ESSENTIALS OF INTRODUCTORY CLASSICAL MECHANICS

Sixth Edition

CHAPTER 4

ENERGY

OVERVIEW

In this chapter we introduce the concept of ***energy***. Unlike the concepts we have previously encountered (position, velocity, force, etc.), energy can take a number of different forms. It can be stored and transferred from one object to another or from one form to another, but the overall energy of a closed system remains constant—we say that energy is ***conserved***. This is of great practical value in solving problems, but it is also a fundamental property of nature, which remains true even under conditions when Newton's laws must be modified by relativity or quantum mechanics.

When you have completed this chapter you should:

- ✔ understand what is meant by conservation of energy;
- ✔ be able to calculate the kinetic energy of a body and use it in problems;
- ✔ be able to calculate the work done by a given force and relate this to a transfer of energy;
- ✔ understand the concept of potential energy and be able to calculate and use the potential energy associated with gravity, spring forces, and electrostatic forces;
- ✔ recognize the existence and importance of other forms of energy;
- ✔ understand how conservative forces can be derived from a potential energy function, and relate this to the condition of equilibrium.

ESSENTIALS

We found in Chapter 1 that for a body moving in one dimension with a constant acceleration, the equations for $v(t)$ and $x(t)$ can be combined to give the useful formula

$$v^2 = v_0^2 + 2ax \ ,$$

where v_0 is the velocity at the beginning of some time interval, v is the velocity at the end of the interval, x the distance the body travels during the time interval, and a is the acceleration. If m is the body's mass, we can rewrite this as

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 + Fx \ ,$$

where $F = ma$ is the (constant) force acting. If F is the uniform force of gravity near the surface of the earth, then it is common practice to measure x as positive upwards, and to call it h, for height. Since $F = -mg$, the equation can be written as

$$\boxed{\frac{1}{2}mv^2 + mgh = \frac{1}{2}mv_0^2 \ .}$$

The equation states that the quantity on the left-hand side maintains a constant value throughout the motion. Furthermore, the equation can be shown to hold regardless of whether the body is falling vertically, sliding down an inclined plane, moving as a projectile with horizontal and vertical velocity, etc., provided no air resistance or frictional forces are involved. Problem 3.10

This equation is extremely useful for solving problems where the detailed time dependence of the motion is not important. More importantly, the quantity on the left-hand side, $\frac{1}{2}mv^2 + mgh$, is an example of a quantity that has important implications in all fields of physics. This quantity is a new basic concept: the ***energy*** of the body. Energy is a scalar quantity, measured in joules: 1 J = 1 N·m = 1 kg·m^2/s^2. Supplementary Notes.

As can be seen from the equations above, energy occurs in several different forms. The quantity $\frac{1}{2}mv^2$ is the ***kinetic energy*** of the body, while mgh is the (gravitational) ***potential energy*** that it has by virtue of being a vertical distance h above some reference level, such as the floor or the ground. Potential energy can be regarded as energy which is temporarily stored and which is capable of being released. For example, energy which is stored by lifting a book above the ground can be released by dropping the book, which will then accelerate and gain kinetic energy. The potential energy is really a property of a particular configuration of the whole system (e.g., a compressed spring, or two separated objects exerting a gravitational or electrostatic force on each other). Note that the reference level from which h is measured can be chosen arbitrarily—only the difference in potential energy between two configurations is meaningful. Problems 4A and 4B.1

Supplementary Notes.

Energy can be transformed from potential energy to kinetic energy and vice versa (e.g., the motion of a projectile or pendulum), but it cannot be created or destroyed. We call a system ***closed*** if it is not communicating in any way with the rest of the universe; i.e., if no objects or forces are entering or influencing the system from the outside. Experiments show that the ***total*** energy of a closed system never changes. We say that ***energy is conserved***. Conservation of energy has played an extremely important role in the development of physics, because it appears to be an exact principle that applies in all known circumstances.

If a net force acts on a particle, it accelerates, so its kinetic energy must change. To quantify this, consider a force $\vec{\mathbf{F}}$ acting on a particle of mass m and initial velocity $\vec{\mathbf{v}}_0$. For simplicity, let us restrict ourselves to the case of one-dimensional motion. Suppose the force acts for a short time Δt and produces an acceleration a, so that the final speed is $v = v_0 + a\Delta t$. Then the final kinetic energy is

$$\begin{aligned}\tfrac{1}{2}mv^2 &= \tfrac{1}{2}m\left(v_0 + a\Delta t\right)^2\\ &= \tfrac{1}{2}mv_0^2 + ma\left(v_0\Delta t + \tfrac{1}{2}a\Delta t^2\right)\\ &= \tfrac{1}{2}mv_0^2 + F\Delta x,\end{aligned}$$

where $\Delta x = v_0\Delta t + \frac{1}{2}a\Delta t^2$ is the distance the particle has moved in time Δt. The change in kinetic energy is given by the applied force times the displacement of the particle. We call this the ***work*** done by the force on the particle: thus

$$\boxed{W \equiv F\Delta x.}$$

Problem 4B.2

From the derivation above it is clear that in the absence of other forces ***the work done by a force on an object is equal to the amount of kinetic energy transferred to the object***. If more than one force acts, we can repeat the calculation using the net acceleration produced by all the forces, $a_{\text{net}} = \sum F_i/m$, and we find that the same relation holds: the total work done $(\sum F\Delta x)$ is equal to the net change in kinetic energy. This is the ***work-energy theorem***:

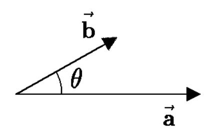
$$\boxed{W = K_f - K_i,}$$

where K_i is the initial kinetic energy, K_f the final kinetic energy, and W the total work done by all the forces acting on the body.

In situations involving a continuous transfer of energy, it is often useful to think in terms of the work done per unit time. This is called the ***power***, and is measured in watts: 1 W = 1 J/s. Problem 4C.2

Since work is a scalar while force and displacement are vectors, the definition of work in more than one dimension requires a modification of the $W \equiv F\,\Delta x$ definition given above. We have already seen, however, that the kinetic energy of a projectile varies with its height, but is not affected by the horizontal motion which is perpendicular to the gravitational force. To generalize this principle to arbitrary forces, we introduce the ***dot product***, or scalar product, between two vectors, which can be defined by

$$\boxed{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \equiv |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \cos\theta \ ,}$$



where $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are arbitrary vectors, $|\vec{\mathbf{a}}|$ and $|\vec{\mathbf{b}}|$ denote their magnitudes, and θ is the angle between them. Equivalently, one can write

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}| \times \text{component of } \vec{\mathbf{b}} \text{ in direction of } \vec{\mathbf{a}}.$$

$$= |\vec{\mathbf{b}}| \times \text{component of } \vec{\mathbf{a}} \text{ in direction of } \vec{\mathbf{b}}.$$

It can also be shown that in terms of the components of the vectors, $\vec{\mathbf{a}} = [a_x, a_y, a_z]$ and $\vec{\mathbf{b}} = [b_x, b_y, b_z]$, the dot product is given by

$$\boxed{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_x b_x + a_y b_y + a_z b_z \ .}$$

In three dimensions it is only the component of the force in the direction of the displacement $\overrightarrow{\Delta\mathbf{r}}$ that contributes to the work done, so for a constant force Problems 4C

$$\boxed{W \equiv \vec{\mathbf{F}} \cdot \overrightarrow{\Delta\mathbf{r}} \ .}$$

If the force acts at right angles to the displacement, the dot product is zero, so no work is done and no energy is transferred.

If the force is not constant, we must integrate over the displacement to find the work done. In one dimension this gives

$$\boxed{W \equiv \int F(x)\,\mathrm{d}x \ ,}$$

while in three dimensions

$$\boxed{W \equiv \int_{\vec{\mathbf{r}}_1}^{\vec{\mathbf{r}}_2} \vec{\mathbf{F}} \cdot \overrightarrow{\mathbf{dr}} \ .}$$

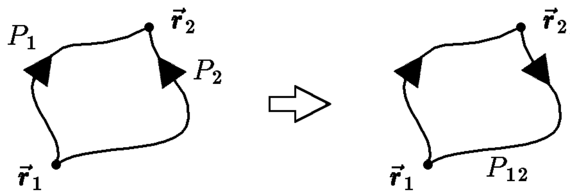
It can be shown that the work-energy theorem holds for non-constant forces, and for three-dimensional motion, with W defined by the above equation.

Supplementary Notes.

Does it matter ***how*** we get from $\vec{\mathbf{r}}_1$ to $\vec{\mathbf{r}}_2$? Even in one dimension there are clearly an infinite number of ways to get from x_1 to x_2: we can go directly there and stop, or we can overshoot and come back; we can travel slowly or rapidly, at constant speed or with varying speed. Does this make a difference to the work done?

The answer depends on the nature of the force. In the one-dimensional case, if the force exerted on a body depends only on the body's position, and not on its velocity or the history of its past motion, then there can be no dependence on the precise route from x_1 to x_2: if the force does not depend on velocity, clearly the velocity does not affect the work done, and if we overshoot, the work $F(x)\Delta x$ that we do in going from x to $x+\Delta x$ is cancelled by the work $F\cdot(-\Delta x)$ that we do coming back from $x+\Delta x$ to x. So in this case the body at position x_2 has a well-defined ***potential energy*** compared to the body at x_1: if we move from x_1 to x_2, no matter how we do it, we will do a certain well-defined amount of work. A force which behaves in this way is called a ***conservative force***. Gravity, the electrostatic force and the spring force are all conservative forces. Friction, which we will discuss in Chapter 6, is an example of a non-conservative force: as we know from everyday experience, friction acts to resist the relative motion of two surfaces, and therefore the frictional force exerted on a body at position x depends on whether its velocity is positive or negative. Hence it is no longer true that the work done in moving from x to $x+\Delta x$ cancels the work done moving from $x+\Delta x$ to x, and the route we take from x_1 to x_2 does matter. For non-conservative forces, therefore, we cannot calculate the work done just from the initial and final positions of the particle.

In two or three dimensions, the situation seems more complicated, because the body can take an infinite number of routes from $\vec{\mathbf{r}}_1$ to $\vec{\mathbf{r}}_2$ without retracing its path. It is therefore not sufficient to conclude that retracing our steps from $\vec{\mathbf{r}} + \overrightarrow{\Delta\mathbf{r}}$ back to $\vec{\mathbf{r}}$ will cancel the work done in moving from $\vec{\mathbf{r}}$ to $\vec{\mathbf{r}} + \overrightarrow{\Delta\mathbf{r}}$. Note, however, that any two routes P_1 and P_2 can be combined to form a closed loop P_{12}, consisting of P_1 followed by the reverse of P_2:



Since the work done along the path P_2 changes sign if the path is traversed in the opposite direction, the statement that the work along P_1 and P_2 are equal is equivalent to the statement that the work done along the closed path is zero. Thus, a force is conservative if the work done on an object when it travels around a closed loop is exactly zero. In this chapter, we consider only conservative forces.

The definition of the potential energy, specified above to be equal to mgh for a uniform gravitational field, can be generalized to describe any conservative force. The potential energy of an object at a point $\vec{\mathbf{r}}_p$ is defined to be the work that must be done to bring the object to that point. Since only potential energy differences are meaningful, one arbitrarily chooses a reference point $\vec{\mathbf{r}}_0$, defining the potential energy at $\vec{\mathbf{r}}_0$ to be an arbitrarily chosen number U_0. The potential energy at any other point $\vec{\mathbf{r}}_p$ is then the potential energy at $\vec{\mathbf{r}}_0$, plus the work that is required to bring the object from $\vec{\mathbf{r}}_0$ to $\vec{\mathbf{r}}_p$:

$$\boxed{U(\vec{\mathbf{r}}_p) \equiv U_0 - \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}_p} \vec{\mathbf{F}} \cdot \overrightarrow{\mathbf{dr}}\ .}$$

The minus sign appears in front of the force term because we need to calculate the work that would have to be done by a force from outside the system to ***oppose*** the force $\vec{\mathbf{F}}$ exerted by the objects in the system. In one dimension, this relation reduces to

$$\boxed{U(x_p) \equiv U_0 - \int_{x_0}^{x_p} F\, \mathrm{d}x\ .}$$

In this chapter we will be concerned mainly with the following four cases:

Problems 4B.3, 4B.4, and 4B.5

Force	Force Law	Potential Energy
Uniform gravity	$\vec{\mathbf{F}} = -mg\hat{\boldsymbol{y}}$	$U = mgy$
Gravity of sphere	$\vec{\mathbf{F}} = -\frac{GMm}{r^2}\hat{\boldsymbol{r}}$	$U = -\frac{GMm}{r}$
Electrostatics	$\vec{\mathbf{F}} = \frac{Qq}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}}$	$U = \frac{Qq}{4\pi\epsilon_0 r}$
Spring	$F = -kx$	$U = \frac{1}{2}kx^2$

Using the path-independence of the integral, one can show that for any two points,

$$U(\vec{\mathbf{r}}_2) - U(\vec{\mathbf{r}}_1) = -\int_{\vec{\mathbf{r}}_1}^{\vec{\mathbf{r}}_2} \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}} .$$

Since the work-energy theorem guarantees that the integral on the right (without the minus sign) is the amount by which the kinetic energy of the object increases as it moves from $\vec{\mathbf{r}}_1$ to $\vec{\mathbf{r}}_2$, it follows that

$$\boxed{K_2 + U(\vec{\mathbf{r}}_2) = K_1 + U(\vec{\mathbf{r}}_1) = \text{constant} ,}$$

Problems 4D

where K_1 and K_2 denote the kinetic energy at $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$, respectively. So, for conservative forces, the total ***mechanical energy***, the sum of the kinetic and potential energies, does not change with time. (In this chapter we will not consider nonconservative forces (e.g., friction), but it is worth mentioning that such forces can convert energy into other forms (e.g., heat, or electromagnetic waves).)

The above discussion has assumed that only one particle in our system is moving, and the potential energy of the system is then a function of the position of that one object—we can reasonably, if loosely, say that it is a property of the object. This is a good approximation in many real situations: for example, in projectile problems the Earth can be regarded as stationary, so the potential energy of the Earth-projectile system is determined by the position of the projectile and we say that "the projectile has gravitational potential energy mgh." However, this is not always true: if we consider, say, the electrostatic forces between two charged particles of equal mass,

Problem 4D.6

in general both of them will move. In such cases the potential energy difference between two configurations of the system is the work required to move all the particles from one configuration to the other.

For motion in one dimension, the change in potential energy as an object moves from x to $x + \Delta x$, for infinitesimal Δx, is given by $\Delta U = -F\,\Delta x$. It follows that

$$\boxed{F = -\frac{\mathrm{d}U}{\mathrm{d}x}\ .}$$

In three dimensions the analogous relation is

$$\boxed{\vec{\mathbf{F}} = \left[-\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z}\right]\ .}$$

The symbol $\partial U/\partial x$, the ***partial derivative of U with respect to x***, is used to describe a derivative of a function of more than one variable: it means to differentiate U with respect to x, treating the other variables (in this case y and z) as constants.

Since the force can be determined from the potential energy, and vice versa, a physical situation can be specified equally well by specifying the force law or the potential energy function. In many situations it is more convenient to specify the potential energy.

From $F = -\mathrm{d}U/\mathrm{d}x$ it is apparent that positions of equilibrium for a force correspond to places where the derivative of U is zero, i.e. (local) minima or maxima, or occasionally saddle points, of the potential. Stable equilibria correspond to local minima.

Problem 4E.1

Any smooth function $y(x)$ can be expressed in the form of a power series

Problems 4E.2, 4E.3, 4E.4, and 4E.5

$$y(x) = y(0) + k_1 x + k_2 x^2 + k_3 x^3 + \dots\ ,$$

where k_1, k_2, etc., are constants. If we do this for a potential energy function, then

$$U(x) = U(0) + k_1 x + k_2 x^2 + k_3 x^3 + \dots\ .$$

Since only potential differences are defined, we are free to choose $U(0) = 0$, which shifts the function by an overall constant. Furthermore, we can define the x coordinate so that $x = 0$ is a local minimum of $U(x)$. Differentiating with respect to x gives

$$F(x) = -\mathrm{d}U/\mathrm{d}x = -k_1 - 2k_2 x - 3k_3 x^2 - \dots\ ,$$

but the statement that $x = 0$ is a local minimum implies that $k_1 = 0$ (the derivative vanishes at a local minimum) and that $k_2 \geq 0$ (because if it were negative $x = 0$ would be a local maximum). Hence if x is very small, so that the terms in x^2 and higher powers of x are negligible compared to the term in x, we have

$$F(x) \approx -2k_2 x\,,$$

which (as we saw in Chapter 2) is the type of force that leads to simple harmonic motion. Thus, in nearly all cases, a sufficiently small displacement from any local minimum of the potential energy function will lead to simple harmonic motion. The rare exceptions are cases where $k_2 = 0$ at a local minimum, so that the force is proportional to $-x^3$—this will still cause oscillations, but not simple harmonic oscillations—and certain types of functions which cannot be expressed as power series. Such functions are not found to arise physically, although sometimes they do appear in idealized physics problems.

SUMMARY

* The quantities $\frac{1}{2}mv^2$ and mgh which arise in some formulations of projectile and related problems are not just calculational conveniences, but different forms of a new physical concept, the ***energy*** of the body.

* Unlike force, velocity, etc., energy can occur in different guises: $\frac{1}{2}mv^2$ is the ***kinetic energy*** of a moving object, and mgh is the ***gravitational potential energy*** of a body near the Earth's surface, at a height h relative to some reference level.

* Forces can do work on an object. The work done by a constant force on an object is defined as the dot product of the force $\vec{\mathbf{F}}$ and the displacement vector $\overrightarrow{\Delta r}$ through which the object has moved, i.e., $\Delta W = \vec{\mathbf{F}} \cdot \overrightarrow{\Delta r}$. The ***work-energy theorem*** states that kinetic energy of the object increases by the total amount of work done on it by all the forces acting on it. (If the total work done is negative, then the kinetic energy of the object decreases.) The amount of work done per unit time is called the ***power***.

* In the case of a non-constant force, the above definition of work still holds for an ***infinitesimally small*** displacement $\overrightarrow{\Delta r}$. The total work over a large displacement is obtained by integrating $\vec{\mathbf{F}} \cdot \overrightarrow{\Delta r}$ along the path taken by the object, so $W = \int_{\text{Path}} \vec{\mathbf{F}} \cdot \overrightarrow{\mathbf{d}r}$.

* The total amount of energy present in any closed system is unchanged by these transformations: energy is ***conserved***. As far as we can tell the principle of energy conservation is an exact truth, which remains valid even under circumstances that require Newtonian mechanics to be replaced by relativity and/or quantum theory.

* Energy can be stored in the configuration of a system, e.g. the compression of a spring, or the separation of two gravitationally interacting bodies (the Sun and the Earth, or the Earth and a projectile, for instance). If the work done by a force depends only on the initial and final configurations, but not on the path between them, then the force is called ***conservative***. For such forces, which include gravitational, electrostatic, and ideal spring forces, one can define a ***potential energy*** function to describe the energy stored in the configuration. For the gravitational force between two spheres, or between a sphere and a point mass, the potential energy is $U = -GMm/r$, where G is Newton's constant, M and m are the masses, and r is the distance between their centers. The potential energy of a spring is $\frac{1}{2}kx^2$, where k is the spring constant and x is the displacement from the natural length.

* For conservative forces, the total ***mechanical energy*** (kinetic energy plus potential energy) is conserved. For non-conservative forces, such as friction, other forms of energy, such as heat, have to be included in the conservation law.

* For conservative forces, the force can be expressed as the derivative of the position-dependent potential energy, and the positions of equilibrium correspond to local maxima or minima of the potential energy. A stable equilibrium corresponds to a local minimum of the potential energy.

* Physical concepts introduced in this chapter: energy, kinetic energy, potential energy, conservation of energy, work, power.

* Mathematical concepts introduced in this chapter: dot (or scalar) product, partial derivative.

$*$ Many of the equations introduced in this chapter are summarized in the following table:

1 Dimension	**3 Dimensions**	**Description**
$W \equiv F\Delta x$	$W \equiv \vec{\mathbf{F}} \cdot \overrightarrow{\Delta\mathbf{r}}$	Work done by a constant force $\vec{\mathbf{F}}$
$W \equiv \int F(x)\,\mathrm{d}x$	$W \equiv \int_{\vec{\mathbf{r}}_1}^{\vec{\mathbf{r}}_2} \vec{\mathbf{F}} \cdot \overrightarrow{\mathbf{dr}}$	Work done by a varying force $\vec{\mathbf{F}}$
$U(x_p) \equiv U_0 - \int_{x_0}^{x_p} F\,\mathrm{d}x$	$U(\vec{\mathbf{r}}_p) \equiv U_0 - \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}_p} \vec{\mathbf{F}} \cdot \overrightarrow{\mathbf{dr}}$	Potential energy derived from force $\vec{\mathbf{F}}$
$F = -\dfrac{\mathrm{d}U}{\mathrm{d}x}$	$\vec{\mathbf{F}} = \left[-\dfrac{\partial U}{\partial x}, -\dfrac{\partial U}{\partial y}, -\dfrac{\partial U}{\partial z}\right]$	Force derived from potential energy

$*$ Other equations introduced in this chapter:

$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \cos\theta$ (scalar (or dot) product of two vectors)

$\quad = a_x b_x + a_y b_y + a_z b_z$

$\frac{1}{2}mv^2 + mgh = \frac{1}{2}mv_0^2$ (kinetic & potential energy for projectile)

$\frac{1}{2}mv^2 + U(x) = \text{constant}$ (energy conservation)

$W = K_f - K_i$ (work-energy theorem)

$U = \frac{1}{2}kx^2$ (potential energy for spring force)

$U = mgh$ (gravitational potential energy, near Earth)

$U = -\frac{GMm}{r}$ (gravitational potential energy, spherical bodies)

PROBLEMS AND QUESTIONS

By the end of this chapter you should be able to answer or solve the types of questions or problems stated below.

Note: throughout the book, in multiple-choice problems, the answers have been rounded off to 2 significant figures, unless otherwise stated.

At the end of the chapter there are answers to all the problems. In addition, for problems with an (H) or (S) after the number, there are respectively hints on how to solve the problems or completely worked-out solutions.

4A KINETIC ENERGY

4A.1 A particle of mass 0.3 kg moves with velocity [0.6, 1.2, - 0.5] m/s. What is its kinetic energy?

(a) 0.25 J; (b) 0.79 J; (c) 0.31 J; (d) 0.52 J.

4A.2 A projectile of mass m is launched over level ground with an initial speed of v_0 at an angle θ to the horizontal. Neglecting air resistance, what is its kinetic energy (a) just after launch, (b) at the top of its flight, (c) when it lands?

4B WORK AND POTENTIAL ENERGY IN ONE DIMENSION

4B.1 A book of mass 0.5 kg is lifted at constant speed from the floor to the top of a filing cabinet 1.5 m high. By how much has its potential energy changed?

(a) 0.75 J; (b) 7.4 J; (c) 0 J; (d) none of these.

4B.2 (S) You lift a box of mass m from the floor and place it on a shelf a height h above floor level.

(a) How much work have you done on the box? What is the total work done on the box? What happens to the energy of the box? Do the details of the motion of the box during the lift make a difference to your answers?

(b) At the instant when the box has reached height $\frac{1}{2}h$, is the work you have done on it so far more or less than half of the total work you will do during the lift? Explain your reasoning.

4B.3 Suppose that a spring obeys Hooke's law with spring constant k. Show that the potential energy stored in it when it is compressed (or extended) by an amount x is given by $\frac{1}{2}kx^2$, taking the potential energy of the unstretched spring to be zero.

4B.4 (S) The gravitational potential energy of a mass m outside a spherical body of mass M is given by $U(r) = -GMm/r$, where r is the distance of the mass m from the center of the body M. Show that this expression can be reduced to the form $U = mgh$ for a mass m near the Earth's surface.

4B.5 (H) Derive expressions for, and plot, the potential energy

(a) of a space probe of mass m as a function of distance from the Sun, which has mass M (neglect the effect of the planets);

(b) of a positive charge q as a function of distance from a positive charge Q;

4B.5, continued:

(c) of a negative charge $-q$ as a function of distance from a positive charge Q.

It is convenient to be able to draw the plots in a way that is independent of the specific values of M, m, Q, and q. To do this, introduce an arbitrary reference distance r_0. In case (a) the scale on your y-axis should be in terms of GMm/r_0, while in cases (b) and (c) the scale is in terms of $Qq/(4\pi\epsilon_0 r_0)$. In all cases the horizontal axis can be taken as r/r_0, and U_0 should be defined so that U is zero when the distance is infinitely large. For simplicity you may treat this as a one-dimensional problem, considering motion only along a line joining the two objects. (The correct answer to the one-dimensional problem is also valid in three dimensions, but you are not asked to show this.)

4B.6 In 100 words or less explain what is meant by the term "potential energy".

4C WORK IN THREE DIMENSIONS

4C.1 A skier of mass 70 kg skis 40 m down a frictionless ski-slope inclined at 30° to the horizontal. How much work has been done by gravity on the skier?

(a) 2.8 kJ; (b) 27 kJ; (c) 14 kJ; (d) 24 kJ.

4C.2 (S) A horse tows a barge along a canal. The horse exerts a constant force of 750 N and the tow-rope makes an angle of 30° with the direction of motion of the barge. How much work is done by the horse on the barge, via the tow-rope, over a distance of 1 km? If the horse is traveling at 1 m/s, what power does it supply to the barge? Assume that this stretch of the canal is straight.

4C.3 (H) A particle moves with constant velocity [4, −2, 1] m/s. If its mass is 3 kg, what is its kinetic energy? One of the forces acting on it is given by $\vec{\mathbf{F}} = [-1, 2, 2]$ N. Find the work done by this force on the particle as it moves a distance of 3 m. What is the total work done on the particle, and why?

4C.4 (S) A boy swings a pail of water on the end of a rope in a vertical circle. If the rope is 90 cm long, the mass of the pail plus water is 1.5 kg, and the rope is just taut at the top of the swing, what is the speed of the pail at the bottom of the swing, what work is done as the pail moves, and what force is responsible for the work? Assume that the boy's hand does not move during the period considered in this problem.

4D CONSERVATION OF ENERGY

4D.1 (S) **Energy Conservation Riddles**

(a) John is holding a 17 inch computer monitor, with a mass of 25 kg, at a constant height. He complains that it is hard work, and he is becoming exhausted. Jim, who is taking physics, tells him that since there is no displacement in the direction of the applied force, he is not doing any work, and therefore should not be tired. Who is right, and why?

(b) Mary is riding an elevator from the 4th to the 7th floor, wearing a backpack of mass M. Between the 5th and 6th floors the elevator is moving at constant speed, through a distance ℓ. Joan, another physics student, argues that since the velocity of Mary's

4D.1, continued:

backpack is constant, the total force must be zero, and therefore Mary is applying an upward force just enough to cancel that of gravity, Mg. Since the displacement is upward by a distance ℓ, the work that Mary has done on the backpack is $W = Mg\ell$. Mary, on the other hand, points out that the weight she feels is just the same as it would be if she were stationary on the sidewalk, in which case there would be no displacement and therefore no work. Since she is burning no more calories than she would if she were on the sidewalk, and since energy cannot be created from nothing, there is no energy available for her to do work on the backpack. Who is right, and why?

(c) A skater on a frictionless ice rink is initially stationary. Holding onto a rope attached to the wall, he gives a yank and starts himself moving toward the wall. Jean, a physics student who is watching, tells her friend Joe that since the kinetic energy of the skater has increased, work must have been done. It was the rope that applied the force, Jean explains, so it was the rope that did the work on the skater. "Nonsense,“ Joe replied, "ropes can't do work! Ropes don't have any source of energy, so the principle of energy conservation implies that they can't do work. Obviously it was the skater who did the work." "But the skater can't possibly exert a net force on himself,“ retorted Jean, "so he can't have done the work." Who is right here, and why?

4D.2 A construction worker uses a hoist to lift a 50 kg bag of cement at constant speed from the ground to the top of a scaffold 10 m high.

(a) What is the ***total*** work done on the cement?

(b) What is the work done on the cement by the hoist?

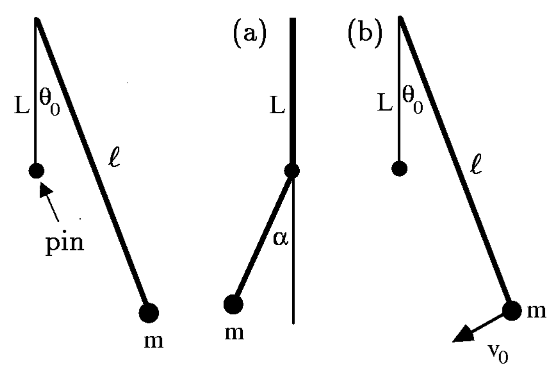
(c) What is the change in the potential energy of the cement?

(d) Discuss your answers in terms of the principle of energy conservation.

(e) Would your answers be different if the speed were not constant?

4D.3 (S) A cannon is capable of firing cannonballs with a fixed speed v_0 but at a range of elevation angles. What will be the speed of the ball at a height h?

4D.4 (H) A simple pendulum consisting of a mass m attached to a string of length ℓ is released from rest at an angle θ_0. A pin is located at a distance L below the pivot point. When the pendulum swings down, the string hits the pin as shown.



(a) What is the maximum angle α that the string makes with the vertical after hitting the pin?

(b) If the bob had been released with an initial velocity v_0 as shown, what would be the maximum value of α? How would this be affected if v_0 were in the opposite direction?

4D.5 (S) A spring with spring constant k is used to power a spring gun which launches a ball of mass m vertically upwards. If the spring is compressed by an amount x beyond its equilibrium position to launch the ball, derive an expression for the maximum height reached by the ball, measured from its position when the spring is compressed. If k is 200 N/m, m = 50 g, and the spring is initially compressed by 4 cm, how high will the ball go?

4D.6 (S) Two objects of equal mass m are connected by a spring of negligible mass, unstretched length ℓ_0, and spring constant k. They are initially at rest on a frictionless air table. An experimenter pulls the objects apart until they are separated by a distance $\ell > \ell_0$. What is the potential energy of the system of masses and spring? If the experimenter now lets go, at what speed is each mass moving when the spring has contracted back to its unstretched length? Compare this with the situation where one end of the spring is attached to a body of mass m and the other to a fixed point on the air table.

4E EQUILIBRIUM AND OSCILLATIONS

4E.1 The potential energy of a particle is given in terms of its position x by $U = -x + 2x^2$, where U is measured in joules and x in meters. At what value of x is the particle in equilibrium?

(a) 0.25 m; (b) 0 m; (c) 0.5 m; (d) none of these.

4E.2 (S) (a) The interior of an ornamental bowl forms a hemisphere of radius 10 cm. The surface is smooth and highly polished. If I place a small object at a distance s from the bottom of the bowl (where s is measured along the inner surface of the bowl, so that s varies from 0 to $\pi R/2$, R = 10 cm), how fast will it be moving when it reaches the bottom of the bowl?

(b) What is the form of the potential energy of the object if s is small? Deduce the force acting on the object if it is displaced slightly from the bottom of the bowl and then released, and describe the resulting motion.

4E.3 (S) A small charge of mass 10 g moves in one dimension in a complicated linear distribution of fixed charges. Over a certain range of distances, the potential energy of the charge is found to be described to a good approximation by

$$U(r) = U_0 + ar + br^2 + cr^3 \ ,$$

where U_0 = 10 J, $a = -1$ J/m, $b = 2$ J/m^2, and $c = -1$ J/m^3. At what values of r would the particle be in equilibrium, and what would the period of small oscillations (if any) around these equilibrium positions be?

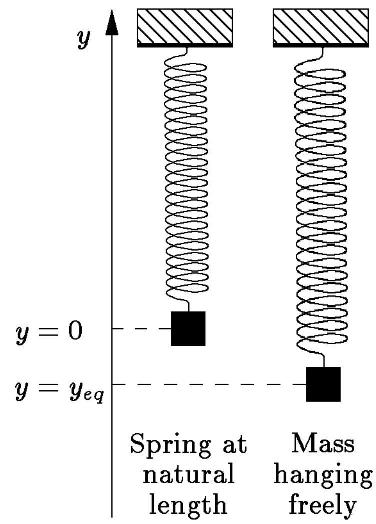
4E.4 (H) A slider of mass 50 g moves on a frictionless linear air track built in the form of a sine wave, so that the height of the track at any point x is given by $y = A \sin kx$, where A = 0.3 m and k = 2 m^{-1} (i.e. the track describes a complete cycle of the sine function over a distance of π meters in x, reaching a maximum height of 30 cm and a minimum height of -30 cm). The length of the track is 4 m, so $0 \leq x \leq 4$ m.

(a) If the potential energy of the slider is defined to be zero where the track reaches its minimum height what is its value at an arbitrary point x?

(b) At what values of x would a stationary slider be in equilibrium? Are small oscillations possible about any of these points, and if so what is the period of these oscillations?

(You may find it useful to recall that for small angles $\cos\theta = 1 - \frac{1}{2}\theta^2$.)

4E.5 (H) A 1 kg mass is suspended from a spring of spring constant 120 N/m. The coordinate system is defined so that y is directed vertically upwards and $y = 0$ when the spring is at its natural length. The mass is first positioned at $y = 0$, and then lowered gently until it is hanging freely from the spring.



Where does the mass come to rest? What then is its gravitational potential energy, and what is the potential energy stored in the spring, compared to their values when the mass is at $y = 0$? Explain why these two quantities are not equal. What work was done (i) by gravity, (ii) by the spring force, as the mass was lowered from $y = 0$ to its equilibrium position? If instead of being lowered gently the mass had simply been released at $y = 0$ and allowed to fall under the combined effects of gravity and the spring force, at what speed would it be moving when it crosses the equilibrium value of y, and what would be its subsequent motion (be quantitative as far as possible)?

COMPLETE SOLUTIONS TO PROBLEMS WITH AN (S)

4B.2 (a) *You lift a box of mass m from the floor and place it on a shelf a height h above floor level. How much work have you done on the box? What is the total work done on the box? What happens to the energy of the box? Do the details of the motion of the box during the lift make a difference to your answers?*

Conceptualize

There are two basic ways to deal with energy problems, the work approach and the conservation of energy approach. In the work approach, we focus on a ***single object***, the forces acting on it, and its displacement. Since potential energy is a ***system*** property, it plays no role in the work approach to energy problems. In contrast, the conservation of energy approach focuses on the ***total*** energy of a ***system***: in this case the potential energy of the system must be included, but we do not need to consider forces acting and work done internally within the system (although if the system is not closed we may need to consider the work done by an external force to calculate the transfer of energy into or out of the system).

In this case most of the question is phrased in terms of work, so we should use the work approach. The object in question is clearly the box, its displacement is h, vertically upwards, and the forces acting on it are gravity and the force supplied by you. Both of these are acting vertically, i.e. parallel to the displacement of the box, and therefore both can do work on the box.

Formulate

The work-energy theorem states that the total work done on the box is

$$W_{fi} = K_f - K_i \ .$$

This is clearly zero, as initial and final kinetic energy are both zero.

At first sight it is not obvious how to calculate the work you do on the box, because we are not told what force you exerted. However, we do know the gravitational force on the box, mg directed downwards, and can therefore calculate the work done by gravity on the box,

$$\vec{\mathbf{F}} \cdot \overrightarrow{\Delta \mathbf{y}} = -mgh$$

(the minus sign comes from the fact that the force is directed downwards, whereas the displacement is directed upwards). Since we know that the total work done is zero, we can easily determine the work done by the only other force acting, namely your muscular effort.

To calculate the energy of the box, or more exactly of the box-Earth system, we need to include the gravitational potential energy. Since the box's height above the Earth's surface has increased by h, the system potential energy has increased by mgh.

Solve

The work you do on the box is $0 - (-mgh) = mgh$. As stated above, the total work done on the box is zero, independent of the details of the motion (the only stipulation is that the box starts from rest and finishes at rest). As we will see in part (b), however,

4B.2, continued:

the work done at an arbitrary point during the lift ***does*** depend to some extent on the details of the motion, specifically on the speed at which the box is lifted.

The total energy of the box-Earth system has increased by mgh, the increase in its gravitational potential energy. This is equal to the work done by you, which represents an energy transfer into the system from outside (in contrast, the work done by gravity is internal to the box-Earth system, and does not change the total energy of the system).

Scrutinize and Learn

Our solution appears to be self-consistent, since the work you do accounts for the increase in the gravitational potential energy of the box. But where has this energy come from? Your kinetic energy has not decreased, and nor has your gravitational potential energy. What has happened is that exothermic chemical reactions in your muscles release chemical potential energy, some of which goes into supplying energy to the box. (Much of it unfortunately goes into raising your body temperature—muscle effort is not a very efficient form of energy transfer.) Chemical potential energy and heat are two more forms that the energy of a body can take. We consider heat as a form of energy later in the book, in Chapter 11.

4B.2 (b) *At the instant when the box has reached a height $\frac{1}{2}h$, is the work you have done on it so far more or less than half of the total work you will do during the lift? Explain your reasoning.*

Conceptualize

Most of the conceptualization carries over from part (a). We are still considering work done on the box, and the forces doing work are gravity and the force exerted by you. This time the displacement is $\frac{1}{2}h$. At first sight it looks as though this problem is trivial, but there is another factor we must consider: the lift is still in progress at this point, and therefore the box is ***moving*** when it passes the point $\frac{1}{2}h$. Hence there is some kinetic energy involved, and this must be included when we apply the work-energy theorem.

Formulate and Solve

The conceptualization has essentially solved the problem. At height $\frac{1}{2}h$, gravity has done work $-\frac{1}{2}mgh$ on the box. The total work done on the box is equal to the kinetic energy it has gained, $\frac{1}{2}mv^2$, where v is its speed at the moment we are considering. Therefore the work you have done is $\frac{1}{2}mgh + \frac{1}{2}mv^2$, which is more than half of the total work you will do.

Scrutinize and Learn

If we assume that you lift the box at constant speed, it's clear that the "extra" work $\frac{1}{2}mv^2$ is done in the first instant of the lift. Right at the end of the lift you exert an upward force of magnitude less than mg, allowing the box to come to rest on the shelf: during this period there is a net downward force which does work $-\frac{1}{2}mv^2$, giving zero total work as we calculated in part (a). This does accord with our experience that the hardest part of the lift is getting the box off the ground in the first place. Other aspects of our formal definition of work, however, are not so consistent with everyday experience: see, for example, the energy conservation riddle 4D.1(a).

4B.4 *The gravitational potential energy of a mass m outside a spherical body of mass M is given by $U(r) = -GMm/r$, where r is the distance of the mass m from the center of the body M. Show that this expression can be reduced to the form $U = mgh$ for a mass m near the Earth's surface.*

Conceptualize

The gravitational potential energy mgh is the difference between the potential energy at height h and the potential energy at a reference point, which we take as the surface of the Earth, $h = 0$. We can calculate this by considering the difference between the potential energy at radius $r_E + h$ and the potential energy at radius r_E, where r_E denotes the radius of the Earth. Since we are dealing with a mass "near the Earth's surface" the value of h will be small compared to the radius of the Earth, or $h \ll r_E$.

Formulate

We can use the form of $U(r)$ given in the question, namely $U(r) = -GMm/r$, where M is the mass of the Earth and r the distance from its center. This gives

$$\Delta U(h) = -\frac{GMm}{r_E + h} - \left(-\frac{GMm}{r_E}\right) .$$

This equation alone cannot be enough, because we have G and M here and g in the expression we are trying to justify. So we also need an equation for g, which we can obtain by applying $F = ma$ to a freely falling body near the Earth's surface:

$$\frac{GMm}{r_E^2} = mg .$$

With these two equations we can solve the problem: we will use the second equation to eliminate GM from our expression for $\Delta U(h)$. It is not immediately obvious that we will also be able to eliminate r_E.

Solve

Simplifying the expression for $\Delta U(h)$,

$$\Delta U(h) = -GMm\left(\frac{1}{r_E + h} - \frac{1}{r_E}\right) = -GMm\left(\frac{-h}{r_E(r_E + h)}\right) .$$

Since $h \ll r_E$ (i.e., we are near the Earth's surface), $r_E(r_E + h)$ is not significantly different from r_E^2. We can therefore write:

$$\Delta U(h) = \frac{GMm}{r_E^2} h ,$$

which becomes, on using our expression for mg,

$$\Delta U(h) = mgh ,$$

and this is exactly what we were asked to show.

4B.4, continued:

Scrutinize

Since we were given the final expression in the question, the dimensions are "obviously" correct, but we can check anyway: mgh represents mass $\times$ acceleration $\times$ length, i.e. force $\times$ distance, which is what we expect for work or energy.

The crucial step in our argument is that $r_E(r_E + h) \approx r_E^2$; this is what allows us to eliminate r_E from our final equation. Is this reasonable? If we recall that the radius of the Earth is about 6400 km, and consider the largest values of h we are likely to use, for example a commercial airliner flying at $h = 11$ km, the difference between $r_E(r_E + h)$ and r_E^2 is 0.17%. It does seem reasonable to argue that this difference is negligible.

Learn

We seem to have achieved the impossible here: we have taken an initial equation with four parameters, i.e. G, M, r_E and h, used only one additional equation, and ended up with an expression with only two parameters, g and h! Where did the other variables go? The answer is that in this case G, M and r_E are not behaving like separate variables—they occur in the same combination in both equations, so they can all be eliminated as if they were a single variable.

The technique we applied here, namely neglecting a small quantity in combination with a much larger one (setting $r_E + h \approx r_E$) is very widely used in physics and engineering. If you use it, however, you must be sure that the large quantity is not going to cancel out at some later stage in the calculation—for example, in this problem it would ***not*** be helpful to set $r_E + h \approx r_E$ in $1/(r_E + h) - 1/r_E$!

There are realistic problems where h is not much smaller than r_E, such as the calculation of the gravitational potential energy of a communications satellite ($h \approx 36000$ km). For such calculations we must use $U = -GMm/r$ rather than $U = mgh$.

4C.2 *A horse tows a barge along a canal. The horse exerts a constant force of* 750 N *and the tow-rope makes an angle of 30° with the direction of motion of the barge. How much work is done by the horse on the barge, via the tow-rope, over a distance of* 1 km*? If the horse is traveling at* 1 m/s, *what power does it supply to the barge? Assume that this stretch of the canal is straight.*

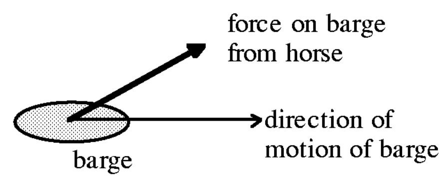
Conceptualize

It seems natural to accept that the force applied by the horse does work on the barge, but, as we will see in Problem 4D.1, "common sense" is not always a good guide when dealing with the physicist's definition of work. We had better start by analyzing this problem using the conservation-of-energy approach.

The horse pulls on the rope, applying a force to it. The rope moves, and therefore work has been done on it ($\vec{\mathbf{F}} \cdot \overrightarrow{\Delta \mathbf{r}} \neq 0$). However, the rope's kinetic energy does not increase, nor does it stretch, so the energy of the rope has not increased. Hence the energy supplied to the rope system by the horse must be balanced by the energy supplied by the rope to something else, namely the barge: the rope does work on the barge equal to the work done on the rope by the horse.

4C.2, continued:

It is, therefore, correct to equate the work done by the horse to the work done on the barge. Since the force that the horse is applying to the barge is not parallel to the direction of motion of the barge, we will need the dot product $\vec{\mathbf{F}} \cdot \vec{\mathbf{r}}$, which corresponds to taking the projection of the force along the barge direction.

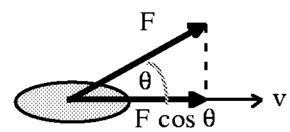


The power supplied to the barge is simply the work done per unit time.

Formulate

The projection of the force along the direction of motion is $F\cos\theta$, where F is the magnitude of the force and θ is the angle between the force and the direction of motion. The work done is therefore

$$W = Fr\cos\theta \ ,$$



where r is the displacement of the barge.

The work done per unit time is

$$\frac{\mathrm{d}W}{\mathrm{d}t} = F\cos\theta \cdot \frac{\mathrm{d}r}{\mathrm{d}t} = Fv\cos\theta \ ,$$

where v is the speed of the barge (both F and θ are constant, remember).

Solve

Our formulation has already produced the algebraic solutions. All that remains is to put in the numbers:

Work done over 1 km = 750 N $\times$ 1000 m $\times$ cos 30° = 650 kJ.

Power supplied = 750 N $\times$ 1 m/s $\times$ cos 30° = 650 W.

Scrutinize

The common British and American unit for the power supplied by an engine, the ***horsepower***, does indeed derive from estimates of the power of one (strong) horse. In SI units 1 hp = 746 W. Our value for the power is therefore reasonable. The dimensions are clearly correct, as can be seen in the numerical calculations above.

Learn

In this example we did not need to use the integral formula for the work done, because we were working with a constant force and a constant velocity. If the angle of the towrope had been varying during the period covered by the problem, we would have had to calculate $\int \vec{\mathbf{F}} \cdot \overrightarrow{\mathbf{dr}}$ explicitly, instead of simply multiplying the total displacement by the projection of the force.

Note that the barge continues to move at constant velocity, so the ***total*** work done on it must be zero. The positive work done by the horse is balanced by negative work

4C.2, continued:

done by the water of the canal, which exerts a drag force on the barge in the direction opposite to its direction of motion.

4C.4 *A boy swings a pail of water on the end of a rope in a vertical circle. If the rope is* 90 cm *long, the mass of the pail plus water is* 1.5 kg, *and the rope is just taut at the top of the swing, what is the speed of the pail at the bottom of the swing, what work is done as the pail moves, and what force is responsible for the work? Assume that the boy's hand is stationary.*

Conceptualize

We are asked for the speed of the pail at the bottom of its swing, but we have more information about the forces acting at the top of the swing, since we are told that the rope is only just taut, i.e. the tension in it is zero, at that point. We can therefore break this problem into three parts:

- Calculate the speed of the pail at the top of its swing. This we can do using Newton's laws, since we know the force and the acceleration.
- Calculate the final kinetic energy. We can do this by applying conservation of energy to the Earth-pail system. This gives us the speed of the pail at the bottom of its swing.
- Find the work done on the pail considered as a single object. Since we now know the change in its kinetic energy, we can do this by using the work-energy theorem.

What force is doing the work? It cannot be the tension in the rope, because the pail is moving in a circle, and therefore its velocity is always at right angles to the direction of the rope. A force acting at right angles to the direction of motion cannot do work ($\cos\theta = 0$). Therefore the only force doing work on the pail is gravity.

Formulate

At the top of the swing, the only force acting is gravity, $m\vec{g}$, where m is the mass of the pail. Since the pail is engaged in circular motion, its acceleration is v_0^2/r, where v_0 is the speed at the top of the swing and r is the length of the rope. So we have

$$mg = m\frac{v_0^2}{r} \ . \tag{1}$$

When the pail reaches the bottom of its swing, it has traveled through a vertical distance $2r$ and has therefore lost potential energy $2mgr$, so its kinetic energy must have increased by this amount. Calling the speed at the bottom v, conservation of energy gives us

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 + 2mgr \ . \tag{2}$$

Solve

From Eq. (1), the speed of the pail at the top of its swing is $v_0^2 = rg$. Putting this into the conservation of energy equation (2) and dividing through by $\frac{1}{2}m$ gives

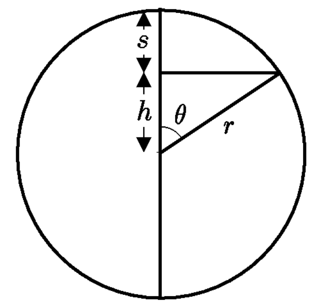
$$v^2 = rg + 4rg = 5rg$$

4C.4, continued:

for the speed of the pail at the bottom of the swing. For the numerical values in the question

$$v = 6.6 \text{ m/s}.$$

We have already concluded that the only force that can do work on the pail is gravity. We know gravity is a conservative force, and therefore the work done depends only on the vertical distance through which the pail moves (see problem 3.10 for some demonstrations of this). Starting, for example, when the pail is at the top of its motion, the work done on the pail by the time it makes an angle θ with respect to the vertical is mgs, where $s = r(1 - \cos\theta)$ is the amount by which the height has decreased from its maximum, as shown in the diagram.



$s = r - h$

The total work done increases (positive work is being done; kinetic energy is increasing) as the pail descends and decreases (negative work being done; kinetic energy decreasing) as it ascends again. Over a full circle no net work is done; the work done over a half circle is $2mgr = 26.5$ J.

Scrutinize

The dimensions of the result are certainly correct, as it involves equations we have checked in earlier problems. We can check the value of v^2 by recognizing that the speed of the pail would be the same if it had simply fallen freely through a distance $2r$, so we can use the equation $v^2 = v_0^2 + 2as$ from Chapter 1:

$$\begin{aligned} v^2 &= v_0^2 - 2g(-2r) \\ &= rg + 4gr = 5rg. \end{aligned}$$

Learn

This is an example of a problem which is easy to solve using energy conservation but very difficult using $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$. To use $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ directly, we would have had to decompose the forces acting into components parallel and perpendicular to the rope and then integrate the perpendicular component over the path of the pail. This is a non-trivial integration, because the tension force is changing in both magnitude and direction as the pail moves.

4D.1 (a) *John is holding a 17 inch computer monitor, with a mass of* 25 kg*, at a constant height. He complains that it is hard work, and he is becoming exhausted. Jim, who is taking physics, tells him that since there is no displacement in the direction of the applied force, he is not doing any work, and therefore should not be tired. Who is right, and why?*

Conceptualize

John is of course telling the truth when he says that he is becoming exhausted, but he is wrong in saying that he is doing work. Jim has correctly applied the physicist's definition of work—the dot product of the force and the displacement—which clearly

4D.1, continued:

vanishes if there is no displacement. It is hard to believe that John is doing no work, since he is getting so tired, but keep in mind that the same job of holding the monitor can be done by a desk, which can hold the monitor for many years without getting tired. Apparently John becomes tired because human muscles are very inefficient in applying a static force. Even though the static force requires no energy, in fact John's muscles cannot exert this force unless they are metabolizing chemical energy and converting it into heat. All of this energy is "wasted," in that all of it goes into heat, not work.

Solve

(This is really a conceptual problem, so we can now just state the solution.) Neither John nor Jim was entirely right. Jim was correct in saying that John had done no work, but wrong when he concluded that John should not expect to become tired. The human body can convert chemical energy to heat even when it is not doing work.

Learn

In this problem we were asked to choose between Jim's formal logic based on the definition of "work," and John's application of "common sense." Once one finds the right answer, it is usually apparent that it is consistent with both. If either is lacking, it is always a good idea to think again.

4D.1 (b) *Mary is riding an elevator from the 4th to the 7th floor, wearing a backpack of mass M. Between the 5th and 6th floors the elevator is moving at constant speed, through a distance ℓ. Joan, another physics student, argues that since the velocity of Mary's backpack is constant, the total force must be zero, and therefore Mary is applying an upward force just enough to cancel that of gravity, Mg. Since the displacement is upward by a distance ℓ, the work that Mary has done on the backpack is $W = Mg\ell$. Mary, on the other hand, points out that the weight she feels is just the same as it would be if she were stationary on the sidewalk, in which case there would be no displacement and therefore no work. Since she is burning no more calories than she would if she were on the sidewalk, and since energy cannot be created from nothing, there is no energy available for her to do work on the backpack. Who is right, and why?*

Conceptualize

We can't find anything wrong with Joan's logic. The force that Mary is applying to the backpack is certainly Mg upward, and the displacement of the backpack during the time interval under discussion is certainly ℓ, also upward. The work done, the dot product of the force and the displacement, is unavoidably $W = Mg\ell$. Looking more carefully at Mary's argument, however, we can find a flaw: while she argues correctly that she is burning no more calories than if she were on the sidewalk, she has not considered the possibility of other sources of energy. What about the force of the elevator floor on her feet? That force is larger than it would be if she were not carrying the backpack, by an amount Mg. Since her displacement during this time period is also ℓ, the extra work that the floor does on her, because she is carrying the backpack, is precisely $Mg\ell$. The source of the energy has been found.

Solve

Joan was correct. Mary was wrong because she failed to realize that the floor was doing more work on her than it would have if she were not carrying the backpack.

4D.1, continued:

Learn

When one needs to find all the forces acting on an object, one should never forget to do the simple exercise of looking around the boundary to see what the object is touching. Within the context of this book, the only forces that are not transmitted by direct contact are gravity and the electrostatic force.

This problem also illustrates another important principle, an elaboration of the principle of energy conservation: Energy is not merely conserved, but it is "locally" conserved. This means that it is not only impossible for the total energy of a closed system to change, it is also impossible for energy to disappear in one place and reappear somewhere else. Energy has to be transmitted. It can be transmitted through contact forces, and also by long range effects such as gravity or electromagnetic fields. The Earth obtains energy from the Sun, for example, mainly by electromagnetic waves. In this problem the energy comes from the motor that drives the elevator, and it is transmitted to Mary by the force of the elevator on her feet, and then to the backpack through the force that Mary's shoulders exert on it.

4D.1 (c) *A skater on a frictionless ice rink is initially stationary. Holding onto a rope attached to the wall, he gives a yank and starts himself moving toward the wall. Jean, a physics student who is watching, tells her friend Joe that since the kinetic energy of the skater has increased, work must have been done. It was the rope that applied the force, Jean explains, so it was the rope that did the work on the skater. "Nonsense," Joe replied, "ropes can't do work! Ropes don't have any source of energy, so the principle of energy conservation implies that they can't do work. Obviously it was the skater who did the work." "But the skater can't possibly exert a net force on himself," retorted Jean, "so he can't have done the work." Who is right here, and why?*

Conceptualize

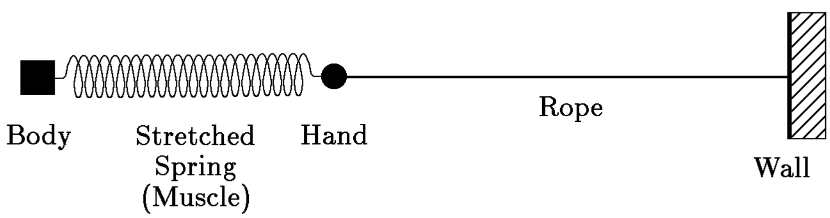
This problem is very subtle, so we will try to puzzle it through one step at a time. Let's begin by examining the statements made by the characters in the problem. Was Jean right when she said that work must have been done on the skater? Of course she was, because the kinetic energy of the skater clearly increased, and the work-energy theorem guarantees that the work done on an object is equal to the change in kinetic energy. Was Jean also right when she claimed that it was the rope that applied the force that started the skater moving? Yes, again she is on target. In the absence of friction from the ice, there is no way that a skater could start himself moving without some external agent applying a force. (This issue will be discussed in more detail in the next chapter, but in a nutshell we can say that it is possible for the skater's shoulder, for example, to apply a force to his arm, but his arm will always apply an equal and opposite force to his shoulder. The net force that the skater exerts on himself is always zero. The net force that any object exerts on itself is always zero.) Finally, was Joe right when he said that the rope has no source of energy, and therefore could not possibly be the agent which did the work on the skater? Yes, Joe was right, too. It would be different if someone were at the other end of the rope pulling, but in this case the rope was tied to the motionless wall. There was no input of energy to the rope, so the principle of energy conservation implies that the rope could not transfer energy to the skater. So,

4D.1, continued:

we have concluded that the skater exerted no net force on himself, but that work was done on the skater by something other than the rope. Where do we go next?

Solve

In this case conceptualization did not get us as far as usual, as we still seem to be missing something crucial. When the solution to a problem seems out of sight, the clever puzzle-solver will try to warm up with a simpler problem that has most of the same ingredients. A skater is complicated, so let's imagine replacing him by a massless pointlike hand attached to a massless arm, which is in turn attached to a pointlike body, which contains all the mass. Muscles are complicated, too, so let's replace the massless arm by a spring. We will assume that the spring starts in a stretched state, so it exerts a force of contraction, much like the muscle it is replacing. The problem then looks like the following:



When the spring contracts it exerts a force on the hand, toward the left, and a force of equal magnitude on the body, toward the right. The net force of the spring on the "skater" is zero, as we expected. Now focus on the hand. The spring is pulling it to the left, but it clasps rigidly to the rope so that it cannot move to the left. As long as the hand holds tightly to the rope, the rope will develop a tension equal to the force of the spring on the hand, so that the net force on the hand will be zero. The rope is therefore exerting a force on the skater, but is it doing work? No, because the displacement is zero! Even as the body starts to slide to the right, the hand that is holding the rope does not move. Since the rope is in contact with the hand, not the body, the work done by the rope is calculated from the displacement of the hand—so it vanishes.

In response to the force applied by the spring, the body will start to move to the right. As it undergoes a displacement to the right, the spring will be doing work on it. Since the spring is part of the skater, the skater is doing work on himself. The net ***force*** that the skater exerts on himself is zero, since the spring pulls the hand to the left with the same force that it pulls the body to the right. But the hand is stationary and the body moves, so the net ***work*** done by the skater on himself is positive. It is crucial here that the skater is not rigid, so that his body can move while his hand is standing still. If the skater were rigid, he would not be able to pull on the rope.

Since the question is qualitative rather than quantitative, the answer to the simplified problem carries over to the real problem. Although the skater can exert no net force on himself, he can nonetheless do net work on himself. As long as he is not rigid, so that different parts of his body can have different displacements, then the net work that he

4D.1, continued:

does on himself can be nonzero, even though the net force he exerts on himself is always zero.

Learn

The secret of solving a hard problem is to isolate the relevant features, stripping away the complexity that can confuse the issue. The spring and point mass model of the skater shown above is a classic example of how oversimplifications can sometimes be very helpful in analyzing a complicated situation. There is an often-told joke that when a theoretical physicist was asked to examine the problem of how dairy farmers could increase milk production, she began her analysis with "Consider a spherical cow." If you want to think more about this, we suggest that you start by considering a spherical joke.

4D.3 *A cannon is capable of firing cannonballs with a fixed speed v_0 but at a range of elevation angles. What will be the speed of the ball at a height h?*

Conceptualize

This problem is clearly a candidate for the energy conservation approach. We consider the Earth-cannonball system, which is a closed system at any time during the cannonball's flight. The total energy is therefore conserved, and the gain in system potential energy produced when the cannonball's height above the ground increases by h must be balanced by a loss in kinetic energy. We work in the reference frame where the Earth is stationary, and so the change in kinetic energy must come from a change in the speed of the cannonball.

Formulate

The initial kinetic energy is $\frac{1}{2}mv_0^2$, and the kinetic energy at height h is $\frac{1}{2}mv_h^2$, where v_h is the speed at height h. The potential energy at height h is mgh. We do not know the mass m of the ball, but note that it occurs linearly in all three expressions and will therefore cancel out.

Solve

We simply write $\frac{1}{2}mv_h^2 = \frac{1}{2}mv_0^2 - mgh$

and thus $v_h = \sqrt{v_0^2 - 2gh}$.

Scrutinize

An acceleration multiplied by a length has dimensions of [length]^2/[time]^2, which is the same as [velocity]^2, so the dimensions of the answer are correct.

We can verify the exact form (and demonstrate the comparative simplicity of the energy approach!) by using the techniques for analyzing projectile motion that we learned in Chapter 1. The velocity of the cannonball at time t is given by

$$v_y = v_0 \sin\theta - gt \qquad\qquad v_x = v_0 \cos\theta \ ,$$

where θ is the angle the initial velocity makes with the horizontal. The height h is

$$h = v_0 t \sin\theta - \tfrac{1}{2}gt^2 \ .$$

4D.3, continued:

This is a quadratic equation for t which we can solve to find

$$t = \frac{v_0 \sin\theta}{g}\left(1 \pm \sqrt{1 - \frac{2gh}{v_0^2 \sin^2\theta}}\right) \ .$$

The minus sign represents the time when the ball reaches this height on its way up, and the plus sign when it's on the way down. If the expression under the square root is negative, the ball never reaches this height (notice that this is extra information that we didn't get from the energy solution). Now substitute this value for t into v_x and v_y and combine them to get

$$\begin{aligned} v_h^2 = v_x^2 + v_y^2 &= v_0^2 \sin^2\theta - 2v_0 g t \sin\theta + g^2 t^2 + v_0^2 \cos^2\theta \\ &= v_0^2 + g^2 t^2 - 2v_0 g t \sin\theta \\ &= v_0^2 + v_0^2 \sin^2\theta \left(1 \pm \sqrt{1 - \frac{2gh}{v_0^2 \sin^2\theta}}\right)^2 - 2v_0^2 \sin^2\theta \left(1 \pm \sqrt{1 - \frac{2gh}{v_0^2 \sin^2\theta}}\right) \ . \end{aligned}$$

Expanding this truly horrible-looking expression gives us

$$\begin{aligned} v_h^2 = v_0^2 + v_0^2 \sin^2\theta + v_0^2 \sin^2\theta - 2gh \pm 2v_0^2 \sin^2\theta \sqrt{1 - \frac{2gh}{v_0^2 \sin^2\theta}} \\ - 2v_0^2 \sin^2\theta \mp 2v_0^2 \sin^2\theta \sqrt{1 - \frac{2gh}{v_0^2 \sin^2\theta}} \end{aligned}$$

and all the terms in $v_0^2 \sin^2\theta$ cancel out to leave $v_h^2 = v_0^2 - 2gh$ as before.

Learn

It is clear that the energy approach provides a much simpler path to the solution! To find whether the cannonball could actually reach the specified height for a given angle, we could have combined the two techniques by using Newton's laws to analyze the horizontal motion of the ball—which, in the absence of air resistance, is very simple—and energy conservation to deal with the vertical motion. Our solution to the horizontal motion would assure us that the kinetic energy associated with this is constant, and so we can ignore it in applying energy conservation to the vertical motion.

The fact that the force approach confirms the speed to be independent of the angle at which the cannonball was launched is another demonstration that gravity is a conservative force: the kinetic energy gained by the ball (and the potential energy lost) in reaching height h is independent of the path by which it reached that height.

4D.5 *A spring with spring constant* 200 N/m *is used to power a spring gun which launches a* 50 g *ball vertically upwards. If the spring is compressed by 4* cm *beyond its equilibrium position to launch the ball, how high will the ball go?*

4D.5, continued:

Conceptualize

Working in terms of energy conservation, we consider the closed system of Earth, spring and ball. The total energy of this system is made up of three parts: the potential energy stored in the configuration of the spring, the gravitational potential energy of the ball-Earth system, and the ball's kinetic energy. We could choose various reference points from which to define the gravitational potential energy: the most obvious is the starting point, at which the spring is compressed (and hence has some potential energy) and the ball is stationary. The total energy of the system is then equal to the initial potential energy of the spring. [Note that we could also conceptualize this problem in terms of forces, as we did in Chapter 2, but as we saw in Problem 4D.4 the energy approach is often very much simpler than the force approach.]

Formulate

First we must find the potential energy associated with the compression x_c of a spring. We did this in problem 4B.3; the result was

$$U(x_c) = \int_0^{x_c} kx\,\mathrm{d}x = \frac{1}{2}kx_c^2 \ .$$

The kinetic energy is given by $\frac{1}{2}mv^2$, and the gravitational potential energy by mgh, where h is the height reached above the starting point. Thus the total energy of the system of spring plus ball at the start of the problem is

$$E_i = \frac{1}{2}kx_c^2 \ .$$

At maximum height, since the spring is no longer compressed and the ball's velocity is zero, the total energy is

$$E_f = mgh \ .$$

Energy conservation tells us that the initial and final energies must be equal, so combining these two equations will solve the problem.

Solve

Combining our two equations,

$$mgh = \frac{1}{2}kx_c^2 \ .$$

The maximum height that the ball reaches above its starting point is

$$h = \frac{kx_c^2}{2mg} = \frac{200 \text{ N/m} \times (.04 \text{ m})^2}{2 \times 0.05 \text{ kg} \times 9.8 \text{ m/s}^2} = 0.33 \text{ m} = 33 \text{ cm} \ .$$

Scrutinize

The dimensions of k are [force]/[length], since the spring force is $-kx$, and we know that mg is a force. So $kx^2/2mg$ is a length, as it should be. The ball reaches greater heights for larger spring forces or greater spring compression, which is sensible, and if

4D.5, continued:

you increase the mass of the ball (or imagine increasing g) it will not go so high, which is also sensible.

Note that this problem is simplified by the fact that the spring gun is directed vertically. If it had instead been directed at some angle to the vertical, we would have had to include the ball's horizontal kinetic energy when calculating E_f. We could determine this by solving the equations of motion for the horizontal component only, as discussed in the previous solution.

Learn

Like Problem 4D.4, this is an example of a problem which can be solved by $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, but whose solution is very much simpler using energy methods. This is generally true: if a problem can be solved either way, the solution using conservation laws is almost always easier than that using the full differential equation. On the other hand, the energy solution generally gives less information than the force solution, because it is insensitive to details of the motion. For example, the energy approach does not tell us that the velocity in this case is directed vertically, in contrast to the horizontal motion in problem 4C.4.

4D.6 *Two objects of equal mass m are connected by a spring of negligible mass, unstretched length ℓ_0, and spring constant k. They are initially at rest on a frictionless air table. An experimenter pulls the objects apart until they are separated by a distance $\ell > \ell_0$. What is the potential energy of the system of masses and spring? If the experimenter now lets go, at what speed is each mass moving when the spring has contracted back to its unstretched length? Compare this with the situation where one end of the spring is attached to a body of mass m and the other to a fixed point on the air table.*

Conceptualize

Our system consists of the two equal masses and the spring. Since all the motion takes place on a level table, we do not need to consider gravitational potential energy, so the total energy of the system is the potential energy of the spring and the kinetic energy of the two masses. This is initially zero: the string is at its unstretched length and the masses are stationary. When the experimenter separates the two masses, she does work on at least one of them, and it in turn does work on the spring. The stretched system, held still, has zero kinetic energy, but positive potential energy stored in the configuration of the spring. When the experimenter lets go, the masses-and-spring system is closed—no net external force is acting—and so its total energy must remain constant. As the spring contracts back to its unstretched length, both masses will gain kinetic energy. Since the system is symmetrical, the kinetic energy gain is split equally between the two masses. In contrast, if one end of the spring is fixed, all the kinetic energy must go to the mass on the free end.

Formulate

As we saw in 4B.4, the potential energy of a spring stretched by an amount x is $\frac{1}{2}kx^2$, so the total energy of the system is $\frac{1}{2}k(\ell - \ell_0)^2$. When the spring has contracted back to its unstretched length, the system's potential energy is zero, so its kinetic energy $2 \times (\frac{1}{2}mv^2)$ must be equal to the total energy.

4D.6, continued:

Solve

The kinetic energy of each mass when the spring is back to its unstretched length is

$$\tfrac{1}{2}mv^2 = \tfrac{1}{4}k(\ell - \ell_0)^2,$$

so the speed of each mass at this point is

$$v = \sqrt{\frac{k}{2m}}(\ell - \ell_0).$$

If one end of the spring is held fixed and the other attached to a mass m, the kinetic energy of that mass is

$$\tfrac{1}{2}mv_1^2 = \tfrac{1}{2}k(\ell - \ell_0)^2,$$

so its speed is

$$v_1 = \sqrt{\frac{k}{m}}(\ell - \ell_0).$$

Scrutinize

Recalling that the dimensions of k are $[\text{mass}]/[\text{time}]^2$, we see that the two expressions for v are dimensionally correct. They also behave sensibly if we change the input parameters: if the spring is not stretched initially ($\ell = \ell_0$), the velocity is zero; if the spring has no resistance to stretching and therefore stores no potential energy ($k = 0$, so $\frac{1}{2}kx^2 = 0$), the velocity is again zero.

Learn

We might have expected that the two-mass system would look the same as the one-mass system in the reference frame where one of the masses is stationary. However, that is not so: in such a frame the speed of the second mass is

$$v_2 = 2v = \sqrt{\frac{2k}{m}}(\ell - \ell_0),$$

and this is not equal to v_1. In a system where more than one mass moves, we must keep track of all the moving bodies, not just the one we might be interested in. In the next chapter we will develop the concepts needed to do this in a systematic way.

4E.2 (a) *The interior of an ornamental bowl forms a hemisphere of radius* 10 cm. *The surface is smooth and highly polished. If I place a small object at a distance s from the bottom of the bowl (where s is measured along the inner surface of the bowl, so that s varies from* 0 *to* $\pi R/2$, $R = 10$ cm*), how fast will it be moving when it reaches the bottom of the bowl?*

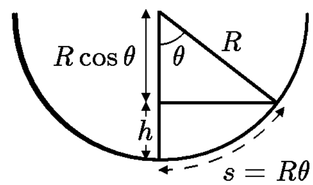
Conceptualize

This is another problem in which energy conservation implies that a loss of gravitational potential energy mgh is balanced by a gain in kinetic energy $\frac{1}{2}mv^2$.

4E.2, continued:

Formulate and Solve

We need to find an expression for h in terms of s; from the diagram this is



$$h = R\left(1 - \cos\frac{s}{R}\right) \ .$$

Hence the speed of the object at the bottom of the bowl is

$$v = \sqrt{2gh} = \sqrt{2gR\left(1 - \cos\frac{s}{R}\right)} \ .$$

Scrutinize and Learn

Note that angles are dimensionless, so anything we take a cosine or sine of must have no dimensions—one can't take cos(1 meter) or sin(0.3 kg). In our expression s/R is a ratio of two lengths, which is OK. We have checked in earlier problems that the dimensions of an acceleration times a length are the same as velocity squared.

Is our expression for v ever going to involve the square root of a negative number? (This usually corresponds to a situation which cannot physically occur.) In this case the answer is no, because the cosine function has a maximum value of 1.

(b) *What is the form of the potential energy of the object if s is small? Deduce the force acting on the object if it is displaced slightly from the bottom of the bowl and then released, and describe the resulting motion.*

Conceptualize

The form of the gravitational potential energy of the object for any s can be easily calculated by taking the standard form mgh and using our expression for h. The effect of taking a ***small*** s is to make the angle θ small, and so we can use the standard approximate forms of the trigonometric functions for small angles. We can then use the potential energy to find the net force acting on the object.

If we visualize the situation described in the problem, we can see that the object should slide partway up the opposite side of the bowl and then slide down again. Energy conservation implies that (in the absence of friction) it will reach the same height above the bottom of the bowl as its starting point (since it should have the same potential energy). Therefore we can see that the ball should repeat the same motions indefinitely—it will oscillate about the bottom of the bowl.

Formulate

Taking $U = mgh$ and using our expression for h gives, in terms of s,

$$U(s) = mgR\left(1 - \cos\frac{s}{R}\right) \ .$$

This defines U such that it is zero at the bottom of the bowl.

4E.2, continued:

For small angles θ, as we noted in Chapter 2,

$$\sin\theta \approx \theta ,$$

so we can use the standard trigonometric relation

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha$$

to deduce that for small θ

$$\cos\theta \approx 1 - \frac{1}{2}\theta^2 .$$

The force can be found by differentiating the potential energy, using

$$F(s) = -\frac{\mathrm{d}U}{\mathrm{d}s} .$$

Our strategy for solving the problem is now clear: we will use the small-angle approximation to obtain an approximate form for $U(s)$ in the case where s is small, and then differentiate this to find the force. We expect to find that the form of the force implies oscillatory motion.

Solve
Substituting $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ into our potential energy expression gives

$$U(s) = \frac{mg}{2R}s^2 \qquad \text{(for } s \ll R\text{)}.$$

Then

$$F(s) = -\frac{\mathrm{d}U}{\mathrm{d}s} = -\frac{mg}{R}s .$$

This has the standard form that we associate with simple harmonic motion: the force is proportional to the distance from a position of equilibrium (the bottom of the bowl) and directed towards it. The equation of motion can be cast in the standard form

$$m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = -\frac{mg}{R}s \quad \Longrightarrow \quad \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = -\omega^2 s , \quad \text{with} \quad \omega = \sqrt{g/R} .$$

We conclude that the object will perform simple harmonic oscillations about the bottom of the bowl, with $\omega = \sqrt{g/R}$ and period $T = 2\pi/\omega = 2\pi\sqrt{R/g} = 0.63$ s.

Learn
You should observe a similarity to our analyses of the simple pendulum. This is just what we should expect, as the motion of the object is ***exactly*** equivalent to pendulum motion—we are just using the surface of the bowl to force it to remain a fixed distance R from the bowl's center, instead of using a string to force it to remain a distance ℓ from the suspension point.

4E.2, continued:

Note that energy conservation implies that the object will oscillate around the bottom of the bowl for any initial value of s, since the argument about returning to the same level of potential energy does not depend on small angles. However, for larger angles the form of the force is more complicated, and the motion is not exactly simple harmonic. This is, of course, also true for the simple pendulum.

4E.3 *A small charge of mass* 10 g *moves in one dimension in a complicated linear distribution of fixed charges. Over a certain range of distances, the potential energy of the charge is found to be described to a good approximation by*

$$U(r) = U_0 + ar + br^2 + cr^3 \ ,$$

where $U_0 = 10$ J, $a = -1$ J/m, $b = 2$ J/m^2 *and* $c = -1$ J/m^3. *At what values of* r *would the particle be in equilibrium, and what would the period of small oscillations (if any) around these equilibrium positions be?*

Conceptualize
An equilibrium position is one at which the charge feels no net force. Therefore, to find the equilibria we must find the force (by differentiating the potential energy) and locate the positions r at which the force is zero.

To determine whether the charge will oscillate, we can consider the force on it if it is moved a small distance from the equilibrium point. If this force points back towards the equilibrium, there will be oscillations; if not, there won't be. For sufficiently small displacements, the oscillations will have the standard simple harmonic form and we will be able to determine the period.

Formulate
The force acting on the charge is:

$$F(r) = -\frac{\mathrm{d}U}{\mathrm{d}r} = -a - 2br - 3cr^2 \ .$$

This is a case where it is simpler to insert the numerical coefficients, giving

$$F(r) = 1 - 4r + 3r^2 = (1-r)(1-3r) \ ,$$

where r is measured in meters and F in newtons.

Solve
There are clearly two equilibrium points, at $r = 1$ m and $r = \frac{1}{3}$ m. We consider these in turn.

Case r = 1 m:

Formulate
We need to re-express the potential energy in terms of displacements from the equilibrium position. Defining x to be this displacement, so that in this case $x = r - 1$. The potential energy becomes

$$\begin{aligned} U(x) &= U_0 - (x+1) + 2(x+1)^2 - (x+1)^3 \\ &= U_0 - x^2 - x^3 \approx U_0 - x^2 \text{ for small } x \ . \end{aligned}$$

4E.3, continued:

Solve
Differentiating $U(x)$, the force $F(x) = +2x$ for small x. This is ***not*** the form that we expect for simple harmonic motion: the direction of the force is such that if the charge is displaced slightly from equilibrium it will tend to move further away from $x = 0$. This is a position of ***unstable*** equilibrium, and no oscillations are possible. Examination of $U(r)$ shows that $r = 1$ m is a local maximum of the potential.

Case $r = \frac{1}{3}m$:

Formulate and Solve
Following the same procedure, but defining $x = r - \frac{1}{3}$, we obtain

$$U(x) = U_0 - \left(x + \tfrac{1}{3}\right) + 2\left(x + \tfrac{1}{3}\right)^2 - \left(x + \tfrac{1}{3}\right)^3$$
$$= 9\tfrac{23}{27} + x^2 - x^3 \approx 9\tfrac{23}{27} + x^2 \text{ for small } x\,.$$

The force $F(x) = -2x$ for small x. This time we do have the appropriate conditions for oscillation. The equation of motion of the small charge is

$$-2x = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\ ,$$

so the solution will be of the form $x = A \sin \omega t$ with $\omega = \sqrt{2/m}$ and period $2\pi/\omega$, or 0.44 s.

Learn
It is easy to see that in the first case the potential energy ***decreases*** as x moves away from zero, whereas in the second case it ***increases***. These are the conditions for unstable and stable equilibrium respectively. (There are equilibria which are neutral, i.e. neither stable nor unstable, such as a puck resting on a level frictionless surface. What happens to the potential energy when the puck is displaced in such a situation? What is the subsequent motion of the puck?) Notice that in neither case considered here has U reached its absolute or global minimum (or maximum) value: for this particular function U tends to $-\infty$ as r tends to $+\infty$ and vice versa. For oscillations to be possible it is only necessary that the potential energy increase locally in all accessible directions—it can decrease again further away.

HINTS FOR PROBLEMS WITH AN (H)

The number of the hint refers to the number of the problem

4B.5 (a) By how much does the potential energy of the mass change when it is moved radially away from the sun by an infinitesimal distance Δr?

(b) & (c) What is the force between two charges? How does it compare to that between two masses?

4C.3 What is the displacement Δr of this particle in a time Δt? How long does it take to travel 3 m?

What is the net force acting on a particle moving at constant velocity?

4D.4 What is the potential energy of the pendulum as a function of angle before and after the string hits the pin? What can you say about the total energy of the pendulum? [Take the potential energy to be zero at the bottom of the pendulum's swing.]

4E.4 What is the potential energy function for a small displacement s from each of the equilibrium positions?

If you are having trouble with this problem, study the solution to problem 4E.3.

4E.5 Draw a force diagram for the mass. What condition must hold for the mass to be at rest?

What does the word 'gently' imply in the phrasing of this question?

When the mass is dropped, what is the change in its potential energy between its starting position and the equilibrium position? What does this tell you about the change in its kinetic energy?

ANSWERS TO HINTS

4B.5 (a) $\Delta U = -\vec{\mathbf{F}} \cdot \overrightarrow{\mathbf{\Delta r}} = \dfrac{GMm}{r^2}\Delta r$

[Note that the outward movement corresponds to a gain in potential energy.]

(b) & (c) See Chapter 2, *Summary*.

4B.6 An acceptable answer would be

"The potential energy of a particle at position $\vec{\mathbf{r}}$ relative to an arbitrary reference position $\vec{\mathbf{r}}_0$ is the work done by a conservative force in moving the particle from $\vec{\mathbf{r}}_0$ to $\vec{\mathbf{r}}$. Because the force is conservative, this energy is 'stored' in the new configuration and would be released if the particle were to move back from $\vec{\mathbf{r}}$ to $\vec{\mathbf{r}}_0$."

4C.3 $\Delta\vec{\mathbf{r}} = \vec{\mathbf{v}}\Delta t$
$= ([4, -2, 1]\,\mathrm{m}) \times (\Delta t/\mathrm{s})$;
$3/\sqrt{21}$ s.

Zero net force.

4D.4 Before: $mg\ell(1 - \cos\theta)$;

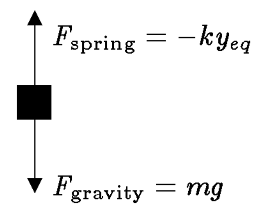
After: $mg(\ell - L)(1 - \cos\alpha)$.

The total energy is conserved, so it is equal to the potential energy at the ends of the swing (when the kinetic energy is zero).

4E.4 $0.294(1 - s^2)$ J (for $x = \pi/4$ m, s in m);

$+0.294s^2$ J (for $x = 3\pi/4$ m, s in m).

4E.5



$mg = -ky$

At negligible velocity, so acceleration and kinetic energy are effectively zero.

-0.40 J; its kinetic energy must increase by 0.40 J.

ANSWERS TO ALL PROBLEMS

4A.1 c

4A.2 $\frac{1}{2}mv_0^2$; $\frac{1}{2}mv_0^2\cos^2\theta$; $\frac{1}{2}mv_0^2$

4B.1 b

4B.2 See complete solution.

4B.3 If the spring is compressed (or extended) by an amount s, the force it exerts is $F(s) = -ks$. The potential energy $U(x)$ corresponding to a compression (or extension) x is, by definition,

$$U(x) = U_0 - \int_0^x F(s)\,\mathrm{d}s = -\int_0^x (-ks)\,\mathrm{d}s,$$

where we have set $U_0 = 0$ (i.e. the potential energy is zero when the spring is at its natural length). Evaluating this integral gives

$$U(x) = \frac{1}{2}kx^2$$

as required.

4B.4 See complete solution.

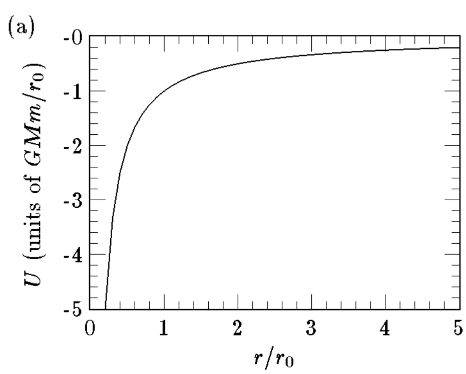
4B.5 Derivations:

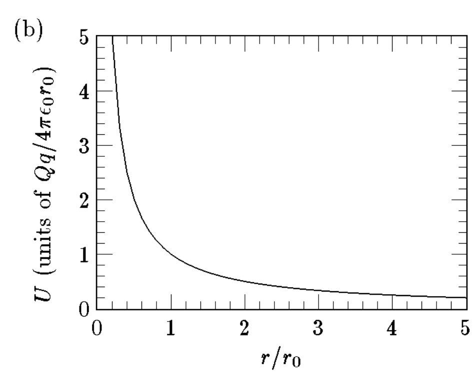
(a) $U(r_p) = U_0 - \int_\infty^{r_p} F\,\mathrm{d}r = -\int_\infty^{r_p}\left[-\frac{GMm}{r^2}\right]\mathrm{d}r = -\frac{GMm}{r}\bigg|_\infty^{r_p} = -\frac{GMm}{r_p}$.

(b) $U(r_p) = U_0 - \int_\infty^{r_p} F\,\mathrm{d}r = -\int_\infty^{r_p}\left[\frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}\right]\mathrm{d}r = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r}\bigg|_\infty^{r_p} = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r_p}$.

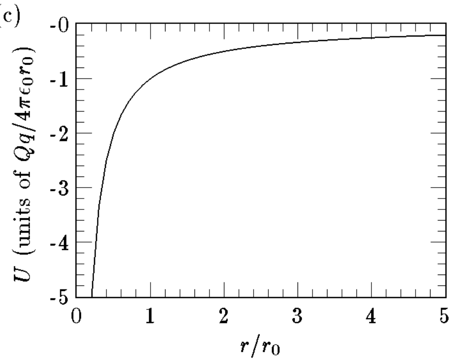
(c) Same as (b), but with opposite sign, so $U(r_p) = -\frac{1}{4\pi\epsilon_0}\frac{Qq}{r_p}$.

Graphs:





(c)



4C.1 c

4C.2 See complete solution.

4C.3 31.5 J; -3.9 J; 0 J (constant velocity implies no net force, so no net work).

4C.4 See complete solution.

4D.1 See complete solution.

4D.2 0 J; 4.9 kJ; 4.9 kJ.

Since the cement moves at constant speed, there is no change in kinetic energy, and hence no work is done. Gravity does negative work on the cement, since the force of gravity is opposite the direction of motion. The hoist, therefore, must do an equal magnitude of positive work to compensate. The potential energy of the cement is increased, and that energy must be balanced by a loss of chemical potential energy in the body of the worker.

A changing speed would have no effect on the overall work done, provided that the final kinetic energy was the same as the initial kinetic energy. During periods of acceleration upwards, the hoist is exerting a greater force (doing more work) than gravity; when the cement is slowing down (acceleration directed downwards) the reverse is true.

4D.3 See complete solution.

4D.4(a) $\cos\alpha = \dfrac{\ell\cos\theta_0 - L}{\ell - L}$;

(b) $\cos\alpha = \dfrac{\ell\cos\theta_0 - L - v_0^2/2g}{\ell - L}$;

Not changed.

4D.5 See complete solution.

4D.6 See complete solution.

4E.1 a

4E.2 See complete solution.

4E.3 See complete solution.

4E.4(i) $0.147(1+\sin 2x)$ J, where x is in meters.

(ii) At $x = \pi/4$, $3\pi/4$; oscillations possible about the second of these, with period 1.8 s.

4E.5 $y = -8.2\,\mathrm{cm}$; -0.80 J; 0.40 J; not equal because the spring force decreases as $|y|$ decreases, so in the region $0 \leq |y| \leq 8.2$ cm the spring force is smaller in magnitude than the gravitational force, and hence is doing less work, so the 'stored' potential energy must also be less. No work is done overall (the mass's kinetic energy does not change) because the person lowering the mass "gently" to its equilibrium point exerts an additional upward force to prevent it from falling freely.

(i) 0.80 J; (ii) -0.40 J.

0.89 m/s; simple harmonic motion with $\omega = \sqrt{k/m} = 11$ rad/s and period 0.57 s.

CHAPTER 4

SUPPLEMENTARY NOTES

CONSERVATION LAWS

The law of conservation of energy is different in character from Newton's laws. Newton's laws provide precise descriptions of the motion of an object, provided that we can completely specify the situation (i.e. the mass and current motion of the object and the size, direction and behavior of all forces acting on it). Often we cannot do this—for example, it may be very difficult in practice to specify the action of air resistance on an object of complicated shape—and in this case we must make approximations which may seriously affect the accuracy of our calculations. In other cases the situation may be relatively simple to specify, but the resulting differential equations may have no easy solution. An example of this is gravitational interactions involving more than two bodies. We already know how to set up the equations (we just use $Gm_i m\hat{\mathbf{r}}_i/r_i^2$ for the force on mass m due to each of the other masses m_i and then add the resulting forces vectorially), but the system of equations we get can't be solved generally for even the simplest case of three objects. Astronomers studying, for instance, the question of whether Jupiter's moons may be captured asteroids must use large computers to tackle the problem by numerical methods.

Energy conservation is different. Except for simple problems involving motion in one dimension, energy conservation does not give us enough information to predict how a system will behave. For the problem of three particles interacting gravitationally with each other, for example, energy conservation will allow us to determine the speed of any one particle, if somebody tells us the positions of all the particles and the speeds of the other two particles. On the other hand, the partial information provided by energy conservation can sometimes be very useful in understanding important properties of a system. For example, if a chemical reaction occurs inside an insulated container, conservation of energy will allow us to calculate how much the temperature will rise, even though it would be unimaginable to calculate the trajectories of the colossal number of molecules involved in the reaction. Furthermore, physicists have developed so much confidence in the principle of energy conservation that they are likely to rely on it even in situations when the underlying physics is unknown. One example is the story of the neutrino, which began in 1914 when the British physicist, James Chadwick, discovered that energy appeared to be lost in a process known as beta decay. In this process an electron (which at that time was called a beta particle) is emitted by the decay of a radioactive nucleus, which then increases in atomic number by one unit. Chadwick found that the electron could be emitted with a range of energies, extending up to the value that would be expected by energy conservation. For a decade and half physicists searched for some way that the energy could have escaped, but none could be found. Finally, in 1931 the Austrian-born physicist Wolfgang Pauli proposed what he called a "desperate solution." Perhaps, he suggested, a new type of particle is given off in the reaction—a particle that interacts so weakly that it could not be detected. Two years later the particle was named a neutrino by the Italian physicist Enrico Fermi , but the particle remained undetected for another 23 years! Finally, in 1956 Pauli received a telegram from Clyde L. Cowan, Jr. and Frederick Reines, informing him that the elusive neutrino had finally been detected, at the Savannah River reactor in South Carolina.

The application of energy conservation is easy on the microscopic scale, because on this scale all forms of energy can be well identified. The only forms in which energy occurs are kinetic energy, potential energy associated with the forces, and mass (which is a form of energy described by probably the most famous equation in all of science, Einstein's $E = mc^2$). On the macroscopic scale we face difficulties associated with the fact that when we are dealing with real objects made up of atoms a great deal of the energy of a system can be 'hidden' in the kinetic and potential energy of these atoms, which we do not measure directly. This leads to the apparent 'loss' of energy associated with forces such as friction. Even though the energy is not really lost, just hidden, in practical calculations we have to recognize that the energy which makes the hood and tires of your car feel hot after a long drive has not gone into providing kinetic energy for the car, and is thus lost from the point of view of an automobile engine designer. We will look at situations like this in more detail in Chapter 6.

Energy conservation is not the only conservation law. In the next chapter we will meet another, the conservation of ***momentum***. Many problems can be solved completely or almost completely using only momentum and energy conservation, without ever referring directly to the forces acting on the system.

One of the subtle ideas to arise from the study of physics is a deep connection between conservation laws and symmetries. This connection is not obvious; it was not seen by Newton, and indeed the connection is not even logically necessary when the laws of motion are expressed in Newtonian language, as $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$. The science of mechanics, however, was rewritten in the eighteenth century. Many mathematicians and physicists contributed to this development, including Johann Bernoulli (1667-1748), Leonhard Euler (1707-1783), and Jean Le Rond d'Alembert (1717-1783). The work culminated in the publication in 1788 of *Mécanique analytique*, by Comte Joseph Louis Lagrange (1736-1813).* The key idea was to reformulate the equations of classical mechanics in a way that makes no direct reference to forces, but instead uses a "principle of least action." The action S is defined as the integral over time of the ***Lagrangian*** L:

$$S = \int_{t_1}^{t_2} L\,\mathrm{d}t \ ,$$

where L is the kinetic energy ***minus*** the potential energy. For a specified value of the initial position (at time t_1) and final position (at time t_2) of a particle (or particles), Lagrange invented a way to find the trajectory that minimizes S. This trajectory is called the path of least action, and Lagrange found that it was identical to the trajectory implied by Newton's $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ equations. The method is less general than Newton's, because it works only when the forces can be described by a potential energy function, so nonconservative forces are excluded from the beginning. The Lagrangian method turns out, however, to be a very powerful technique for treating difficult problems, and it also provides a formulation of classical mechanics that leads to a natural bridge to quantum theory.

Since the motion of any system is determined by the principle of least action, everything that one needs to know about a system is contained in its Lagrangian. This idea transcends classical

* Contrary to the style of this book, Lagrange boasted in his introduction that "No diagrams will be found in this book. The methods that I explain in it require neither constructions nor geometrical or mechanical arguments, but only the algebraic operations inherent to a regular and uniform process."

mechanics, so today particle theorists describe relativistic quantum field theories by writing an expression for the Lagrangian. It seems plausible that if we knew the right Lagrangian, we would know all of the laws of physics.

Finally—and this was our purpose in introducing Lagrangians here—the principle of least action leads to a fundamental connection between symmetries and conservation laws. If the laws of physics do not change with time—and as far as we can tell they do not—then the Lagrangian does not depend on time, and the principle of least action can be used to show that energy is necessarily conserved. If the Lagrangian does not depend on position in space—and as far as we can tell it does not, as long as all parts of an experiment are moved together, so that their relative positions do not change—then the principle of least action implies that momentum is conserved. Not surprisingly, the conservation of both energy and momentum still hold even when we consider extreme situations, such as speeds close to that of light, when Newton's laws are no longer valid. In the context of Lagrangians, any transformation of the positions and velocities of particles that leaves the Lagrangian unchanged is called a symmetry. For any symmetry, the principle of least action guarantees that there will be a corresponding conservation law.

Another conservation law that we will meet later in the book is the analog of momentum conservation for ***rotational*** motion, as opposed to motion in a straight line. This relates to the fact that the Lagrangian is unchanged by a rotation of the entire system, since the results of an experiment do not depend on the orientation of the laboratory in space. When you study electromagnetism you will discover that electric charge is conserved, and if you take a course in nuclear or particle physics you will be introduced to exotic conserved quantities like baryon number and lepton flavor. Conservation laws pervade physics, each of them pointing to some underlying symmetry or invariance of the structure of the universe. The existence, or sometimes the failure, of established conservation laws strongly constrains the form of theories put forward to describe natural phenomena. For example, oddly enough, the results of experiments involving neutrinos ***may*** depend on whether the experiment is described using a left-handed or a right-handed coordinate system. This is reflected in the ***non***-conservation of a quantity called parity. The non-conservation of parity is an experimental fact of nature, and has been incorporated into the structure of modern theories of particle physics.

FORCES AND FIELDS

We have defined the gravitational force between two bodies in terms of the mass of each body and their separation. We do not have a definition for the gravitational force exerted by a single object—we insist on another body for the force to act on. However, our intuitive feeling is that the gravity of an object is somehow 'there' even if the object exists in isolation: a single proton floating in intergalactic space would still have the potential to exert gravitational and electrostatic forces even though in fact there may be no other particle close enough for us to measure such a force. We might picture the space surrounding the proton as being affected by the presence of the proton in such a way that any other stray proton or electron wandering into the vicinity would automatically feel the appropriate gravitational and electrostatic forces.

This is a case where the mathematical development of physics coincides with our common sense. The physical concept which corresponds to our 'effect of the particle on its surrounding space' is called a ***field***. We say that the proton creates a gravitational field described at any point by $-GM\hat{\mathbf{r}}/r^2$, where $\hat{\mathbf{r}}$ is the position vector of the point relative to the proton. The force on a particle of mass m due to the gravity of the proton is then given by the value of the field times the mass of the particle. We can treat the electrostatic force in the same way, defining an electrostatic field $\vec{\mathbf{E}}$ with which the particle interacts according to its charge: $\vec{\mathbf{F}} = q\vec{\mathbf{E}}$, where q is the charge of the particle.

Just as the gravitational force can also be specified in terms of the gravitational potential energy, so the gravitational field can be specified in terms of a gravitational ***potential***. The potential is the integral of the field with respect to displacement, just as the potential energy is the integral of the force. Field and potential are different ways of describing the same thing: we use whichever is more convenient for a given situation. We meet the electric potential in everyday life whenever we check the voltage supplied by a battery: this is simply the difference in electrical potential between the two terminals of the battery.

The field picture is enormously useful because it lets us separate the problem into two parts: the creation of the field by object A, and the action of the field on object B. For example, in dealing with the motion of an electron in a TV set we can first calculate the electric (and magnetic) fields set up by the components of the set, and then worry about the effect of those fields on the motion of the electron. If we later modify the set so that the electrons enter our area of interest with a different velocity, we don't have to redo the first part of the calculation. However, fields are more than a calculational convenience. They carry energy: we have described potential energy as 'stored energy'—now we can interpret this as energy stored in the field. In the study of electromagnetism we find that fields can also carry momentum, leading to cases where momentum conservation seems not to hold if we consider the interaction of particle A directly with particle B, but does hold if we consider the separate interactions of A with the local electromagnetic field and B with ***its*** local electromagnetic field.

Finally, the field picture can be viewed as the classical, macroscopic description of the modern microscopic view of the fundamental forces. For all the forces that we think we understand at a quantum mechanical level, the action of the force is transmitted by a physical particle emitted by object A and absorbed by object B (or vice versa), with momentum and energy conserved in both emission and absorption. The particles, however, are really quantum mechanical objects called "virtual particles," which do not necessarily have the same relation between energy and momentum or between velocity and momentum that classical particles have. (The only fundamental force we don't understand at this level is gravity, since general relativity is not formulated in terms of quantum mechanics.) In the absence of any object B, object A still emits force particles, so the field is still there, but it subsequently re-absorbs them itself, thus maintaining constant energy and momentum. This reinforces the reality of fields as physical entities rather than mathematical conveniences.

The field concept is a subtle one, and in this book we will never meet a situation where it is ***essential*** to think in terms of fields. Fields are, however, essential to understanding many phenomena associated with electromagnetism, including such important effects as the transmission of energy from the Sun to the Earth in the form of light, which is a traveling wave of electric and magnetic fields. It was only when the field concept was invented in the nineteenth century that the modern theory of electromagnetism could be developed.

ENERGY CONSERVATION AND THE ORIGIN OF THE UNIVERSE

Energy conservation, and particularly the concept of gravitational potential energy, has interesting consequences when applied to the largest known system, the entire universe. We know that the visible part of the universe contains about 100 billion galaxies, each consisting of typically 100 billion stars with masses roughly similar to the mass of the Sun. According to the theory of relativity, any object of mass M has an equivalent "rest" energy Mc^2, where c is the speed of light. Since c is a large velocity, these rest energies tend to be huge. Further, all of these galaxies are moving apart as the universe expands, and thus they all have associated kinetic energy. We may be tempted to conclude that the energy of the whole universe is incalculably huge, and that its theoretical origin in a big bang represents a gigantic violation of our wonderful new law of energy conservation.

This would not necessarily be a disaster. The conditions prevailing at the origin of the universe are so extraordinary in both physical and mathematical terms that it is conceivable that our presently known laws of physics do not apply. However, in fact we are not forced to resort to this rather drastic remedy. It is perfectly possible to argue that ***the total energy of the entire universe is zero!***

The justification for this apparently nonsensical suggestion is that we have not yet included the gravitational potential energy of the universe in our calculations. Clearly, the mass in the universe must produce a gravitational potential, and energy is stored in this potential. The point to recognize is that the stored energy has a ***negative*** sign (we saw the effects of this when we looked at planetary orbits). The total energy of the universe, taking into account the positive contributions from the matter and its kinetic energy and the negative contribution of the gravitational potential energy, could be positive, negative or (if they just cancel) zero.

While many features of cosmology can be described well by Newtonian physics, the best understanding that we currently have is based on Einstein's theory of general relativity. To discuss a concept as global as the total energy of the universe, one needs to use the general relativistic description. General relativity is really a theory of gravity—one which was designed to be consistent with Einstein's theory of special relativity. General relativity describes gravity not as a force, but instead as a distortion of space and time. As discussed in the Supplementary Notes at the end of Chapter 1, the assumption that the universe is homogeneous and isotropic (i.e., the same in all places and in all directions) leads to three possible geometries for the universe: open, closed, and flat. For a closed universe the total energy is always exactly zero, with the positive energy of matter exactly canceled by a negative contribution from the gravitational potential energy. For an open universe, the total energy is infinite. The total energy of a flat universe is less well-defined, since it sits right on the borderline between an open universe of infinite energy and a closed universe of zero energy. However, a flat universe is always observationally indistinguishable from a very large closed universe, and therefore a universe that appears to be flat is always consistent with the possibility of zero total energy. Consequently, the evidence for a flat universe discussed in Chapter 1 can be interpreted as evidence that the total energy of the universe could be zero. Thus, the creation of the universe—however it happened—need not have constituted a violation of the principle of energy conservation.