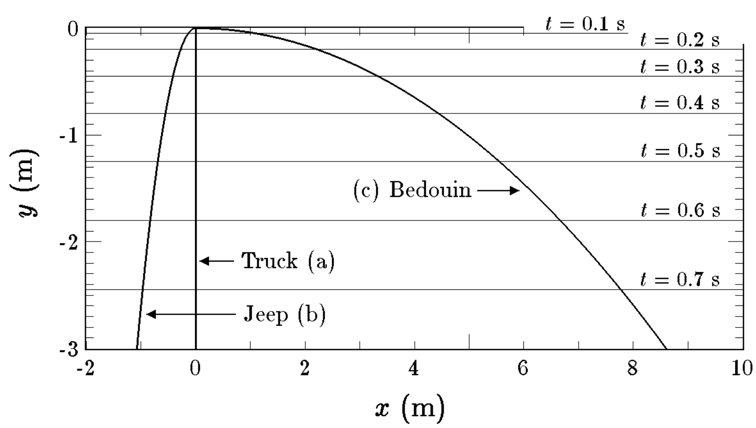


3.4 See complete solution.

3.5 (a) 3.2 m/s

(b) Front of jeep 3.1 m behind front of truck.

(c)



3.6 No: the beam balance will give a larger reading than the spring balance.

3.7 $T = 2\pi\sqrt{\dfrac{r^3}{gR^2}}$

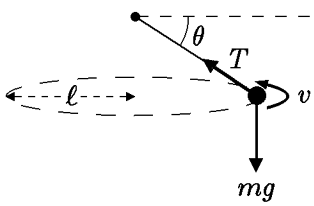
The difficulty in measuring G arises from the fact that the only way to accurately determine the mass of a celestial body such as the Earth is to use $|\vec{\mathbf{F}}| = GMm/r^2$. We know m, the mass of our test object, accurately, and we can measure r accurately also. But we can only determine M if we already know G, and vice versa. Hence we must determine G by measuring the gravitational force between two ***known*** masses M and m, and this inevitably means that M is much smaller than the mass of a planet. As gravity is so weak (see Problem 2B.6), the gravitational force between two relatively small masses is tiny, and very difficult to measure accurately, so the resulting value of G is not very precise.

3.8 See complete solution.

3.9 Speed: twice as large for 10°; period: 0.6% shorter for 10°. Periods of conventional and conical pendulum are equal in the small angle approximation.

3.10 See complete solution.

3.11



(a) $\tan\theta = g\ell/v^2$;

(b) $T = \dfrac{m}{\ell}\sqrt{v^4 + g^2\ell^2}$.

(c) $x = \sqrt{\frac{2h}{g}\,v^2 + \ell^2}$.

3.12 (a) $\rho\ell w\left(d\dfrac{V}{v} + h\right)$.

(b) Very small v: becomes very large. Very large v: becomes $\rho\ell wh$. Best strategy will involve large v.

(c) Bend forward at 31° to vertical and run at 12 km/h.

3.13 See complete solution.

3.14 4.2×10^{20} N; 1.9×10^{20} N.

3.2×10^{-3} m/s^2 towards Sun;

8.6×10^{-3} m/s^2 towards Sun;

6.5×10^{-3} m/s^2 at 25° Earthwards of the line joining Sun and Moon.

That due to the Sun; because Earth and Moon share effectively the same orbit, and thus the same centripetal acceleration, relative to the Sun, so we see only the Moon's motion relative to us.

3.15 kx/M, directed to the left.