ESSENTIALS OF INTRODUCTORY CLASSICAL MECHANICS

Sixth Edition

CHAPTER 5

SYSTEMS OF PARTICLES

OVERVIEW

So far we have considered the motion of a single particle acted on by an external force. In many situations this is an oversimplified view, since we are actually dealing with several objects which exert forces ***on each other***, such as one pool ball hitting another. Each pool ball, in turn, is composed of perhaps 10^{24} atoms, all exerting forces on each other. In this chapter we will see how to extend our analyses to such cases, starting with the simplest system of two bodies. We will use the word ***system*** to refer to ***any*** specified set of objects, such as a set of two pool balls, the set of all the atoms in one pool ball, or the set of all atoms in two pool balls.

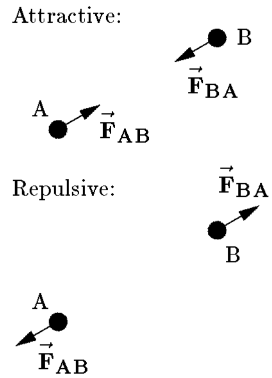
When you have completed this chapter you should:

- ✔ understand what is meant by the concept of momentum;
- ✔ know that momentum, like energy, is a conserved quantity, and be able to relate this to Newton's third law;
- ✔ understand how to relate forces to the rate of transfer of momentum;
- ✔ be able to use the conservation of momentum and energy to solve problems involving the interaction of two bodies;
- ✔ understand the concept of center of mass; be able to relate the total momentum of a system to the velocity of the center of mass, and the total applied force to the acceleration of the center of mass;
- ✔ distinguish between internal and external forces and their effects on the motion of the center of mass;
- ✔ understand the concept of impulse and the relation between impulse and momentum.

ESSENTIALS

The forms of the force laws of gravitation and electrostatics are symmetrical—the force exerted on body A by body B is the same magnitude as the force exerted on B by A, but opposite in direction. An attractive force (e.g., gravity or the electrostatic force between opposite charges) causes A to accelerate towards B, and B to accelerate towards A. A repulsive force (e.g., the electrostatic force between charges of the same sign) causes A to accelerate away from B, and B away from A. Thus we can write

Supplementary Notes. Problems 5A.1 and 5A.3



$$\boxed{\vec{\mathbf{F}}_{\mathrm{AB}} = -\vec{\mathbf{F}}_{\mathrm{BA}} \ ,}$$

where $\vec{\mathbf{F}}_{\mathrm{AB}}$ is the force exerted on A by B and vice versa. ***Newton's third law*** states that this is ***always*** true, for any force (though one must be careful in formulating the law in some cases, particularly those involving moving electric charges; you will find out about this when you study electromagnetism). Thus the third law states that ***the force exerted on body A by body B is equal in magnitude and opposite in direction to that exerted on B by A.***

(This is often expressed as "action equals reaction", but in fact the law is completely symmetrical between the two forces. There is no motivation to call one the 'action' and the other the 'reaction'.)

Consider a system of particles that has no forces acting on it from the outside, and let $\vec{\mathbf{f}}_i$ denote the total force exerted on the i^{th} particle by all the other particles in the system. It follows from the third law that

$$\sum_i \vec{\mathbf{f}}_i = 0 \ ,$$

since all the internal forces between pairs of particles cancel. Using the ***second*** law we can rewrite this as

$$\sum_i m_i \frac{\mathrm{d}\vec{\mathbf{v}}_i}{\mathrm{d}t} = 0 \ .$$

Since individual masses do not change with time, we can further manipulate the relation to give

$$\sum_i \frac{\mathrm{d}(m_i \vec{\mathbf{v}}_i)}{\mathrm{d}t} = 0 \ .$$

By the mathematical identity $\dfrac{\mathrm{d}}{\mathrm{d}t}(a+b) = \dfrac{\mathrm{d}a}{\mathrm{d}t} + \dfrac{\mathrm{d}b}{\mathrm{d}t}$, this is the same as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i m_i \vec{\mathbf{v}}_i\right) = 0 \ .$$

The equation above has the form of a conservation law, since it says that the quantity in parentheses is independent of time. To express this conservation law in a simple way, we define the ***momentum*** $\vec{\mathbf{p}}$ of a particle by

$$\boxed{\vec{\mathbf{p}} \equiv m\vec{\mathbf{v}} \ ,}$$

where m is its mass and $\vec{\mathbf{v}}$ is its velocity. The SI unit of momentum is therefore kg·m/s. The total momentum of a system of particles is

$$\boxed{\vec{\mathbf{P}}_{\mathrm{tot}} \equiv \sum_i \vec{\mathbf{p}}_i = \sum_i m_i \vec{\mathbf{v}}_i \ .}$$

Problem 5A.2

Using these definitions, the equation derived above by applying Newton's third law to a system with no external forces can be rewritten as

$$\boxed{\frac{\mathrm{d}\vec{\mathbf{P}}_{\mathrm{tot}}}{\mathrm{d}t} = 0 \ .}$$

Problems 5B

Thus, ***momentum is conserved***, meaning that the total momentum $\vec{\mathbf{P}}_{\mathrm{tot}}$ of a system of particles does not change if there are no external forces acting on the system. (Note that the total momentum ***must*** be calculated by adding the individual momenta as ***vectors***. One does not add the magnitudes.) The momenta of individual particles in the system can of course change (for example, two charged particles may accelerate towards each other from rest under their mutual electrostatic attraction: their individual momenta increase in magnitude, but are oppositely directed, so the total momentum of the system is unchanged).

Momentum conservation, like energy conservation, is a fundamental law of physics which holds in all known circumstances (though when speeds approach that of light, we must be more careful about how we actually calculate the momentum). In fact, modern physicists view the third law as a consequence of momentum conservation, rather than the other way round.

The net force on a particle is, by Newton's second law, the rate of change of its momentum,

$$\boxed{\vec{\mathbf{F}} = m\frac{\mathrm{d}\vec{\mathbf{v}}}{\mathrm{d}t} = \frac{\mathrm{d}\vec{\mathbf{p}}}{\mathrm{d}t} \ .}$$

Problem 5A.4

What happens if there are ***external*** forces acting on a system of particles? Then for each mass m_i,

$$\vec{\mathbf{F}}_i^{\text{ext}} + \vec{\mathbf{f}}_i = m_i \vec{\mathbf{a}}_i \ ,$$

where $\vec{\mathbf{F}}_i^{\text{ext}}$ is the ***external*** force on this mass and $\vec{\mathbf{f}}_i$ is the ***internal*** force from all the other particles in the system. We can add all the equations for individual particles to get, for the whole system,

$$\sum_i \vec{\mathbf{F}}_i^{\text{ext}} + \sum_i \vec{\mathbf{f}}_i = \sum_i m_i \vec{\mathbf{a}}_i \ .$$

The previous argument for the vanishing of the sum of the internal forces still applies, so the second sum on the left-hand side is zero. Thus, denoting the total external force by $\vec{\mathbf{F}}_{\text{tot}}^{\text{ext}}$,

$$\boxed{\vec{\mathbf{F}}_{\text{tot}}^{\text{ext}} \equiv \sum_i \vec{\mathbf{F}}_i^{\text{ext}} = \sum_i m_i \vec{\mathbf{a}}_i = \frac{\mathrm{d}\vec{\mathbf{P}}_{\text{tot}}}{\mathrm{d}t} \ .}$$

The fact that the total force on a system is equal to the rate of change of its momentum allows us, in many cases, to treat large objects (e.g. planets) as if they were point particles. We must still explore whether we can relate the total momentum of a system to some definition of its average velocity. We could proceed using vector notation, but instead we will manipulate individual components of vectors.

Starting with the definition of the total momentum, write the x-component as

$$\mathrm{P}_{\text{tot},x} = \sum_i m_i v_{i,x} = \sum_i m_i \frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \sum_i m_i x_i \ ,$$

where x_i is the x-coordinate of the i^{th} particle. Denoting the total mass of the system by $M_{\text{tot}} \equiv \sum_i m_i$, we can divide and multiply by M_{tot} to obtain

$$\mathrm{P}_{\text{tot},x} = M_{\text{tot}} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{M_{\text{tot}}} \sum_i m_i x_i \right) \ .$$

Defining the ***center of mass*** of the system by

$$\boxed{x_{\text{cm}} \equiv \frac{1}{M_{\text{tot}}} \sum_i m_i x_i \qquad \text{(and likewise for y and z),}}$$

Problems 5C

the previous equation can be written as

$$\boxed{\mathrm{P}_{\mathrm{tot},x} = M_{\mathrm{tot}} \frac{\mathrm{d}x_{\mathrm{cm}}}{\mathrm{d}t} \qquad \text{(and likewise for y and z).}}$$

Thus, the momentum of the system can be calculated as if it were a point particle carrying the total mass of the system and moving at the velocity of the center of mass. Note that if the particles have equal mass, x_{cm} is just the average of the x-coordinates. Returning to vector notation,

$$\boxed{\vec{\mathbf{P}}_{\mathrm{tot}} = M_{\mathrm{tot}} \vec{\mathbf{v}}_{\mathrm{cm}} \ ,}$$

where

$$\boxed{\vec{\mathbf{v}}_{\mathrm{cm}} \equiv \frac{\mathrm{d}\vec{\mathbf{r}}_{\mathrm{cm}}}{\mathrm{d}t}} \quad \text{and} \quad \boxed{\vec{\mathbf{r}}_{\mathrm{cm}} \equiv \frac{1}{M_{\mathrm{tot}}} \sum_i m_i \vec{\mathbf{r}}_i \ ,}$$

and $\vec{\mathbf{r}}_i$ is the position vector of the i^{th} particle.

Since the rate of change of the momentum is the total externally applied force, we have immediately

$$\boxed{\vec{\mathbf{F}}^{\mathrm{ext}}_{\mathrm{tot}} = \frac{\mathrm{d}\vec{\mathbf{P}}_{\mathrm{tot}}}{\mathrm{d}t} = M_{\mathrm{tot}} \frac{\mathrm{d}\vec{\mathbf{v}}_{\mathrm{cm}}}{\mathrm{d}t} = M_{\mathrm{tot}} \frac{\mathrm{d}^2\vec{\mathbf{r}}_{\mathrm{cm}}}{\mathrm{d}t^2} = M_{\mathrm{tot}} \vec{\mathbf{a}}_{\mathrm{cm}} \ .}$$

The center of mass moves as if it were a point particle carrying the total mass of the system and acted upon by the sum of the external forces.

This is the reason for the name ‘center of mass’. It also explains why in previous chapters we have been able to treat large objects such as the moon as if they were single point particles. As long as we know the sum of the external forces acting on each atom of a large object, we can calculate the acceleration of its center of mass. Note that even if the system consists of many disconnected chunks of matter, the center of mass moves as if it were a point particle accelerating under the influence of the total force.

In many applications one needs to calculate the center of mass of a system that decomposes simply into several parts, such as the Earth-moon system. In principle the center of mass is defined by applying the general formula above, summing over all the atoms in both celestial objects. It is shown in the solution to Problem 5C.3, however, that the problem simplifies: The center of mass can be calculated as if the system had only two particles, the Earth and the moon, each treated as a point particle located at its own center of mass.

Problems 5C.3 and 5C.4

The relation between the ***total kinetic energy*** of a system and its center of mass motion is a bit more complicated. As shown in the solution to Problem 5C.5, it is

$$\boxed{K_{\text{tot}} = \underbrace{\frac{1}{2} M_{\text{tot}} v_{\text{cm}}^2}_{\text{Center of Mass Energy}} + \underbrace{\sum_i \frac{1}{2} m_i \left(\vec{\mathbf{v}}_i - \vec{\mathbf{v}}_{\text{cm}}\right)^2}_{\text{Internal Energy}} \;,}$$

Problem 5C.5

(not, as we might have guessed, $\frac{1}{2} M_{\text{tot}} v_{\text{cm}}^2$). Because kinetic energy involves a sum of squares, the internal velocities of the pieces do not cancel, so we get the second term (which is just the sum of the kinetic energies of the individual particles as seen from the center of mass). This ***internal energy*** of the system is the reason that energy conservation, in terms of the sum of kinetic and potential energy, often seems to fail in real-life situations—a solid object is a system of atoms, not a single particle, and can have internal energy. If we could measure the motions of the individual atoms, energy conservation would surely hold; we will see in Chapter 11 that we can demonstrate this indirectly by looking at the ***temperature*** of an object as a measure of its internal energy.

Although internal forces cannot change the total momentum of a system of particles, they may change its total kinetic energy: as we saw in the solution to Problem 4D.1(c), an internal force may do net work on some part of a system of particles. This means that it is not always straightforward to apply energy conservation arguments when considering the overall motion of the system. However, there are some special cases where it is clear that the work done by internal forces must cancel. One example is the normal force between two bodies in contact. By Newton's third law, $\vec{\mathbf{F}}_{21} = -\vec{\mathbf{F}}_{12}$. The displacement $\overrightarrow{\Delta \mathbf{r}}$ can be decomposed into a component ***parallel*** to the surfaces in contact, which does not contribute to the work done since it is, by definition, perpendicular to the normal forces (so

$\vec{\mathbf{F}} \cdot \overrightarrow{\Delta\mathbf{r}}_{||} = 0$), and a component ***perpendicular*** to the surfaces. If the bodies remain in contact, this component must be the same for both objects, and so $\Delta W_{21} = -\Delta W_{12}$. (In the next chapter we will see that if there is friction between the two surfaces, then the work done by friction is related to the parallel component of $\overrightarrow{\Delta\mathbf{r}}$. This will not, however, alter the conclusion that the normal force does not do any net work.) Much the same logic applies to two objects connected by a massless, inextensible rope: if one end of the rope is displaced by an amount Δs in the same sense as the tension (i.e. towards the middle of the rope), the other end must move the same distance against the direction of the tension (away from the middle of the rope), and so the net work done is $T\Delta s + (-T\Delta s) = 0$.

The equations we have derived are valid for a system of any number of particles moving in any arbitrary way with respect to each other. However, ***solving*** the equations for complicated systems is usually impractical (the rest of this book will deal with ways of simplifying such problems in various special but useful cases). In this chapter we restrict ourselves to systems containing only two bodies, where the equations can be solved more easily.

Collisions of two objects are an obvious example of two-body problems. Is kinetic energy conserved in a collision? The answer in the case of point particles is "yes": we measure the part of the internal kinetic energy that corresponds to the motion of the two point particles with respect to the center of mass, and the point particles themselves have no internal structure, so this leaves nothing unaccounted for. Such collisions are called ***elastic***. Elastic collisions are a good approximation to many real situations (e.g. bouncing a very springy ball, or the motion of the molecules of a gas).

Problems 5B.1, 5B.2, 5B.3, 5B.5, and 5B.6

If the colliding objects are not point particles, but have some internal structure which can be deformed or rearranged, measuring the speeds of the two colliding objects does not account for all the internal kinetic energy. In such cases, the measured kinetic energy after the collision is ***not*** the same as before the collision. Collisions of this type are called ***inelastic***. The kinetic energy of the bodies can decrease in an inelastic collision, for example if the two bodies stick together; it can also increase, as it might if a compressed spring were released in the collision. Inelastic collisions are not really two-body collisions, since we cannot treat the colliding objects as point particles (their internal energies have become relevant to the problem).

Problems 5B.4, 5B.5, and 5B.7

Elastic collisions and other two-body interactions can often be most easily solved by working in the ***center-of-mass frame***, i.e. the frame of reference which is at rest relative to the center of mass. In this frame there is zero net momentum, so at any given time the two bodies must have equal and opposite momenta.

Problems 5B.3 and 5C.6

In collisions and similar situations where a complicated and unknown force acts for a very short time, the total transfer of momentum is more relevant to practical aspects of the problem than the details of the force. The transfer of momentum by a force $\vec{\mathbf{F}}$ acting over a time interval t_1 to t_2 is called the ***impulse*** $\vec{\mathbf{J}}$:

Problems 5D

$$\boxed{\vec{\mathbf{J}} = \int_{t_1}^{t_2} \vec{\mathbf{F}}\, \mathrm{d}t = \int_{t_1}^{t_2} \frac{\mathrm{d}\vec{\mathbf{p}}}{\mathrm{d}t}\, \mathrm{d}t = \vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1 \ .}$$

This is called the impulse-momentum theorem, and is clearly analogous to the work-energy theorem. The integral of a net force over a ***displacement*** gives the ***kinetic energy*** transferred: the integral of a net force over ***time*** gives the ***momentum*** transferred.

Impulse, like force and momentum, is a vector quantity.

SUMMARY

* The force exerted on a body A by another body B is equal in magnitude and opposite in direction to that exerted on B by A (Newton's third law).
* The ***momentum*** of a particle is defined as its mass times its velocity. Like velocity, it is a vector. The net force on a particle is equal to the rate of change of its momentum, and the total force on a system of particles is equal to the rate of change of its total momentum. Momentum, like energy, ***is conserved in any closed system***.
* For any system of particles we can calculate a ***center of mass***. If external forces are applied to a system of particles, the center of mass moves as if it were a point particle carrying the total mass of the system and acted upon by the sum of the external forces. The net momentum of the system is equal to the momentum that this point particle would have.
* The kinetic energy of a system of particles is equal to the kinetic energy of a single equivalent particle having the same total mass and located at the center of mass ***plus*** the kinetic energies of the particles making up the system ***as measured relative to the center of mass***. The latter part is called the ***internal energy*** of the system.
* Collisions between pairs of particles may be ***elastic*** (the net kinetic energy is conserved) or ***inelastic*** (there is some conversion between kinetic energy and internal energy). Momentum is conserved in both cases.
* The change in an object's momentum over a given time interval is equal to the integral of the net force acting over that time interval, and is called the ***impulse***.
* Physical concepts introduced in this chapter: momentum, center of mass, impulse.
* Mathematical concepts introduced in this chapter: none (but you may need to review some calculus and the meaning of the summation symbol $\sum$).
* Equations introduced in this chapter:

$$\vec{\mathbf{F}}_{\mathrm{AB}} = -\vec{\mathbf{F}}_{\mathrm{BA}}$$ (Newton's third law);

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}}$$ (momentum);

$$\frac{\mathrm{d}\vec{\mathbf{P}}_{\mathrm{tot}}}{\mathrm{d}t} = 0$$ (conservation of momentum in absence of external force);

$$\vec{\mathbf{F}} = \frac{\mathrm{d}\vec{\mathbf{p}}}{\mathrm{d}t}$$ (Newton's second law in terms of momentum);

$$\vec{\mathbf{r}}_{\mathrm{cm}} \equiv \frac{1}{M_{\mathrm{tot}}}\sum_i m_i\vec{\mathbf{r}}_i$$ (position of center of mass);

$$\vec{\mathbf{v}}_{\mathrm{cm}} \equiv \frac{\mathrm{d}\vec{\mathbf{r}}_{\mathrm{cm}}}{\mathrm{d}t} = \frac{1}{M_{\mathrm{tot}}}\sum_i m_i\vec{\mathbf{v}}_i$$ (velocity of center of mass);

$$\vec{\mathbf{F}}^{\mathrm{ext}}_{\mathrm{tot}} = M_{\mathrm{tot}}\vec{\mathbf{a}}_{\mathrm{cm}} = \frac{\mathrm{d}\vec{\mathbf{P}}_{\mathrm{tot}}}{\mathrm{d}t}$$ (acceleration of a system of particles);

$$\vec{\mathbf{P}}_{\mathrm{tot}} = \sum_i m_i\vec{\mathbf{v}}_i = M_{\mathrm{tot}}\vec{\mathbf{v}}_{\mathrm{cm}}$$ (momentum of a system of particles);

$$K_{\text{tot}} = \frac{1}{2} M_{\text{tot}} v_{\text{cm}}^2 + \sum_i \frac{1}{2} m_i \left(\vec{\mathbf{v}}_i - \vec{\mathbf{v}}_{\text{cm}}\right)^2 \qquad \text{(K.E. of a system of particles);}$$

$$\vec{\mathbf{J}} = \int_{t_1}^{t_2} \vec{\mathbf{F}}\, \mathrm{d}t = \int_{t_1}^{t_2} \frac{\mathrm{d}\vec{\mathbf{p}}}{\mathrm{d}t}\, \mathrm{d}t = \vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1 \qquad \text{(impulse-momentum theorem).}$$

PROBLEMS AND QUESTIONS

By the end of this chapter you should be able to answer or solve the types of questions or problems stated below.

Note: throughout the book, in multiple-choice problems, the answers have been rounded off to 2 significant figures, unless otherwise stated.

At the end of the chapter there are answers to all the problems. In addition, for problems with an (H) or (S) after the number, there are respectively hints on how to solve the problems or completely worked-out solutions.

5A FUNDAMENTAL CONCEPTS (MOMENTUM AND NEWTON'S THIRD LAW)

5A.1 You are sitting in a chair. The forces acting include several forming equal and opposite pairs, e.g. (i) your weight and the normal force exerted on you by the chair; (ii) the weight of the chair and the gravitational force exerted by the chair on the Earth (neglecting the effect of the Earth's rotation). Which of these pairs are equal and opposite owing to Newton's third law?

(a) both (i) and (ii); (b) (i) only; (c) (ii) only; (d) neither (i) nor (ii).

5A.2 An 80 kg ice-hockey player traveling at 10 m/s collides head-on with an opposing player traveling at 7 m/s in the opposite direction. If the second player has mass 75 kg, what is the magnitude of the net momentum of the two players, to two significant figures?

(a) 280 kg·m/s; (b) 1300 kg·m/s; (c) 460 kg·m/s; (d) none of these.

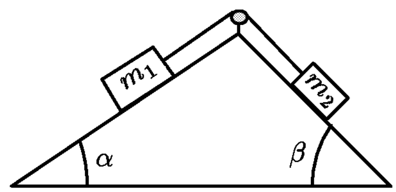
5A.3 (S) A book with mass m is lying on a table of mass M. What are the forces acting (a) on the book and (b) on the table? Which pairs of forces are equal and opposite ***by Newton's third law***?

5A.4 (H) A fire hydrant delivers water at a volume flow rate (i.e. volume/time) L. The water travels vertically upwards through the hydrant at speed v and then does a 90° turn to emerge horizontally at the same speed v. Assuming the pipe and nozzle have uniform cross-sections throughout, obtain an expression for the force exerted by the water on the corner of the hydrant. If the rate of delivery is $L = 800$ liters per minute at $v = 25$ m/s, what is the magnitude of the force that the structure of the hydrant has to withstand? What is the direction of that force? (One liter of water has a mass of one kilogram.)

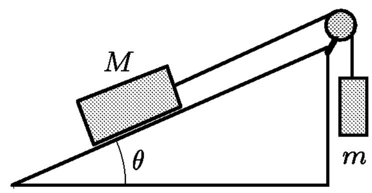
5A.5 (S) A uniform rope of mass m and length ℓ is attached to a hook in the ceiling, and hanging from it is a mass M. Assuming that m is *not* negligible compared to M, what is the tension in the rope (i) at the hook; (ii) at the mass M; (iii) at some arbitrary point a distance y below the hook? If the rope is now removed from the hook and used to tow the mass M horizontally along a frictionless surface, what force must be applied to the end of the rope to give the mass M an acceleration a? Assume that the rope does not stretch.

5A.6 In 50 words or less, explain the difference between ***internal*** and ***external*** forces.

5A.7 (S) Two blocks of masses m_1 and m_2 are connected by a massless inextensible rope as shown. At the apex of the frictionless triangular support, the rope passes over a frictionless pulley. Find the acceleration of the blocks and the tension in the rope.



5A.8 Two blocks are connected by a massless inextensible rope over a frictionless pulley as shown. The block of mass m hangs freely, and there is no friction between the block of mass M and the slope. The experimenter finds that when she places the blocks carefully in this position they remain stationary. Find (a) the tension in the rope and (b) an expression for the mass m in terms of θ and the mass M. What will happen if the experimenter now gives the block of mass M a gentle push down the slope?



5B CONSERVATION OF MOMENTUM

5B.1 (i) A pool ball traveling at 2 m/s hits another (stationary) ball of the same mass dead center, so that the extended trajectory of the first ball passes through the center of the second. What is the velocity of the first ball after the elastic collision?

(a) 1 m/s in the original direction;

(b) 2 m/s at some angle to the original direction;

(c) less than 2 m/s at some angle to the original direction;

(d) zero.

(ii) What is the velocity of the second ball?

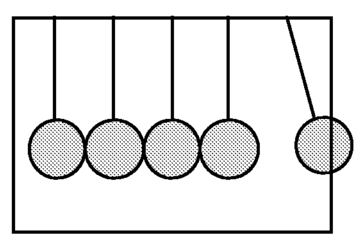
(a) 1 m/s in the same direction as the first;

(b) 2 m/s in the original direction of the first ball;

(c) less than 2 m/s at some angle to the direction of the first ball;

(d) 2 m/s at some angle to the direction of the first ball.

5B.2 A Newton's cradle consists of five steel ball-bearings of equal mass suspended in a frame. You take the end ball, displace it slightly, and let it go, so that it swings into the other four balls with speed v. What happens, and why? Assume all collisions are elastic.



5B.3 (H) Two sliders on a linear air track are fitted with spring-loaded fenders so that their collisions will be perfectly elastic. If one has mass m and speed v and the other has mass M and is stationary, what will be the velocity of each one after the collision? Deduce from this what will be the result if the first slider collides elastically with an immovable wall.

Discuss what will happen if we extend this situation to two dimensions, so that an air puck on a frictionless table collides elastically with a wall. Assume the puck's initial velocity makes an angle θ with the wall.

5B.4 (H) You are roller-skating peacefully down a street at a constant speed of 3 m/s when someone suddenly throws a football at you from directly ahead. If your mass is 65 kg and the football's is 0.40 kg, what is your speed afterwards if (a) you catch the ball, which was thrown at you with a horizontal velocity of 15 m/s; (b) you miss it and it bounces off you with a velocity of 10 m/s (relative to the street) in the opposite direction? In each case, what is the total kinetic energy of the system comprising you and the ball before and after your interaction?

5B.5 (S) You have two sliders on a frictionless linear air track, each of mass m. One is stationary, and you slide the other one into it with a velocity v_0. What happens to the momentum and kinetic energy of each slider if (a) you have attached spring-loaded fenders so that the collision is perfectly elastic; (b) you have stuck on blobs of putty so that the two sliders stick together and move off together after the collision?

5B.6 (S) In the sport of curling, teams take turns to slide polished granite stones across an ice rink, aiming at a designated target zone. A common tactic is to dislodge well-placed opposition stones by hitting them with your own stone. Assuming that all the stones are of a standard mass, that friction can be neglected, and that the collision is elastic, is it possible to determine what will happen to a stationary stone B if it is hit by another stone A moving with velocity v? If it is not possible to determine exactly what will happen, what can be determined, and what additional information is required?

5B.7 (H) You and a friend sit motionless on sleds on frictionless ice. You slide a 10 kg block across the ice to her at 2 m/s relative to your sled (i.e. after the ice block is released, the relative velocity of the block with respect to your sled is 2 m/s), and she catches it and slides it back to you at the same speed (relative to her own sled). If you and your sled, without the 10 kg mass, together have a mass of 90 kg, and she and her sled have a mass of 70 kg, what is your speed, and that of your friend, after you catch the returned block?

5C THE CENTER OF MASS

5C.1 A binary star system consists of two stars separated by 10^{10} km. Star 1 is three times as massive as star 2. How far from star 1 on the line joining the two stars is the center of mass of the system?

(a) 2.5×10^9 km; (b) 3.3×10^9 km; (c) 5.0×10^9 km; (d) 7.5×10^9 km.

5C.2 (S) A girl is teaching her younger brother to skate by towing him around on a rope. They finish their practice session by hauling in on the rope from each end until they meet. If her mass is 40 kg, his is 30 kg, and the rope is 5 m long, how far from her original position will they end up? Assume that they were stationary to begin with, that the rope has negligible mass, and that there is no friction.

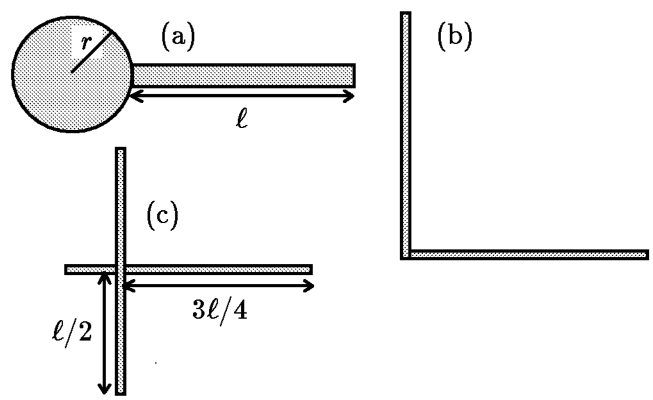
5C.3 (S) Find the position of the center of mass of:

(a) the Earth-Moon system (masses of 6.0×10^{24} and 7.4×10^{22} kg) separated by 384 thousand kilometers;

(b) a small spherical mass m attached to the end of a long thin rod of length ℓ and mass m;

(c) three thin rods, each of mass m and length ℓ, arranged to form three sides of a square.

5C.4 (H) Find the position of the center of mass of (a) a sphere of mass m and radius r attached to a rod of mass m and length ℓ; (b) two rods of mass m and length ℓ joined at right angles; (c) two rods of mass m and length ℓ crossed as shown in the diagram.



5C.5 (S) Use the definition of the center of mass to prove the expression given in the *Essentials* for the kinetic energy of a system of particles.

5C.6 (H) Two pucks of mass m_1 and m_2, moving on a level frictionless surface, undergo an elastic collision. Prior to the collision, their speeds were v_1 and v_2 respectively, as measured in their center of mass frame (i.e. the frame in which their center of mass is at rest). What are their speeds after the collision? Can you make any statement about their directions of motion after the collision?

5D IMPULSE

5D.1 A baseball approaches the batsman at 30 m/s. After he hits it, it is traveling in the opposite direction with a speed of 40 m/s. If the mass of a baseball is 0.145 kg, what was the magnitude of the impulse he applied to the ball, to three significant figures?

(a) 1.45 kg m/s; (b) 5.80 kg m/s; (c) 10.2 kg m/s; (d) none of these.

5D.2 (S) During a tennis rally, the ball approaches a player at a speed of 30 m/s. He returns the shot so that the ball has a speed of 35 m/s at an angle of 160° to the original direction. What impulse did he apply to the ball? If ball and racquet were in contact for 0.01 s, what average force (averaged over time) did he exert? A tennis ball has a mass of 60 g.

5D.3 (H) In a bat-and-ball game, the ball hits the bat at 35 m/s and is projected back in the opposite direction at 50 m/s. What is the impulse applied to the ball and the average force exerted (averaged over time) if the game is (a) tennis, involving a 60 g ball in contact with the racquet for 0.01 s; (b) baseball, with a 0.145 kg ball in contact with the bat for 0.002 s?

COMPLETE SOLUTIONS TO PROBLEMS WITH AN (S)

5A.3 *A book with mass m is lying on a table of mass M. What are the forces acting (a) on the book and (b) on the table? Which pairs of forces are equal and opposite by **Newton's third law**?*

Conceptualize

If we neglect the Earth's rotation, the book and the table are both stationary, so no net force can be acting on either of them. There is obviously a gravitational force $m\vec{\mathbf{g}}$ acting on the book, and $M\vec{\mathbf{g}}$ on the table. These must be balanced by the normal forces $\vec{\mathbf{n}}$ and $\vec{\mathbf{N}}$ respectively, where $\vec{\mathbf{n}}$ is the force exerted by the table on the book, and $\vec{\mathbf{N}}$ is the force exerted by the floor on the table.

In case (a), the book, $m\vec{\mathbf{g}}$ and $\vec{\mathbf{n}}$ are the only forces acting, so for zero net force they must be equal and opposite, $\vec{\mathbf{n}} = -m\vec{\mathbf{g}}$. But are they equal and opposite by ***Newton's third law***? Clearly not: the third law states that "the force exerted by body A on body B is equal in magnitude and opposite in direction to that exerted by B on A". Two forces acting ***on the same object*** cannot, therefore, be a third law pair. The gravitational force $m\vec{\mathbf{g}}$ is exerted on the book by the Earth, so its third law partner is the force exerted on the Earth by the book. Naturally this force does not cause a measurable acceleration when applied to a mass of 6×10^{24} kg, so we usually ignore it when doing practical problems.

Similarly the third law states that the normal force $\vec{\mathbf{n}}$ exerted on the book by the table is balanced by a force $-\vec{\mathbf{n}}$ exerted on the table by the book. These are surface forces caused by the fact that the atoms making up book and table cannot interpenetrate. They are thus manifestations of the electromagnetic fundamental force, and quite unrelated to the gravitational forces operating: the fact that the two sets of forces are numerically equal is due to the geometry of the situation (if the book were lying on a sloping surface $\vec{\mathbf{n}}$ would no longer be $-m\vec{\mathbf{g}}$).

For case (b), the table, we have a gravitational force $M\vec{\mathbf{g}}$ and the normal force exerted by the book, $-\vec{\mathbf{n}} = m\vec{\mathbf{g}}$. Both of these act downwards, and therefore must be balanced by the normal force exerted by the floor, $\vec{\mathbf{N}} = -(m + M)\vec{\mathbf{g}}$, distributed in practice among the legs of the table.

Solve

(This was a conceptual problem: we have already done the small amount of formulation necessary—basically just stating the form of the gravitational force.)

(a) The forces acting on the book are gravity, $m\vec{\mathbf{g}}$, and the normal force from the table, $\vec{\mathbf{n}} = -m\vec{\mathbf{g}}$. The third law partners of these forces are the gravitational force exerted on the Earth by the book, $-m\vec{\mathbf{g}}$, and the normal force exerted by the book on the table, $-\vec{\mathbf{n}} = m\vec{\mathbf{g}}$.

(b) The forces acting on the table are the gravitational force $M\vec{\mathbf{g}}$, the normal force from the book, $-\vec{\mathbf{n}} = m\vec{\mathbf{g}}$, and the normal force from the floor, $\vec{\mathbf{N}} = -(m+M)\vec{\mathbf{g}}$. Their third law partners are respectively the gravitational force exerted by the table on the Earth, the normal force exerted by the table on the book, and the normal force exerted by the table on the floor.

5A.3, continued:

Of the forces acting on the book and the table, the only third law pair is the normal forces they exert on each other. All the other third law partners act on different objects (the Earth and the floor).

Learn

Note that not all equal and opposite force pairs are manifestations of the third law! In particular, forces acting on the ***same*** object ***cannot*** be related in this way, because the third law explicitly relates a force exerted ***on*** an object to a force exerted ***by*** the object.

5A.5 *A uniform rope of mass m and length ℓ is attached to a hook in the ceiling, and hanging from it is a mass M. Assuming that m is not negligible compared to M, what is the tension in the rope (i) at the hook; (ii) at the mass M; (iii) at some arbitrary point a distance y below the hook? If the rope is now removed from the hook and used to tow the mass M horizontally along a frictionless surface, what force must be applied to the end of the rope to give the mass M an acceleration a? Assume that the rope does not stretch.*

Conceptualize

Hanging rope

Up to now we have always considered the tension in a ***massless*** rope, and have stated without proof that it points along the rope and is the same at all points on the rope. This is a consequence of Newton's second and third laws: if the rope is massless, there must be no net force on it (otherwise, applying $\vec{\mathbf{a}} = \vec{\mathbf{F}}/m$, it would have an infinite acceleration!), and by Newton's third law the force exerted ***by*** the rope on an object tied to its end is equal and opposite to the force exerted ***on*** the rope by the object. Thus if you pull with force F on one end of a massless rope, the same force F is exerted by the rope on whatever is attached to its other end.

In this case the rope is not massless, and so the argument that there can be no net force acting on it does not hold. We must apply Newton's second and third laws directly to the points we are interested in—initially, the top and bottom of the hanging rope. The forces acting are gravity (the weight of the rope and the weight of the mass M), a normal force from the hook, and the tension in the rope. All of these act in a single vertical line, along the rope, so we have only a single component equation.

Formulate

At the hook, the force exerted ***on*** the rope ***by*** the hook is the normal force N. Since the rope-plus-mass system is not accelerating, this must cancel the downward forces $mg + Mg$ acting on the system (the tension is an internal force and will cancel). The tension in the rope is the force exerted ***by*** the rope ***on*** the hook, and by Newton's third law this must be $-N$, where we define forces to be positive downward.

We do a similar analysis at the mass: the force exerted by the mass on the rope is its weight, Mg, so the tension in the rope at this point must be $-Mg$.

For part (iii) we consider a small section of rope at a distance y below the hook. We divide the rest of the rope into two pieces: the part between us and the hook, which has length y and mass my/ℓ—since the rope is uniform, its mass must be distributed

5A.5, continued:

evenly along its length—and the part between us and the mass, which has length $\ell - y$ and mass $m\left(1-\frac{y}{\ell}\right)$. This part of the problem is then equivalent to part (ii), with a mass $M + m\left(1-\frac{y}{\ell}\right)$ suspended from a rope of length y.

Solve

For part (i), we have $N = -(m+M)g$, and so $T(y=0) = (m+M)g$.

In part (ii), we have already worked out that $T(y=\ell) = -Mg$.

Part (iii) gives $T(y) = -\left(M + m\left(1-\frac{y}{\ell}\right)\right)g$.

Comparing these three equations, we see that the magnitude of the tension decreases linearly from $(m+M)g$ at $y=0$ to Mg at $y=\ell$.

Scrutinize

This answer appears to make sense: if we put $m = 0$, returning to our familiar massless rope, the magnitude of the tension at all three points comes out to be Mg as we expect. As we worked out when conceptualizing this problem, the constancy of the tension in a massless rope is a consequence of the rope's masslessness, and not a general property of tension.

Conversely, if we let the rope hang under its own weight, setting $M = 0$, we find that the tension at the bottom end is zero. This is essential physically—if it were not true, the rope would be exerting an upward force on its own end, and would miraculously rise into the air! The tension at the top, of course, is not zero: the top of the rope feels a downward force from the rope's own mass.

We now go on to consider the second part of the problem.

Conceptualize

Tow-rope

The principles of our initial conceptualization still hold, but in this case we are using F = mass $\times$ acceleration, Newton's second law, whereas in the first part we were really using Newton's first law (the system was not accelerating, so no net force was acting on it). The vertical forces are zero (the weights are cancelled by a normal force from the surface), so we need consider only the horizontal force.

Formulate and Solve

The mass M has acceleration a, so the net force acting on it is Ma. This must be the tension in the rope at that end.

Since the rope does not stretch, it too must have an acceleration a, so the net force on the rope must be ma. As the rope is applying a force Ma to the mass, the force required on the other end of the rope is $(m+M)a$.

Scrutinize

We can check this answer by considering a system in which a mass m and a mass M are connected by massless ropes. To give such a system acceleration a, we would have to apply a force $(m+M)a$ on the massless string attached to mass m, and the string

5A.5, continued:

connecting m and M would have constant tension $T = Ma$ and would apply a force Ma to mass M. This agrees with our answers for the massive rope. Once again, allowing the rope to be massless gives the tension the constant value Ma, and allowing the mass on the end to go to zero sends the tension at that end to zero.

Learn

The tension in the rope behaves in exactly the same way in both parts of the problem. We could check the tension at any point by the same method we used above, dividing the rope into smaller and smaller pieces. This suggests an alternative, calculus-based, way of calculating the tension in a massive rope:

Consider a segment of the rope of length Δx, and hence mass $\Delta m = m\Delta x/\ell$. located at a distance x from the free end of the rope. If it has acceleration a, it must be subject to a net force $F = \Delta m a$. This can only be the difference between the tension $T(x)$ pulling it forward and the tension $T(x + \Delta x) = T + \Delta T$ pulling it back towards the mass. Hence we have

$$\Delta T = -\Delta m a = -\frac{ma}{\ell}\Delta x,$$

which in the limit where $\Delta x \to 0$ becomes

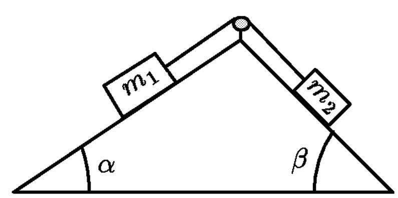
$$\frac{\mathrm{d}T}{\mathrm{d}x} = -\frac{ma}{\ell}.$$

The tension decreases linearly along the rope, just as we calculated earlier. If we integrate this equation from $x = 0$ to $x = x_0$, we get

$$T(x_0) - T(0) = -\frac{ma}{\ell}x_0,$$

where $T(0)$ is the tension at $x = 0$. Putting this equal to $(M + m)a$ gives us the same equation that we had for the hanging rope above.

5A.7 *Two blocks of masses m_1 and m_2 are connected by a massless inextensible rope as shown. At the apex of the frictionless triangular support, the rope passes over a frictionless pulley. Find the acceleration of the blocks and the tension in the rope.*



Conceptualize

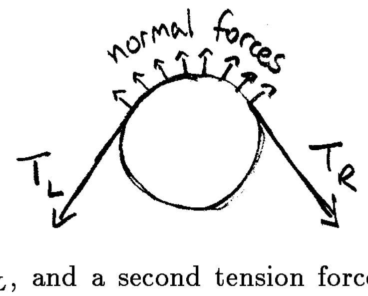
The difference between this problem and those we have met before is that in this case the rope changes direction when it passes over the pulley. Before we can set up the problem we need to understand the consequences of this.

By applying Newton's third law, we see that the left-hand part of the rope exerts a force on the pulley equal to the force exerted by the pulley on the left-hand rope. Similarly, the rope exerts a force on the block equal to the force that the block exerts on the rope. As the rope is massless, there can be no net force on it (otherwise it would have infinite acceleration), and thus the force exerted by the rope on the pulley is equal in magnitude

5A.7, continued:

(though opposite in direction) to the force exerted by the rope on the block. The same argument applies to the right-hand block.

Now let's consider the pulley, which has a rope tension T_L exerted on it from the left, and a tension T_R from the right. The pulley itself is a disk, and the rope runs around the rim of the disk: the pulley can therefore exert a contact force on the rope. This force is entirely radial, i.e. normal to the rim: we are told that the pulley is frictionless, so it cannot exert any tangential force. Therefore, where the rope loses contact with the pulley rim on the left, we have a radial force n, a tangential force T_L, and a second tension force T acting to the right. This second force must have a tangential component equal in magnitude to T_L, since the normal force can supply no tangential component, and there can be no net force on our massless rope. We can repeat this argument all the way round the pulley until we come to the point at which the rope loses contact again, at which point we balance the tangential component of T (which is still equal to T_L) against T_R. Our conclusion is that ***the frictionless pulley changes the direction of the tension, but does not change its magnitude***. (In a real pulley, friction between the pulley rim and the rope causes the pulley to turn. We will not be able to handle this situation until Chapter 8. However, the idealization of a frictionless pulley is a good approximation if the mass of the pulley wheel is very small compared to either of the blocks, and there is little friction in the bearings which allow the wheel to rotate.)

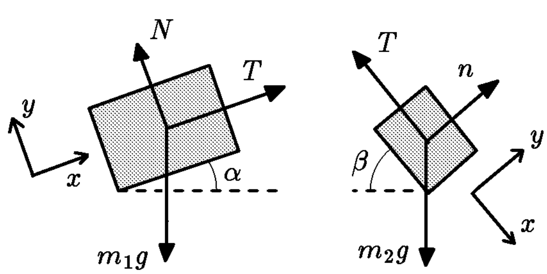


We are now in a position to formulate the problem. We can treat each block as a separate system, with two constraints:

- the magnitude of the tension T is the same for both blocks, as we have just shown;
- as the rope does not stretch, the magnitude of the acceleration $\vec{\mathbf{a}}$ must be the same for each block (they remain the same distance apart).

Formulate

Treating the blocks as separate systems, we can use different coordinate systems. It is most convenient to choose a coordinate system with the x-axis parallel to the slope: in order to keep the ***sign*** of the acceleration the same for both blocks, we let the x-axis point uphill for the left-hand block, and downhill for the right-hand block. Letting a denote the x-component of the acceleration for each block (which must be the same for both blocks), we have



$$F_{x,L} = m_1 a = T - m_1 g \sin\alpha;$$

$$F_{x,R} = m_2 a = m_2 g \sin\beta - T \ .$$

5A.7, continued:

We thus have two equations for the two unknowns T and a, so we are ready to solve them. The y-components do not involve either T or a, and are therefore not helpful in this problem.

Solve

To find a we simply add the two equations:

$$(m_1 + m_2)a = m_2 g \sin\beta - m_1 g \sin\alpha;$$
$$\Longrightarrow \quad a = \frac{m_2 \sin\beta - m_1 \sin\alpha}{m_1 + m_2} g.$$

We can then substitute in $F_{x,L}$ to find T:

$$T = m_1 g \left(\frac{m_2 \sin\beta - m_1 \sin\alpha}{m_1 + m_2} + \sin\alpha \right)$$
$$= \frac{m_1 m_2}{m_1 + m_2} g (\sin\alpha + \sin\beta).$$

Scrutinize

The dimensions are correct: a is g multiplied by the ratio of two masses, and T is g (an acceleration) multiplied by a quantity with the dimensions of mass, i.e. a force. If we set $m_2 = 0$, the acceleration comes out to $-g \sin\alpha$ and the tension to 0, which is what we would expect: the acceleration is just the downhill component of g (the normal component being balanced by the normal force exerted by the slope), and the rope is trailing freely behind the block. The minus sign is correct, because "downhill" in this case is the negative x-direction. Taking $m_1 = 0$ gives $a = g \sin\beta$, as expected. This is a particularly useful check here, because it would be quite easy to misplace a minus sign in the algebra, and dimensional analysis would not find such an error. If we wished, we could make a further check on the algebra by using the equation for $F_{x,R}$ to find T and confirming that we get the same answer.

Learn

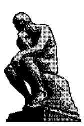
The action of an ideal pulley is simply to change the direction of the tension, without changing its magnitude. Real pulleys are not frictionless, but they are usually very light compared to the tensions in the ropes attached to them, and thus the difference in tensions required to accelerate the non-frictionless pulley is negligible.

Changing the direction of the tension may seem fairly trivial, but it has important practical applications. For example, a typical human being can exert much more force pulling down than pulling up—consider a 100 kg couch potato, who surely cannot lift a 90 kg block of concrete off the floor, but would have no difficulty exerting a downward force of 1000 N (he just has to put his whole weight on the rope). Furthermore, we will see in Problem 7.7 that by combining several pulleys we can actually arrange to lift our 90 kg concrete block by exerting a force of much less than $g \times 90$ kg, provided that we

5A.7, continued:

exert that force over a proportionately longer distance (thereby doing the same amount of work: energy conservation is not violated).

5B.5 *You have two sliders on a frictionless linear air track, each of mass m. One is stationary, and you slide the other one into it with a velocity v_0. What happens to the momentum and kinetic energy of each slider if (a) you have attached spring-loaded fenders so that the collision is perfectly elastic; (b) you have stuck on blobs of putty so that the two sliders stick together and move off together after the collision?*



Conceptualize

Collision problems are applications of conservation laws. What we have to do is work out exactly what is conserved. The rules are:

- total momentum is ***always*** conserved, unless there is an external force acting on the system;
- total kinetic energy is conserved in situations where (i) no work is done on the system by an external force ***and*** (ii) there is no change in the internal energy of any body involved in the collision.

In this problem there is no net external force, so momentum will certainly be conserved. In case (a), where the collision is elastic, kinetic energy will also be conserved, but in case (b) it will not be—the internal energy of the blobs of putty changes when they are deformed by the collision.

Formulate

In case (a) we have two unknowns, the final velocities v_1 and v_2 of the first and second sliders, and two equations, one for conservation of momentum and one for conservation of kinetic energy. (This is a one-dimensional problem, so momentum conservation produces only one equation: for a three-dimensional problem we would have three equations, one for each component.) In case (b) we have only one equation, but also only one unknown, because the final velocities are known to be equal (the sliders are stuck together). We can therefore solve the equations in each case.

For case (a) the two equations are

$$mv_0 = mv_1 + mv_2$$

$$\tfrac{1}{2}mv_0^2 = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}mv_2^2$$

(where v_0 is the speed of the first slider before the collision and v_1, v_2 are the speeds of the first and second sliders after collision), while for case (b) we have

$$mv_0 = 2mv_{1+2}$$

where v_{1+2} is the final velocity of the two sliders.

Solve (a)

Squaring the momentum equation gives

$$v_0^2 = v_1^2 + v_2^2 + 2v_1v_2$$

5B.5, continued:

(dividing out the common factor of m^2). The kinetic energy equation tells us that

$$v_0^2 = v_1^2 + v_2^2,$$

and comparing these two gives

$$2v_1 v_2 = 0.$$

This means that either v_1 or v_2 is zero, and given the geometry of the situation it can only be v_1. The first slider stops, and the second slider moves off with the same speed that the first slider originally possessed. You can see this kind of collision in pool or snooker when one ball hits another square on, and also in the motion of the balls in a Newton's cradle.

Solve (b)
Clearly $v_{1+2} = v_0/2$. The final kinetic energy is $\frac{1}{2} \times 2m \times (\frac{1}{2}v_0)^2 = \frac{1}{4}mv_0^2$, half the original kinetic energy: the rest of the kinetic energy has been converted into internal energy of the system, heating and deforming the blobs of putty.

Scrutinize and Learn
The total kinetic energy in case (b) after the collision is less than that before the collision. This is reassuring: neither slider has an obvious source of additional energy, so while we can imagine ***losing*** kinetic energy in the collision—in this case, by doing work on the blobs of putty—it is hard to see how we could ***gain*** any. In case (a), during the collision we ***do*** do work in compressing the springs, and this ***does*** decrease the kinetic energy, but the spring force is conservative, so we get that kinetic energy back when the springs expand again after impact.

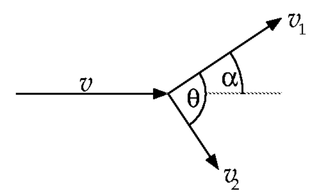
If we consider the center of mass frame, where slider 1 has velocity $\frac{1}{2}v_0$ and slider 2 velocity $-\frac{1}{2}v_0$ (the frame's velocity relative to the stationary frame is $\frac{1}{2}v_0$), then case (b) is trivial: the stuck-together sliders must be stationary in this frame, by definition. In case (a), the ***directions*** of the sliders' velocities in the center of mass frame change, but their ***magnitudes*** remain the same. This is also true of elastic collisions in more than one dimension, and involving objects of unequal masses. It arises because the net momentum in the center-of-mass frame is zero by definition, and so $\vec{\mathbf{v}}_2 = -\frac{m_1}{m_2}\vec{\mathbf{v}}_1$, i.e. $v_2 = -\frac{m_1}{m_2}v_1$. This means that the total kinetic energy is a function only of v_1, and so the elastic collision, which does not change the kinetic energy, cannot change v_1, and hence cannot change v_2.

This simplification does not hold in frames other than the center of mass frame, because in such frames $\vec{\mathbf{v}}_2$ is not just a multiple of $\vec{\mathbf{v}}_1$—it depends also on the total momentum $\vec{\mathbf{P}}$. Therefore working in frames other than the center of mass frame requires more variables, and more complicated algebra. Since transforming the initial parameters of the problem into the center of mass frame is usually quite simple, it often pays to take advantage of the simpler algebra and solve collision problems in this frame.

5B.6 *In the sport of curling, teams take turns to slide polished granite stones across an ice rink, aiming at a designated target zone. A common tactic is to dislodge well-placed opposition stones by hitting them with your own stone. Assuming that all the stones are of a standard mass, that friction can be neglected, and that the collision is elastic, is it possible to determine what will happen to a stationary stone B if it is hit by another stone A moving with velocity v? If it is not possible to determine exactly what will happen, what can be determined, and what additional information is required?*

Conceptualize

This is a collision problem in two dimensions. Our system has four unknowns: the two components of the velocity of stone A after the collision, and the two components of the velocity of stone B after the collision. We can construct three equations: two for the x- and y-components of the momentum, and one for kinetic energy (since we are told the collision is elastic). Therefore we ***cannot*** solve this problem completely, because the number of unknowns is more than the number of equations. (In one dimension we had one fewer equation, since the momentum had only one component, but two fewer unknowns, since each velocity also had only one component: so we ***could*** solve the problem in one dimension.)



Formulate

We can, however, reduce the number of unknowns from four to one. First we write down our three equations. It turns out to be convenient to use a coordinate system in which the x-axis is along $\vec{\mathbf{v}}_1$: in this system

$$v\cos\alpha = v_1 + v_2\cos\theta$$

$$v\sin\alpha = v_2\sin\theta$$

$$v^2 = v_1^2 + v_2^2$$

for the x-component of momentum, the y-component of momentum, and the kinetic energy respectively (dividing out the common factor of the mass of the stones). We proceed (as in the one-dimensional case) by squaring the momentum equations:

$$v^2\cos^2\alpha = v_1^2 + v_2^2\cos^2\theta + 2v_1v_2\cos\theta$$

$$v^2\sin^2\alpha = v_2^2\sin^2\theta$$

and then add these together to get

$$v^2 = v_1^2 + v_2^2 + 2v_1v_2\cos\theta.$$

Solve

Comparing the above equation with the expression for the kinetic energy, we can deduce that

$$2v_1v_2\cos\theta = 0,$$

which means that one of three things must be true:

5B.6, continued:

(i) $v_1 = 0$: the first stone hit the second dead center, reducing this to the one-dimensional problem we did earlier. The first stone stops and the second goes off with velocity $\vec{\mathbf{v}}$.

(ii) $v_2 = 0$: you missed! The first stone continues on its way and the second remains stationary.

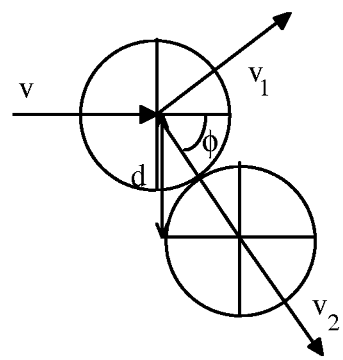
(iii) $\cos\theta = 0$: this is the interesting case. The two stones go off such that their velocities are at right angles to each other. In this case we know that $\theta = 90°$, $v_1 = v\cos\alpha$, and $v_2 = v\sin\alpha$. Our sole remaining unknown is the angle α.

We can therefore solve this problem if we are given any of the final speeds or directions.

Scrutinize

Physically speaking, what we need to know is how glancing the collision was: it's intuitively clear that if the two stones hit head on, there will be no sideways force, and we will get case (i), whereas at the other extreme if they barely touch we will get something approaching case (ii)—the velocity of the first stone will be almost unchanged, and the second stone will go off very slowly at right angles.

The explanation is that the forces between the stones when they collide are contact forces like those which prevent you falling through the floor, and therefore act along the line joining the centers of the stones (as in the diagram). The angle at which the second stone will go off, $\phi = \theta - \alpha$, is determined by $\sin\phi = d/2r$, r being the radius of the stones. So if we know d and r, or just the ratio d/r, we have enough information to solve the problem. The distance d (the component of the distance between the centers of the stones perpendicular to the incoming velocity) is often called the ***impact parameter*** of the collision.



Learn

We could have done the first part of this problem using the ***vector*** form of the momentum equation, $m\vec{\mathbf{v}} = m\vec{\mathbf{v}}_1 + m\vec{\mathbf{v}}_2$. If we divide out the common factor of m and then take the dot product of each side with itself (effectively squaring the equation), we get $v^2 = v_1^2 + v_2^2 + 2\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2$. Comparing this with the energy equation leads us to deduce that $\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 = 0$, yielding the same conclusions as before.

The advantage of this is that we do not need to choose a coordinate system (and in this case choosing the ***wrong*** coordinate system can be unfortunate—try working out this problem with the 'obvious' choice of coordinate system where the x-axis points along the incoming velocity $\vec{\mathbf{v}}$!). Although any problem that can be solved using vectors can also be solved by taking components, it is worthwhile learning to manipulate the vector equations, because the solution is often simpler in this form.

Notice that we could solve the one-dimensional collision problem completely using only the incoming velocities and masses, whereas in two dimensions we needed one additional piece of information regarding either the collision geometry or the outgoing velocities. How many additional pieces of information would we need to solve a ***three***-dimensional collision problem?

5C.2 *A girl is teaching her younger brother to skate by towing him around on a rope. They finish their practice session by hauling in on the rope from each end until they meet. If her mass is* 40 kg, *his is* 30 kg, *and the rope is* 5 m *long, how far from her original position will they end up? Assume that they were stationary to begin with, that the rope has negligible mass, and that there is no friction.*

Conceptualize

This problem is most easily solved by thinking about the center of mass of the children-plus-rope system. No external force is acting on this system, and therefore the center of mass must obey Newton's first law. As it is initially stationary, it will remain stationary while the children haul in on the rope. When both children are in the same place, they must be ***at*** the center of mass, so calculating its position at the beginning of the problem will give the children's position at the end.

We can treat the children as point particles for the purposes of calculating the center of mass, but, as discussed in Problem 4D.1(c), they can't be point particles for the purpose of pulling on the rope.

Formulate

This is effectively a one-dimensional problem, since all the movement takes place along the line of the rope. The location of the center of mass is therefore given by

$$x_{\rm cm} = \frac{\sum m_i x_i}{\sum m_i}.$$

Since we want the final position relative to the initial position of the girl, it makes sense to choose that as the origin of coordinates.

Solve

The position of the center of mass of the system relative to the girl's position when they start to pull is

$$x_{\rm cm} = \frac{(m_{\rm girl} \times 0) + (m_{\rm boy} \times L_{\rm rope})}{m_{\rm girl} + m_{\rm boy}} = \frac{30{\rm kg} \times 5{\rm m}}{70{\rm kg}} = 2.1{\rm m}.$$

Therefore, rounding to the nearest half meter (realistically we can hardly specify to the nearest centimeter the position of a mass consisting of two children!) their final location is 2 m from the girl's original position.

Scrutinize

The dimensions are very simple and obviously correct. The final answer is closer to the girl's starting point, since she is more massive than her brother. If we replace the girl by a pet dog of negligible mass, the formula correctly implies that the system center of mass is at the boy's starting point, and conversely if we replace the boy by the dog we wind up where the girl started (but note that a very small mass ***at a very large distance*** can have a significant effect on the position of the center of mass: the important quantity is not m, but the product mr).

Learn

Note that it doesn't matter who actually hauls in the rope. The force the girl exerts on the rope is always equal and opposite to the force the rope exerts on her, and the same

5C.2, continued:

is true for her brother. Since for a massless rope the tension is the same at each end, it follows that both children exert an equal force on the rope, even if one is actively hauling in and the other merely holding on.

What happens if the rope is not massless? Our center-of-mass argument above still holds, but the equal-tension one does not. We also have to worry about what happens to the pile of rope left when hauling in, because that has an effect on the position of the center of mass. If the children coil the rope up as they go, they will indeed end up at the position of the center of mass; if they leave the rope trailing behind them, they won't (and it will matter who hauled and who hung on). To see this, consider the extreme case of replacing the rope by a large tree-trunk which is much more massive than either child: if the log is initially stationary, it will stay that way, and the children (pulling themselves hand-over-hand along the log) may wind up anywhere along its length. The center of mass of the system still stays fixed, but it is now determined primarily by the position of the tree and not by the positions of the children.

5C.3 *Find the position of the center of mass of:*

(a) the Earth-Moon system (masses of 6.0×10^{24} *and* 7.4×10^{22} kg*) separated by 384 thousand kilometers;*

(b) a small spherical mass m attached to the end of a long thin rod of length ℓ and mass m;

(c) three thin rods, each of mass m and length ℓ, arranged to form three sides of a square.

Conceptualize

This question is so specifically posed that we can consider the conceptualization already done, and proceed to:

Formulate

The equation for the position of the center of mass of ***any*** system is

$$M\vec{\mathbf{r}}_{\rm cm} = \sum_i m_i \vec{\mathbf{r}}_i,$$

where M is the total mass of the system and m_i, $\vec{\mathbf{r}}_i$ are the mass and position of the $i^{\rm th}$ individual mass. For this question, the important point is that we can break this into pieces:

$$\begin{aligned} M\vec{\mathbf{r}}_{\rm cm} &= \sum_{i=1}^{k-1} m_i\vec{\mathbf{r}}_i + \sum_{i=k}^{\ell-1} m_i\vec{\mathbf{r}}_i + \sum_{i=\ell}^{n} m_i\vec{\mathbf{r}}_i, \\ &= M_1\left\{\frac{1}{M_1}\sum_{i=1}^{k-1} m_i\vec{\mathbf{r}}_i\right\} + M_2\left\{\frac{1}{M_2}\sum_{i=k}^{\ell-1} m_i\vec{\mathbf{r}}_i\right\} + M_3\left\{\frac{1}{M_3}\sum_{i=\ell}^{n} m_i\vec{\mathbf{r}}_i\right\}, \\ &= M_1\vec{\mathbf{r}}_{\rm cm,1} + M_2\vec{\mathbf{r}}_{\rm cm,2} + M_3\vec{\mathbf{r}}_{\rm cm,3}\ . \end{aligned}$$

5C.3, continued:

i.e. if we have a system which is made up of several subsystems, we can calculate the center of mass of each subsystem separately and then combine them to find the center of mass of the whole system.

Solve (a)
For the Earth-Moon system we assume that both Earth and Moon are spherical and that their densities either are uniform (a fair approximation for the Moon) or at least depend only on the distance from the center, and not on direction (OK for the Earth). In this case the center of mass of the sphere is located at the center of the sphere, since for each small element of mass at position $\vec{\mathbf{r}}$ relative to the center there is a corresponding piece at $-\vec{\mathbf{r}}$. Hence, taking $\vec{\mathbf{r}} = 0$ at the center of the Earth, our equation reduces to

$$(M_{\text{Earth}} + M_{\text{Moon}})r_{\text{cm}} = (M_{\text{Earth}} \times 0) + (M_{\text{Moon}} \times r),$$

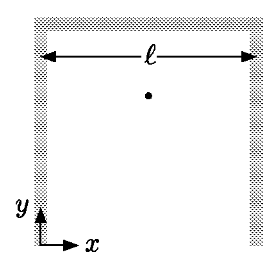
where r is the distance between the Earth's center and the Moon's. This gives us $r_{\text{cm}} = 4700$ km. The Earth-Moon center of mass is 4700 km Moonwards from the center of the Earth, which puts it actually inside the Earth (radius 6400 km).

Solve (b)
The same technique applies to the sphere-and-rod system. We are told the sphere is small, and can thus regard it as a point mass; the center of mass of the rod is at its center, a distance $\ell/2$ from the sphere (this should be obvious from its symmetry; if you're not convinced, see the note to the solution of problem 6.9). Combining these gives us

$$2mr_{\text{cm}} = (m \times 0) + (m \times \tfrac{1}{2}\ell)$$

if we put the origin of coordinates at the center of the small sphere. The center of mass is therefore located at $\ell/4$, or $\frac{1}{4}$ of the way along the rod from the weighted end.

Solve (c)
The third problem is slightly more complicated because it is two-dimensional. The center of mass of each rod is halfway along its length, so we can reduce our structure to three point particles at coordinates $(0, \ell/2)$, $(\ell/2, \ell)$ and $(\ell, \ell/2)$, putting the origin at one corner (see diagram). Using the equation for the center of mass in coordinate form gives us



$$3mx_{\text{cm}} = (m \times 0) + \left(m \times \tfrac{1}{2}\ell\right) + (m \times \ell)$$

$$3my_{\text{cm}} = \left(m \times \tfrac{1}{2}\ell\right) + (m \times \ell) + \left(m \times \tfrac{1}{2}\ell\right)$$

so the center of mass is located at $(\ell/2, 2\ell/3)$.

Scrutinize and Learn
Note that it doesn't matter what coordinate system we choose: if I displace my coordinate origin by an amount $\vec{\mathbf{R}}$, the equation for the center of mass becomes

$$M\vec{\mathbf{r}}_{\text{cm}} = \sum_i m_i \left(\vec{\mathbf{r}}_i + \vec{\mathbf{R}}\right) = \sum_i m_i \vec{\mathbf{r}}_i + M\vec{\mathbf{R}},$$

5C.3, continued:

so the calculated value of $\vec{\mathbf{r}}_{\mathrm{cm}}$ is just displaced by the same amount $\vec{\mathbf{R}}$, as it should be. We are therefore free to choose the most convenient coordinate system to do the calculation.

There is something familiar about the location of the center of mass in these examples. In part (b) especially, it seems to be located at the point where you would support the object if you wanted to balance it on a knife-edge or hang it from a hook. In fact this is absolutely true, although we will not have the technology to prove it until Chapter 8: the force of gravity on an object behaves as if it acts through the center of mass (which is often called the ***center of gravity*** for that reason).

5C.5 *Use the definition of the center of mass to prove the expression given in the* **Essentials** *for the kinetic energy of a system of particles.*

Conceptualize

This is another highly specific problem needing little additional conceptualization. The formulas we will use to solve it are the total kinetic energy of a system of particles and the equation defining the center of mass.

Formulate

The total kinetic energy is just the sum of the individual kinetic energies:

$$K = \sum_i \tfrac{1}{2} m_i v_i^2,$$

and the position of the center of mass is given by

$$M\vec{\mathbf{r}}_{\mathrm{cm}} = \sum_i m_i \vec{\mathbf{r}}_i,$$

where M is the total mass.

Solve

If the center of mass is moving with velocity $\vec{\mathbf{v}}_{\mathrm{cm}}$, we can write K as

$$K = \sum_i \tfrac{1}{2} m_i \left(\vec{\mathbf{v}}_{\mathrm{cm}} + \left(\vec{\mathbf{v}}_i - \vec{\mathbf{v}}_{\mathrm{cm}}\right)\right)^2 .$$

Expanding the square gives us

$$K = \tfrac{1}{2} M v_{\mathrm{cm}}^2 + \sum_i \tfrac{1}{2} m_i \left(\vec{\mathbf{v}}_i - \vec{\mathbf{v}}_{\mathrm{cm}}\right)^2 + \sum_i m_i \vec{\mathbf{v}}_{\mathrm{cm}} \cdot \left(\vec{\mathbf{v}}_i - \vec{\mathbf{v}}_{\mathrm{cm}}\right) .$$

If we compare this with what we want, we see that the first two terms are the "right answer" and the third term is an unwanted addition. Proving the given equation therefore depends on showing that this extra term is actually zero. The only obvious line of

5C.5, continued:

attack is to use the center-of-mass definition again, so let's rewrite our extra piece in terms of $\vec{\mathbf{r}}$ instead of $\vec{\mathbf{v}}$:

$$\sum_i m_i \vec{\mathbf{v}}_{\rm cm} \cdot \left(\vec{\mathbf{v}}_i - \vec{\mathbf{v}}_{\rm cm}\right) = \vec{\mathbf{v}}_{\rm cm} \cdot \sum_i m_i \left(\frac{d\vec{\mathbf{r}}_i}{dt} - \frac{d\vec{\mathbf{r}}_{\rm cm}}{dt}\right).$$

Since the masses don't change with time, we can take the differentiation outside the sum to get

$$\vec{\mathbf{v}}_{\rm cm} \cdot \frac{d}{dt}\left(\sum_i m_i \vec{\mathbf{r}}_i - M\vec{\mathbf{r}}_{\rm cm}\right).$$

But if we recall the definition of the center of mass, the term in the brackets is just $M\vec{\mathbf{r}}_{\rm cm} - M\vec{\mathbf{r}}_{\rm cm}$, which is obviously zero independent of time, so its derivative is zero, and hence this whole term is zero. We are left with what we wanted, namely

$$K = \tfrac{1}{2} M v_{\rm cm}^2 + \sum_i \tfrac{1}{2} m_i \left(\vec{\mathbf{v}}_i - \vec{\mathbf{v}}_{\rm cm}\right)^2.$$

The kinetic energy consists of the kinetic energy that the system would have if it were a point particle, plus the kinetic energies that the individual particles have ***relative*** to their center of mass.

Scrutinize

Our final expression is presumably correct, since it is quoted in the *Essentials,* but as an exercise we should convince ourselves that it has sensible properties. Firstly, if all the particles coalesce into a single point mass, their velocities relative to the center of mass will be zero, and K reduces to the usual formula for a single mass. The same occurs if the particles are all fixed relative to one another, even if they have non-zero separations. This explains why we can often treat extended (but rigid) objects as if they were point particles. Secondly, if we are already working in the center-of-mass frame, our final result is the same as our starting point, namely $K = \sum_i \frac{1}{2} m_i v_i^2$: in this frame the system has no overall motion, so all the kinetic energy comes from the internal movement of its components.

Learn

This derivation is an exercise in manipulating vector equations and derivatives. If you are confused, try rewriting it in component form. It is always possible and legitimate to decompose a vector equation into components, although it is usually easier to keep track of what is going on if you can manipulate the vectors directly.

5D.2 *During a tennis rally, the ball approaches a player at a speed of* 30 m/s. *He returns the shot so that the ball has a speed of* 35 m/s *at an angle of* 160° *to the original direction. What impulse did he apply to the ball? If ball and racquet were in contact for* 0.01 s, *what average force (averaged over time) did he exert? A tennis ball has a mass of* 60 g.

Conceptualize

This problem is primarily concerned with change in momentum. We have the mass of the tennis ball and its initial and final velocities, and can therefore calculate its change in momentum. According to the impulse-momentum theorem, this The force applied will actually vary during the time that the ball and the racquet are in contact (the face of the racquet acts very much like a spring), but the ***average*** force is simply the total impulse delivered divided by the total time for which the force acts.

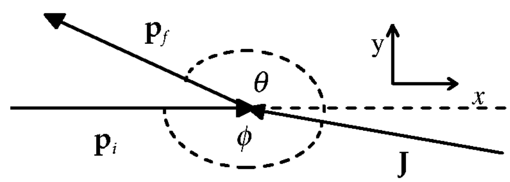
Formulate

If we define the x and y axes as shown (z is perpendicular to the plane defined by the two momentum vectors, and is zero throughout), then the initial and final momenta are

$$\vec{\mathbf{p}}_i = [p_i, 0, 0]$$
$$\vec{\mathbf{p}}_f = [p_f \cos\theta, p_f \sin\theta, 0] .$$

The impulse is

$$\vec{\mathbf{J}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i ,$$



and the average force exerted is

$$\vec{\mathbf{F}}_{\text{ave}} = \frac{\int_{t_i}^{t_f} \vec{\mathbf{F}}\,\mathrm{d}t}{t_f - t_i} = \frac{\vec{\mathbf{J}}}{t_f - t_i} .$$

Solve

The impulse applied was

$$\begin{aligned}\vec{\mathbf{J}} &= [p_f \cos\theta - p_i, p_f \sin\theta, 0] \\ &= [J\cos\phi, J\sin\phi, 0] ,\end{aligned}$$

where

$$p_i = (0.060\,\mathrm{kg}) \times (30\,\mathrm{m/s}) = 1.8\,\mathrm{kg \cdot m/s},$$
$$p_f = (0.060\,\mathrm{kg}) \times (35\,\mathrm{m/s}) = 2.1\,\mathrm{kg \cdot m/s},$$
$$\text{and} \quad \theta = 160^\circ .$$

This gives

$$J = \sqrt{(-3.77)^2 + (0.72)^2}\,\mathrm{kg \cdot m/s} = 3.8\,\mathrm{kg \cdot m/s}$$

$$\text{and} \quad \phi = \tan^{-1}\left(\frac{0.72}{-3.77}\right) = 169^\circ$$

5D.2, continued:

(angles with positive sine and negative cosine are in the second quadrant, between 90° and 180°). The average force is $(3.8\ \text{kg}\cdot\text{m/s})/(0.01\ \text{s}) = 380$ N, in the same direction as the impulse; that is, at 169° to the original direction of the ball.

Scrutinize

Since $1\ \text{N} = 1\ \text{kg}\cdot\text{m/s}^2$, it is clear that mass $\times$ velocity has the same dimensions as force $\times$ time, so equating impulse and (change of) momentum is dimensionally correct. In this problem we have a large change in the x-component of velocity (the ball almost reverses direction) and a comparatively small change in the y-component, so the magnitude of the change is approximately double the incoming momentum. Our values are in agreement with this expectation.

Learn

We have now defined two different integrals of the force: the force integrated over a ***distance*** interval, which is the work done (a dot product of two vectors, therefore a scalar), and the force integrated over ***time***, which is the impulse (a vector multiplied by a scalar, therefore a vector). Note that either one of these can be zero without implying that the other one is also zero: for example, a force applied at right angles to the direction of motion does no work, but can nonetheless apply a non-zero impulse.

HINTS FOR PROBLEMS WITH AN (H)

The number of the hint refers to the number of the problem

> *Note:* In dealing with collision problems, the algebra can become ***very*** tedious if you make a bad choice of reference frame. If you think you have set up the problem properly, but can't see how to extract a useful result from your equations, don't immediately conclude that you have done something hopelessly wrong—try looking for an alternative frame of reference. The frame most likely to be useful is the center of mass frame, or possibly the rest frame of one of the colliding objects.

5A.4 Do you have to apply a force to change the direction of motion of an object? If so, why?

In terms of L and v, what is the magnitude of the momentum carried by the water that flows into the hydrant in 1 s? As a result of flowing round the bend, what is the change in the x component of momentum every second? The y-component? The total momentum?

5B.3 The two sliders form an isolated system. What is meant by 'isolated system'? What is conserved in the motion of the objects which comprise an isolated system?

What is meant by 'elastic collision'? What is conserved in an elastic collision?

To solve this problem, try working in the center of mass frame.

Think of an immovable wall as a slider of infinitely large mass. What happens to your two velocities if you make M very large?

Is there a frame of reference in which the initial situation in your two-dimensional system is identical to the one you have just solved?

5B.4 What quantity (or quantities) is conserved in these collisions?

5B.7 At every step of the problem, think carefully about which objects you should consider as an isolated system. What is conserved in any isolated system? Just after a throw what is the sum of the magnitudes of the velocities of the sled and the block? What is relation between the total momentum of a sled plus block just before a throw and just after? If you are consistently getting the wrong answer, try checking the motion of the center of mass after each catch—what should it be doing?

Note: In most problems it is better to assign symbols for given numerical quantities and then first solve the problem algebraically. This problem is an exception to that "rule". The problem has 4 layers, each of which uses the answers from the previous layer. It gets very complicated to proceed in terms of the original variables. If you are getting the wrong answer, see that you are correctly doing the first step by checking the answer to the following: After you slide the block, what is your speed and what is the speed of the block?

5C.4 How is the center of mass defined? If you're confused, review the solution to problem 5C.3.

5C.6 What is the meaning of 'center of mass frame'? What does this imply about the total momentum of an isolated system considered in its center of mass frame? Use this to find v_2 in terms of v_1. Now do the same thing for ***after*** the collision. Can you now solve the kinetic energy equation?

5D.3 What is the meaning of 'impulse applied to the body'? How does this relate to momentum? In each case, what is the momentum of the ball before and after it is hit?

If you're still stuck, study the solution to problem 5D.2.

ANSWERS TO HINTS

5A.4 Yes, because the direction of the velocity vector changes (i.e., there is an acceleration); equivalently, the momentum changes (an impulse has been applied).

$\rho L v$; $\rho L v$, $-\rho L v$, $\sqrt{2}\rho L v$.

5B.3 One where there is no net external force acting; momentum.

One where there is no change in the internal energy of the colliding objects (no heating or deformation); kinetic energy and momentum.

Velocity of mass m tends to $-v$; velocity of mass M tends to zero.

Yes.

5B.4 Momentum.

5B.7 Momentum; 2 m/s; they are equal; it remains stationary; recoiling at 0.2 m/s; moving forward at 1.8 m/s.

5C.4 Its coordinates are such that

$$x_{\mathrm{cm}} = \frac{\sum_i m_i x_i}{\sum_i m_i},$$

and likewise for y and z.

5C.6 The frame in which the center of mass is stationary; it is always zero.

$$v_2 = \frac{m_1 v_1}{m_2} \qquad \text{(both times)};$$

$$\left[v_1^2\right]_{\text{after}} = \left[v_1^2\right]_{\text{before}}$$

5D.3 $\vec{\mathbf{J}} = \int \vec{\mathbf{F}}\,\mathrm{d}t$; impulse is total change in momentum.

(a) 2.1 and -3.0 kg m/s;

(b) 5.1 and -7.2 kg m/s (taking the positive x-axis in direction of incoming ball in both cases).

ANSWERS TO ALL PROBLEMS

5A.1 c

5A.2 a

5A.3 See complete solution.

5A.4 $\sqrt{2}\rho Lv$, where ρ is the density of water; 470 N, directed upwards 135° from direction of spray.

5A.5 See complete solution.

5A.6 An acceptable answer would be:

"An internal force is exerted by one body in the system under consideration on another body in the system. Such a force does not change the momentum of the system as a whole. An external force is exerted by something outside the system, and does change the system momentum."

5A.7 See complete solution.

5A.8 (a) $T = mg$; (b) $m = M \sin\theta$. The experimenter's gentle push will impart a small downward velocity to the block. After the push has been applied the force equations will resume the form they had before the push, which means that the acceleration will again be zero. The block of mass M will therefore continue down the hill until either it reaches the bottom, or the hanging block reaches the pulley.

5B.1 d; b

5B.2 The displaced ball stops when it hits the other four, and the ball at the other end comes off with the same speed v (only way to conserve both momentum and kinetic energy). This ball then swings out to the same displacement given the first ball (conservation of energy, as in simple pendulum), swings back to hit the rest with the same speed v, and the process repeats (indefinitely, in the absence of friction). Note that the collisions all take place in the horizontal plane, so there is no net external force (gravity is balanced by tension in suspending wires), hence we can use momentum conservation.

5B.3 $\dfrac{m-M}{m+M}\,v$ (slider of mass m) and $\dfrac{2mv}{m+M}$ (slider of mass M).

Slider will rebound from wall with velocity $-v$.

In two dimensions, component of velocity perpendicular to wall will be reversed, but parallel component unchanged. Puck will bounce back from wall at angle $180° - \theta$.

5B.4 (a) 2.89 m/s; (b) 2.85 m/s. (to 2 significant figures, 2.9 and 2.8 m/s, respectively)
Before: 340 J; after: (a) 270 J, (b) 280 J.

5B.5 See complete solution.

5B.6 See complete solution.

5B.7 0.33 m/s and 0.48 m/s, in opposite directions

5C.1 a

5C.2 See complete solution.

5C.3 See complete solution.

5C.4 (a) $\frac{1}{2}\left(r + \frac{1}{2}\ell\right)$ measured from center of sphere;

(b) at point $\left(\frac{1}{4}\ell, \frac{1}{4}\ell\right)$ in a coordinate system where one rod is the x-axis and the other the y-axis;

(c) $3\ell/8$ measured from left-hand end of horizontal rod in diagram.

5C.5 See complete solution.

5C.6 v_1 and v_2 respectively, in opposite directions (but angle with original direction unknown).

5D.1 c

5D.2 See complete solution.

5D.3 (a) 5.1 kg m/s; 510 N.

(b) 12.3 kg m/s; 6.2 kN.

CHAPTER 5

SUPPLEMENTARY NOTES

SOLVING REAL PROBLEMS

In this chapter we begin to see why the simplifications and idealizations we have been using in our problems do not prevent us from gaining physical insight which can be applied to real situations. This works in real life largely because of the set of equations in this chapter giving the acceleration, momentum and kinetic energy of a system of particles. In terms of application to problem solving, these tell us that

- we can calculate the momentum and acceleration of a macroscopic object (e.g. a car) by treating it as a point particle of the same mass;
- momentum is conserved in all situations, even those involving forces like friction where the sum of kinetic and potential energy is not constant (e.g. inelastic collisions);
- apparent failures of energy conservation in situations involving real physical objects can be understood in terms of changes in the internal energy of the system.

The first two points are consequences of the third law, or more generally of momentum conservation, and are vital to the development of physical science since the Renaissance. It is clear that if one had to consider the detailed structure of every object on which a force acts before predicting the motion of the object, it would be impossible to make useful predictions for real objects. (This is a problem which bedevils calculations involving the strong fundamental force, where, for a complicated set of reasons involving the detailed structure of the force, the current theory only yields exact predictions for a very restricted class of simple interactions. As a result most experimental results cannot be interpreted cleanly in terms of the theory, which has consequently been very difficult to develop and test properly.) The third point shows us how to test our theories more rigorously: if it is correct, then if we can find a way of measuring the internal energy of a system of particles (more particularly, of the atoms making up a physical object) we should find that the change in that internal energy after an interaction balances the change in mechanical (kinetic plus potential) energy we observe. Internal energy in fact often manifests itself as heat, and its behavior is studied in a branch of physics called thermodynamics. We will introduce some of the basic concepts of thermodynamics in Chapter 11, but it is far too large a subject in its own right to be considered as a subset of classical mechanics.

The conservation laws for energy and momentum are another important weapon in practical applications. As discussed in the notes to Chapter 4, the great advantage of these principles is that we can often apply them when our knowledge about the system we are studying is incomplete, or when the equations that describe it are mathematically intractable. Although we generally cannot solve a problem completely using conservation laws (for example, we have already seen that a two-body collision in two dimensions is not completely soluble), we can put severe restrictions on the possible solutions. This may in itself be sufficient for the purposes of the practical application (for example, in making sure that a particular effect cannot be large enough to distort the results of an experiment); if not, it at least provides a useful starting point for more detailed calculations, perhaps involving numerical integration.

FIELDS AND THE THIRD LAW

We have quoted the third law in Newton's terms, involving the actions of two objects on each other. In the case of forces like gravity and the electrostatic force, this requires the concept of ***action at a distance***, i.e. body A somehow affects body B despite the fact that they are not in contact. We saw in the Supplementary Notes to Chapter 4 that an alternative viewpoint is to say that body A actually creates a ***field*** with which body B then interacts (and vice versa). Modern particle physics indicates that the microscopic description of this field is a cloud of force-carrying particles which are constantly being emitted and absorbed by the interacting bodies.

This suggests that the proper formulation of the third law would be that the force exerted by body A on the field with which it interacts is equal and opposite to that exerted by the field on A, and likewise for B. For nearly static fields this makes no real difference. Suppose, as an analogy, that body A and body B were connected by a massless rope. We could then argue that the third law applies ***either*** directly to the two tension forces ***or*** individually to the tension force applied to A (or B) by the rope and the force applied to the rope by A (or B). The former is not strictly correct, but as long as the rope is massless it gives the same answer as the latter.

However, if the rope joining A and B is ***not*** massless, then only the second interpretation works, since the tension is no longer constant along the rope. In this case there is a net force on the rope, so the rope itself acquires a nonzero momentum. It turns out that there are circumstances, particularly in the forces felt and exerted by moving electrically charged particles, where the same thing happens for fields, and we ***must*** treat the third law as acting locally, in the interaction of each body with the local electromagnetic field. We will avoid such situations here, but they are quite common in electromagnetic theory.