ESSENTIALS OF INTRODUCTORY CLASSICAL MECHANICS

Sixth Edition

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CHAPTER 6

MACROSCOPIC FORCES AND NON-INERTIAL REFERENCE FRAMES

OVERVIEW

The concept of the internal kinetic energy of a system of particles has several interesting consequences when applied to the motion of physical objects. Since real objects are made up of atoms, even a single macroscopic body can be thought of as a system of many particles. In this chapter we consider the treatment of physical forces which cause unmeasured changes in the internal energy of such a system, and which therefore lead to an apparent non-conservation of mechanical energy. These are known as ***dissipative forces***, the most common being friction. We shall see later that the change in the internal energy of the system caused by the action of a dissipative force can often be detected experimentally as a change in temperature.

We have noted in earlier chapters that Newton's laws are only valid in inertial reference frames. However, in some circumstances it is more natural to work in a non-inertial frame. This can be done by introducing fictitious forces to account for that component of the observed acceleration which is due to the acceleration of the reference frame.

When you have completed this chapter you should:

- ✔ understand what is meant by a dissipative force;
- ✔ recognize the distinction between static and kinetic friction;
- ✔ understand the procedures for determining the forces of static and kinetic friction in various situations;
- ✔ be able to solve problems involving other types of dissipative forces, such as air friction, when the nature of the dissipative force is described in the problem;
- ✔ know what is meant by the term "fictitious force" and be able to use fictitous forces to do calculations involving non-inertial reference frames.

ESSENTIALS

If two physical surfaces come into contact the atoms making up those surfaces interact by the electromagnetic force. We have already met one consequence of this—the normal force which prevents solid objects from sinking into each other.

When a moving body slides on a stationary body, we observe another aspect of these interatomic forces: a frictional force acts to resist the relative motion. The actual mechanism of friction is extremely complex, involving the formation of temporary bonds between surfaces and the deformation of tiny irregularities on each face. Experimentally we find that, to a good approximation, the magnitude of the frictional force exerted by one surface on the other is proportional to that of the normal force exerted by that surface. We can then write

Problems 6B and 6E.7

$$\boxed{\left|\vec{\mathbf{F}}_k\right| = \mu_k \left|\vec{\mathbf{N}}\right| \ ,}$$

where the dimensionless constant μ_k is called the ***coefficient of kinetic friction***, and its value depends on the nature of the surfaces. The subscript k appears on $\vec{\mathbf{F}}_k$ and μ_k because the case discussed here is called ***kinetic friction***, since one surface is moving relative to the other. The force on the moving body is directed opposite its velocity, and then, by Newton's third law, the force on the stationary body is in the direction of the moving body's velocity. Notice that, like Hooke's 'law', this relation is not a physical law: it is an experimentally valid approximation to a very complicated physical process.

A force of friction can also be transmitted between two surfaces when both are stationary, in which case it is called ***static friction***. The method of calculation is somewhat different, since the force of friction will depend on what other forces are acting. The force of static friction is denoted by $\vec{\mathbf{F}}_s$, and is determined by the following rules:

Problems 6A

(1) Newton's third law applies, so the force on one surface is equal in magnitude but opposite in direction to the force on the other surface.

(2) The force will be in the plane of the two surfaces in contact.

(3) Within the plane of the two surfaces, both the direction and magnitude of the frictional force will adjust to cancel all other forces, so that there is no net force that would cause one surface to slide along the other.

(4) The force of static friction obeys the inequality

$$\left|\vec{\mathbf{F}}_s\right| \leq \mu_s \left|\vec{\mathbf{N}}\right| \;,$$

where μ_s is called the ***coefficient of static friction***. If the force determined by criteria (1)–(3) fails to obey this inequality, then it means that friction is not strong enough to hold the surfaces stationary. The surfaces will start to slide, and the rules governing kinetic friction will then apply. Hence for a given situation μ_s can be determined by measuring the force needed to cause the object to begin to move. It is impossible to have $\mu_s < \mu_k$: they can be equal, but usually $\mu_s > \mu_k$, which means that it is usually easier to keep something sliding than to start it sliding.

Occasionally we deal with situations in which neither surface in a friction problem is stationary, such as a suitcase dropped on a moving conveyor belt. For the case of kinetic friction, the magnitude of the force is still given by

$$\left|\vec{\mathbf{F}}_k\right| = \mu_k \left|\vec{\mathbf{N}}\right| \;.$$

The force on each surface is directed opposite to the velocity of that surface ***relative to the other surface***. For the case of static friction, only item (3) needs modification. The force will again be whatever is needed to prevent sliding (subject to the inequality in (4)), but with moving surfaces that does not necessarily mean that the force of friction cancels all other forces. If the conveyor belt is accelerating, for example, the force of static friction acting on the suitcase will cause it to accelerate at the same rate, to prevent sliding.

If we measure only the overall kinetic and potential energy of the body considered as a point particle, we will not record the changes in motion and position of the surface atoms caused by the frictional forces. Therefore, instead of finding Problems 6C

$$K + U = \text{constant} \;,$$

where U is the potential energy associated with any conservative forces acting on the body, we will see that

$$K_{\text{final}} + U_{\text{final}} = K_{\text{initial}} + U_{\text{initial}} + W_{\text{friction}} \;,$$

where W_{friction} is the work done by friction, which corresponds to the unobserved change in the system's internal energy.

The total work W_{friction} done by friction for a closed system is always negative, since friction can convert kinetic energy to heat, but never vice versa. It is possible, nonetheless, for friction to do positive work in cases where we are not discussing a closed system. When one places a suitcase with zero velocity on a moving conveyor belt, kinetic friction does positive work on the suitcase as the suitcase accelerates to the speed of the belt. Once the suitcase reaches the speed of the belt, then the rules of static friction apply. If the belt moves at constant velocity, then no force is necessary to prevent sliding, and the force of static friction will be zero (not $\mu_s|\vec{\mathbf{N}}|$!). If the belt then starts to accelerate, static friction will cause the suitcase to accelerate, so static friction will do positive work on the suitcase. If the conveyor belt accelerates so fast that the force necessary for the suitcase to keep up exceeds the inequality of item (4) above, then the suitcase will start to slip.

Forces which produce apparent changes in the total mechanical energy in this way are called ***dissipative*** (or ***non-conservative***) forces. Another common example is the drag force exerted on an object moving through a liquid or gas. Dissipative forces always arise as experimental approximations to more complex underlying physical forces: at the fundamental level we believe that energy is always conserved.

Problems 6D, 6E.8, 6E.9, and 6E.10

For a conservative force, the work done by the force as an object traverses a closed path is always zero. So if we put energy into a stone by lifting it, we will be able to retrieve the energy when the stone is lowered, because the work done by gravity on the stone must be zero for the closed path. Since the energy put into a stone by lifting it remains available, we defined the concept of potential energy to account for it. For dissipative forces, however, the work done as an object traverses a closed path is not zero. If a body is slid from point A to point B and back, friction opposes the motion in both directions, and the net work done by friction is negative. The energy is converted to heat, but it cannot be retrieved as kinetic energy, and there is no way to describe it as a potential energy.

The approximations involved in treating friction and similar effects in terms of dissipative forces, and the lack of an associated potential energy, do not affect the status of such forces as 'real', well-behaved examples of the concept of force: they obey Newton's laws in inertial reference frames. However, many 'forces' which are familiar in everyday life—for example, the force which pushes you back into the car seat during acceleration—are ***not*** real forces in this sense: instead, they are the observable consequence of making measurements in a non-inertial reference frame.

Consider a particle of mass m moving with constant velocity $\vec{\mathbf{v}}$. Its momentum is $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$, and the net force acting on it is $\frac{\mathrm{d}\vec{\mathbf{p}}}{\mathrm{d}t} = 0$. If we observe from a reference frame moving with constant velocity $\vec{\mathbf{V}}$, the observed momentum becomes $m(\vec{\mathbf{v}} - \vec{\mathbf{V}})$, and if both $\vec{\mathbf{v}}$ and $\vec{\mathbf{V}}$ are constant the observed force is still zero. However, if our reference frame is accelerating, so that $\vec{\mathbf{V}} = \vec{\mathbf{V}}_0 + \vec{\mathbf{a}}t$, the observed momentum is time-dependent, and to the accelerating observer it appears that the particle is being acted on by a force $\frac{\mathrm{d}\vec{\mathbf{p}}}{\mathrm{d}t} = -m\vec{\mathbf{a}}$. If the observer assumes that such a force is indeed acting, her calculations using Newton's laws will give the correct answers: the ***fictitious force*** $-m\vec{\mathbf{a}}$ gives the particle an acceleration $-\vec{\mathbf{a}}$ which compensates for the acceleration $\vec{\mathbf{a}}$ of the observer's frame of reference. Problems 6F.

If the non-inertial frame of reference is undergoing linear acceleration $\vec{\mathbf{a}}(t)$, which could be time dependent, then the fictitious force is simple to write down:

$$\vec{\mathbf{F}}_{\mathrm{fict}} = -m\vec{\mathbf{a}}(t) \ ,$$

where m is the mass of the object on which the force is acting. Beware, however, of rotating frames of reference. In such frames the fictitious force includes not only the ***centrifugal*** force, but also a velocity-dependent term called the *Coriolis* force, which is beyond the scope of this book.

In principle, it is never necessary to use the concept of a fictitious force: we can simply choose not to try to do force calculations in non-inertial reference frames. However, there are applications where the use of an accelerated frame is more natural: if we wish to design a seat belt or an airbag, it makes sense to work in the non-inertial reference frame of the rapidly decelerating car; if we are doing weather forecasts, and therefore want to study the motion of large masses of air relative to the ground, the only sensible reference frame to use is one which is fixed relative to the surface of the rotating Earth. (Note that for small-scale problems such as blocks sliding down inclined planes, it is a very good approximation to regard the Earth's surface as defining an inertial reference frame, but this approximation will not work for large-scale phenomena such as weather patterns.) In these cases we can only use Newton's laws successfully if we introduce appropriate fictitious forces.

Because the purpose of a fictitious force is to compensate for the acceleration of the observer's reference frame, it must produce the same acceleration for particles of different masses, so it must have the form $m\vec{\mathbf{a}}$. We are already familiar with one real force that has exactly this form, namely gravity. Over small length scales, the motion of objects viewed from an accelerating reference frame without gravity is indistinguishable from that seen from an inertial frame with a gravitational field. This is an alternative statement of the Principle of Equivalence (see Chapter 2), that the inertial mass is equal to the gravitational mass—an unexplained coincidence in Newtonian mechanics, but a fundamental principle of General Relativity.

SUMMARY

* Frictional forces are contact forces produced by atomic interactions at the surfaces of physical objects. Since the changes in position and motion of surface atoms are not generally measured, the action of frictional forces when one object slides on another causes an apparent loss of mechanical energy from the system. Forces which behave in this way are called ***dissipative*** (or ***non-conservative***) forces.
* When two surfaces slide along each other, the force of ***kinetic friction*** acts on each. To a good approximation, the magnitude of the force is equal to the ***coefficient of kinetic friction*** times the magnitude of the normal force between the surfaces. The direction of the force on each surface is opposite the direction of the velocity of that surface relative to the other surface.
* If two surfaces are in contact with no relative velocity, the force of ***static friction*** can act between them. This force is in the plane of the two surfaces, and has a direction and magnitude equal to whatever is necessary to prevent the surfaces from sliding, up to a maximum given by the ***coefficient of static friction*** times the magnitude of the normal force. If this maximum is not sufficient to prevent sliding, then the surfaces will slide and the rules of kinetic friction will apply.
* Newton's laws can be applied in non-inertial reference frames by introducing ***fictitious forces*** to account for the acceleration of the reference frame.
* Physical concepts introduced in this chapter: dissipative force, static friction, kinetic friction, coefficient of static friction, coefficient of kinetic friction.
* Mathematical concepts introduced in this chapter: none.
* Equations introduced in this chapter:

$$\left|\vec{\mathbf{F}}_k\right| = \mu_k \left|\vec{\mathbf{N}}\right| \qquad \text{(kinetic friction);}$$

$$\left|\vec{\mathbf{F}}_s\right| \le \mu_s \left|\vec{\mathbf{N}}\right| \qquad \text{(static friction);}$$

$$\vec{\mathbf{F}}_{\text{fict}} = -m\vec{\mathbf{a}}(t) \qquad \text{(fictitious force in linearly accelerating frame).}$$

PROBLEMS AND QUESTIONS

By the end of this chapter you should be able to answer or solve the types of questions or problems stated below.

Note: throughout the book, in multiple-choice problems, the answers have been rounded off to 2 significant figures, unless otherwise stated.

At the end of the chapter there are answers to all the problems. In addition, for problems with an (H) or (S) after the number, there are respectively hints on how to solve the problems or completely worked-out solutions.

6A STATIC FRICTION

6A.1 You are asked to drag a 45 kg crate across a warehouse floor. You find that in order to start the crate moving you have to apply a horizontal force of 250 N. Taking $g = 10$ m/s^2, what is the coefficient of static friction between the crate and the floor?

(a) 0.06; (b) 0.56; (c) 1.8; (d) 0.18.

Your six-year-old cousin tries to help you by dragging the next crate, but she can only apply a force of 50 N. What frictional force opposes her efforts?

(a) 50 N; (b) 250 N; (c) 450 N; (d) none of these.

6A.2 (H) A small weight of mass 50 g is placed on the turntable of a record player, 10 cm from the center. If the player is set for 33.3 rpm, what is the minimum coefficient of static friction required if the weight is to stay put? (Take $g = 10$ m/s^2.)

6B KINETIC FRICTION

6B.1 A train of total mass 100,000 kg is moving at 15 m/s when the engineer spots a cow on the line ahead. He pulls the emergency brake lever and locks the wheels of the train. If the coefficient of kinetic friction between the wheels and the rails is 0.6, how long does the train take to stop? (Take $g = 10$ m/s^2; the track is level.)

(a) 0.25 s; (b) 1.0 s; (c) 2.5 s; (d) 4.0 s.

6B.2 (S) You have just moved into a new apartment, and you are attempting to shift a 60 kg desk from one side of a (fortunately uncarpeted) room to the other. Are you better off pulling horizontally, or at some angle to the horizontal? If the coefficient of kinetic friction between the desk and the floor is 0.45, what is the smallest force you can apply to move the desk at a constant speed?

6B.3 Explain in 50 words or less the distinction between static and kinetic friction.

6C WORK DONE BY FRICTION

6C.1 (S) A child slides down a slide in a children's playground. The slide makes an angle of 40° to the horizontal and is 3 m high. If the coefficient of kinetic friction between the child and the surface of the slide is 0.15, what is the child's speed at the bottom? How much work has been done (a) by gravity; (b) by friction? What is the sum of kinetic and gravitational potential energy of the child, whose mass is 30 kg, at the top and bottom of the slide? What has happened to the 'lost' energy?

6C.2 (H) You pull a mass M up a ramp inclined at an angle θ to the horizontal. If you exert just enough force to move the mass at constant speed and you always pull parallel to the slope, how does (a) the force you exert, (b) the work you do in raising the mass through a vertical height h, depend on the angle of the slope and on the coefficient of friction between the mass and the ramp? Take $g = 10\ \text{m/s}^2$.

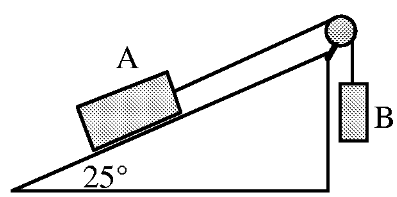
6D OTHER DISSIPATIVE FORCES

6D.1 Air resistance at high speeds can be approximated by $\vec{\mathcal{F}} = -kv^2\hat{\mathbf{v}}$, where $\vec{\mathbf{v}}$ is the velocity vector of the moving object, $\hat{\mathbf{v}}$ is the unit vector parallel to $\vec{\mathbf{v}}$, and k is a constant whose value depends on the object's shape. Using this information, (a) describe qualitatively the behavior of an object falling freely in air and (b) derive an expression for the maximum velocity attained by an object of mass m falling in air through some very long vertical distance h.

6D.2 The exact value of k (as defined above) for a given object depends on many details of its shape, but the most important determining factor is the cross-sectional area perpendicular to the direction of motion (this is why objects designed to minimize air drag tend to be long and slim). In view of this, discuss (a) the action of parachutes; (b) why cats generally survive falls from high-rise apartment balconies whereas human beings generally do not; (c) the result Galileo would have gotten if he really had dropped one wooden and one lead cannonball off the Leaning Tower of Pisa.

6E MOTION WITH DISSIPATIVE FORCES

6E.1 (S) Two blocks are connected by a light rope over a pulley as shown. The pulley is frictionless, but the coefficient of friction between block A and the slope is 0.40 for static friction and 0.30 for kinetic friction. If the mass of block A is 5 kg, what is the smallest mass B needed (a) to start block A sliding up the slope from rest; (b) to keep it moving if it has been started by an external push; (c) to prevent A from sliding ***down*** the slope? (d) What is the frictional force acting on block A if block B has a mass of 2 kg? Take $g = 10\ \text{m/s}^2$.



6E.2 (H) Thutmose, Pharaoh's chief pyramid builder, needs to drag a block of stone of mass 10,000 kg up an earth ramp during the construction of Pharaoh's latest project. The coefficient of static friction between the stone and the ramp is 0.6, and the kinetic friction coefficient is 0.4. He has available a large corps of slaves, each capable of exerting a force of 300 N. If the ramp makes an angle of 20° to the horizontal, how many slaves does he need to get the stone to start to move? Once it has started moving, how many can he divert to work on the next stone? Take $g = 10\ \text{m/s}^2$.

6E.3 In the context of the physics of this chapter, explain why, if you need to stop quickly, it is bad to lock the wheels of your car.

6E.4 An object of mass m rests on a slope making an angle θ with the horizontal. The coefficient of static friction between the object and the slope is μ_s, and the coefficient of kinetic friction is μ_k. As usual, μ_s is larger than μ_k. How steep does the slope have to be before the object starts to slide? If the slope is such that it will just start to slide, what will its acceleration be once it has begun to move?

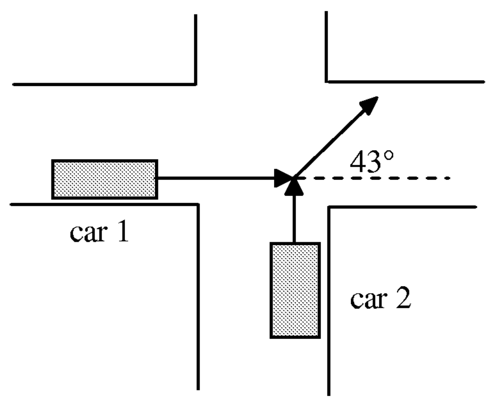
6E.5 (H) A 500 g block rests on a level table. The coefficients of friction between block and table are $\mu_s = 0.35$ and $\mu_k = 0.25$. The block is attached to a wall by means of a horizontal spring of spring constant $k = 100$ N/m. An experimenter pulls on the block to stretch the spring and then lets go, with the block initially at rest. Take $g = 10$ m/s^2.

(a) What is the maximum extension of the spring for which the block will remain stationary when released?

(b) If the block is placed in this position and then given a very gentle push towards the wall, describe in words what will happen. What is the initial acceleration of the block? At what position (relative to its starting point) does it reach its maximum speed?

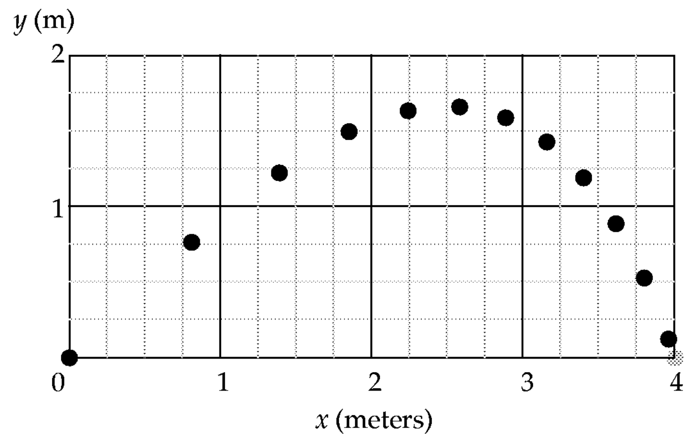
6E.6 (S) A curve on a freeway has radius 300 m, and has been banked for a design speed of 80 km/h (i.e. the inward component of the normal force provides the necessary centripetal acceleration at this speed). The freeway is presently occupied by the getaway car from a bank robbery, with the police in hot pursuit. In dry conditions, how fast can the crooks safely take the bend? What if they had chosen to commit their robbery on the proverbial dark and stormy night? [The coefficients of friction for rubber on dry concrete are $\mu_s = 1.0$, $\mu_k = 0.8$; for wet concrete they are 0.30 and 0.25 respectively.]

6E.7 (S) Two cars collide at an intersection. They remain locked together after the collision, and by measuring the skid marks the police conclude that the wreckage traveled 4.4 m at an angle of 43° to car 1's original direction. Car 1 had mass 1000 kg and car 2 1300 kg. If the accident happened in dry conditions when the coefficient of kinetic friction between rubber and concrete is 0.8, calculate the speeds of the two cars immediately before the collision.



6E.8 (H) At very slow speeds (especially in liquids rather than gases) the drag force is proportional to the speed rather than its square, i.e. , $\vec{f} = -k\vec{\mathbf{v}}$ where k is a constant. Suppose that a small ball of mass m is projected into such a liquid so that it initially has a horizontal velocity $[u, 0, 0]$. If the y-direction is defined to be vertically upwards, what is the ball's velocity vector $\vec{\mathbf{v}}$ at some later time t? Describe in words the motion of the ball. (Assume all effects other than gravity and fluid resistance can be neglected.)

6E.9 The diagram shows the path of a projectile. The interval between adjacent points is 0.1 s, and the diagram was constructed with an initial velocity vector $[10, 10, 0]$ m/s and $k/m = 0.35$ m^{-1} (where m is the projectile mass and the air drag is $\vec{f} = -kv^2\hat{\mathbf{v}}$). Discuss qualitatively how this trajectory differs from the motion of projectiles as analyzed in Chapter 1.



In the absence of air drag, the range of a projectile is maximized by firing it at 45° to the horizontal. To maximize the range for the projectile shown here, should you fire it at (a) 45°, (b) $< 45°$, or (c) $> 45°$? Do not attempt a detailed mathematical analysis, but do try to justify your answer with a reasonably persuasive argument.

6E.10 (S) A particle of mass m is launched vertically upward at time $t = 0$ with initial speed v_0. If the air drag is $\vec{f} = -kv^2\hat{\mathbf{v}}$, show that the particle reaches its maximum height at time

$$t = \sqrt{\frac{m}{kg}}\tan^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right) .$$

What is the maximum height reached? How does this compare with the height reached in the absence of air resistance, for a projectile with $k/m = 0.35\ \mathrm{m}^{-1}$ and $v_0 = 10$ m/s vertically upwards? Take $g = 10\ \mathrm{m/s}^2$.

You may find it useful to know that:

$$\int \frac{\mathrm{d}x}{1+x^2} = \tan^{-1} x \ ;$$

$$\int \tan\theta \mathrm{d}\theta = \ln(\sec\theta) \ .$$

6F NON-INERTIAL REFERENCE FRAMES

6F.1 (H) An experiment aboard a space probe includes a mass M which is attached to a spring of spring constant k and natural length ℓ_0. The other end of the spring is fixed. While the space probe is making a course correction, an astronaut notices that the mass M appears to be stationary, but the spring has stretched to length ℓ. Derive an expression for the acceleration of the space probe:

(a) using an appropriate inertial reference frame

(b) using the rest frame of the accelerating spacecraft.

6F.2 (H) A train is accelerating in a horizontal straight line. A passenger holds a plumb bob consisting of a mass M on a light string of length ℓ.

(a) If the magnitude of the train's acceleration is a, derive an expression for the angle which the string of the plumb bob makes with the vertical.

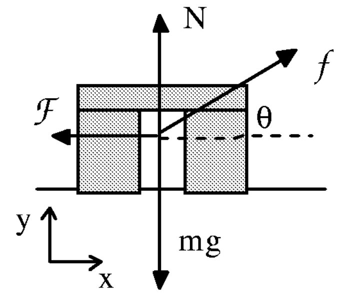
(b) If the bob is slightly displaced from its equilibrium position, what is the period of the resulting oscillations?

COMPLETE SOLUTIONS TO PROBLEMS WITH AN (S)

6B.2 *You have just moved into a new apartment, and you are attempting to shift a* 60 kg *desk from one side of a (fortunately uncarpeted) room to the other. Are you better off pulling horizontally, or at some angle to the horizontal? If the coefficient of kinetic friction between the desk and the floor is 0.45, what is the smallest force you can apply to move the desk at a constant speed?*

Conceptualize

It may seem obvious that you should apply a horizontal force, since you want the desk to move horizontally. However, recall that the frictional force is proportional to the normal force. Looking at the force diagram, we can see that if you apply a force with some upward component, you will decrease the normal force. You can trade a reduced horizontal component of $\vec{f}$ off against a reduced frictional force opposing it. To find the minimum force required, i.e. to minimize f, we will have to construct an equation for the net force in terms of our known quantities (m and μ_k) and the unknown angle θ. We can then differentiate this to find the value of θ which minimizes f.



Formulate

The components of the net force are

$$F_x = f\cos\theta - \mathcal{F}_k$$
$$F_y = f\sin\theta + N - mg\ ,$$

assuming you apply a force of magnitude f at an angle θ to the horizontal.

Since we are considering a situation where the desk is ***moving***, we want ***kinetic*** friction, so the magnitude of the frictional force is $\mathcal{F}_k = \mu_k N$. We want the desk to move horizontally at a constant speed, so there will be no net force acting either horizontally or vertically: $F_x = F_y = 0$

Our unknowns are N, f and θ, three in all, and we have two equations. However, the condition that we want the ***minimum*** value of f will give us a third equation, $\mathrm{d}f/\mathrm{d}\theta = 0$, and so we can solve this problem.

Solve

The condition that $F_y = 0$ tells us that $N = mg - f\sin\theta$. Furthermore $\mathcal{F}_k = \mu_k N$, so $F_x = 0$ implies that

$$f\cos\theta = \mu_k(mg - f\sin\theta)$$
$$\Rightarrow f = \frac{\mu_k\, mg}{\cos\theta + \mu_k\sin\theta}.$$

To find the minimum value of f, we differentiate with respect to θ:

$$\frac{\mathrm{d}f}{\mathrm{d}\theta} = -\frac{\mu_k\, mg}{(\cos\theta + \mu_k\sin\theta)^2}\left(-\sin\theta + \mu_k\cos\theta\right).$$

6B.2, continued:

This is zero when the last term is zero, i.e. when $\tan\theta = \mu_k$. For $\mu_k = 0.45$, our force is best applied at 24° to the horizontal. To find the magnitude of f, we substitute the numerical values into our equation:

$$f = \frac{0.45 \times (60\,\mathrm{kg}) \times (9.8\,\mathrm{m/s}^2)}{\cos 24° + 0.45 \sin 24°} = 240\,\mathrm{N}.$$

If you choose to pull horizontally, you need $f = \mu_k mg = 265$ N, about 10% more.

Scrutinize

Calculating the force required to move the desk pulling horizontally does two things: it confirms that the value we found by setting the derivative to zero is a minimum (remember that the derivative would also be zero for a maximum), and it assures us that the numerical value is reasonable.

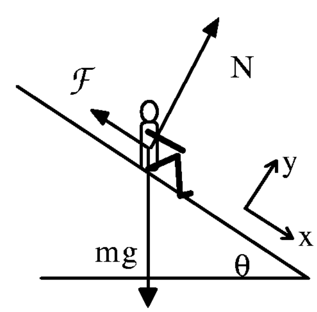
Learn

Experience tells us that another reason not to pull horizontally (especially in the case of a tall object like a bookcase, rather than a desk) is that the object is more likely to topple over than to slide. This is because our force diagram, which shows all the forces acting at the same point, is not very realistic: you would probably pull near the top of the desk, while the frictional force acts on its base. The result of this is that the object develops a tendency to rotate, rather than to remain in the same orientation. This will be the topic of Chapter 8.

6C.1 *A child slides down a slide in a children's playground. The slide makes an angle of 40° to the horizontal and is* 3 m *high. If the coefficient of kinetic friction between the child and the surface of the slide is 0.15, what is the child's speed at the bottom? How much work has been done (a) by gravity; (b) by friction? What is the sum of kinetic and gravitational potential energy of the child, whose mass is* 30 kg*, at the top and bottom of the slide? What has happened to the 'lost' energy?*

Conceptualize

The work done by a force on an object is the dot product of the force and the displacement of the object. Looking at the force diagram for the child, we can see that the normal force will do no work (it is perpendicular to the displacement), friction will do work $-\mathcal{F}x$, where x is the distance the child slides (the friction and the displacement are antiparallel), and gravity will do work mgh, where m is the child's mass and h is the vertical height of the slide.



To calculate the work done by friction, we shall first have to calculate the magnitude of $\mathcal{F}$. Once this is done, we can use energy conservation to calculate the kinetic energy, or we can do it from Newton's laws (since in the process of calculating the friction we shall have found the net force on the child).

6C.1, continued:

Formulate

The free-body diagram for the child is shown above right. The net force is

$$F_x = mg\sin\theta - \mathcal{F}$$
$$F_y = N - mg\cos\theta$$

and in this case, since the child is moving, we are dealing with kinetic friction. Hence $\mathcal{F} = \mu_k N$, and from the fact that the y-component of the net force must be zero we know $N = mg\cos\theta$.

Solve

The distance traveled down the slide is $h/\sin\theta$. Therefore the work done by friction is

$$\begin{aligned} W_{\mathcal{F}} &= -\mu_k mgh\cot\theta \\ &= -(0.15)\times(30\ \text{kg})\times(9.8\ \text{m/s}^2)\times(3\ \text{m})\times\cot 40^\circ = -160\ \text{J}\ . \end{aligned}$$

The work done by gravity is $mgh = 880$ J, which is, by definition, equal to minus the change in the child's gravitational potential energy. We have

$$K_{\text{final}} = U_{\text{initial}} + W_{\mathcal{F}} = mgh + W_{\mathcal{F}}$$

(since the initial kinetic energy is zero, and so is the final gravitational potential energy). So the sum of kinetic and gravitational potential energy is 880 J at the top of the slide and 720 J at the bottom. The 'lost' energy is equal to the work done by friction, and has been transformed into internal energy—specifically, into heating the surfaces involved (this is why you can burn yourself by sliding down a rope, and why a match lights when struck).

Scrutinize

We can check our result using kinematics. The net force on the child is

$$F_x = mg(\sin\theta - \mu_k\cos\theta)\ ,$$

so the child's acceleration is $g(\sin\theta - \mu_k\cos\theta)$. Using $v = at$ and $x = \frac{1}{2}at^2$, we have $x = v^2/2a$, and since $x = h/\sin\theta$ this gives

$$v = \sqrt{2gh(1-\mu_k\cot\theta)} = 6.9\ \text{m/s}\ ,$$

and so $K_{\text{final}} = \frac{1}{2}mv^2 = 720$ J, as before.

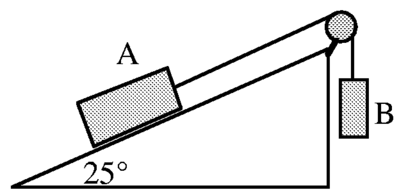
Learn

Note that the work done by gravity is independent of the slope of the slide, but the work done by friction is not—a shallower chute of the same height would involve sliding a longer distance, and thus increase the magnitude of the (negative) work done by friction. The child would end up with a smaller final velocity. It is typical of dissipative forces that the work done in moving from point A to point B ***does*** depend on the path

6C.1, continued:

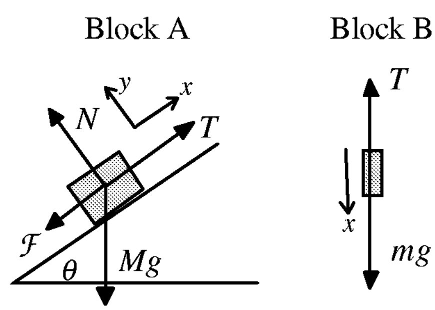
taken between A and B, in contrast to conservative forces. As a result we ***cannot*** define a 'potential energy' associated with a dissipative force, and thus we see an ***apparent*** loss of energy if we only take into account so-called mechanical energy (i.e. kinetic plus potential). In fact this energy is not lost, but simply transformed into a different form (usually heat).

6E.1 *Two blocks are connected by a light rope over a pulley as shown. The pulley is frictionless, but the coefficient of friction between block A and the slope is 0.40 for static friction and 0.30 for kinetic friction. If the mass of block A is* 5 kg, *what is the smallest mass B needed (a) to start block A sliding up the slope from rest; (b) to keep it moving if it has been started by an external push; (c) to prevent A from sliding* ***down*** *the slope? (d) What is the frictional force acting on block A if block B has a mass of* 2 kg*? Take* $g = 10\ \text{m/s}^2$.



Conceptualize

From the point of view of block A, it doesn't really matter that the source of the tension in the rope is the weight of block B. We can therefore divide this problem into two pieces: the forces on block A (which will determine the motion of block A) and those on block B (which will determine the tension). As the rope is "light" (i.e., of negligible mass) and the pulley frictionless, we can assume that the magnitude of the tension is constant.



To solve the problem, we therefore draw ***separate*** force diagrams (***free-body diagrams***) for blaocks A and B, choose appropriate coordinate systems, and write down the component equations of the net force.

Formulate

The free-body diagrams are shown above right. Block B forms a one-dimensional system with net force

$$f_B = mg - T\ .$$

Block A has a somewhat more complicated force diagram. The components of the net force acting on A are

$$F_x = T - \mathcal{F} - Mg\sin\theta$$

$$F_y = N - Mg\cos\theta\ .$$

The two blocks are connected by a rope of negligible mass, so the tension T is the same for both. The acceleration produced, if any, will also be the same for both, since we can assume the rope doesn't stretch (ropes in physics problem land don't stretch unless you are explicitly told that they do!). We choose to draw the friction force pointing ***down*** the slope because we are investigating the case where block A is expected to move ***up*** the slope, and the frictional force will act to oppose this.

6E.1, continued:

Solve (a) & (b)
For case (a), consider the situation when block A is just about to start sliding. The system is still stationary, so we know that (i) the net force on both A and B must be zero and (ii) we are dealing with static friction. The first point tells us (from B's force diagram) that $T = mg$, and the second that (since the block is about to slide) $\mathcal{F}_s = \mu_s N$. From the y component of the force on block A we also know that $N = Mg\cos\theta$. Putting all this information together, we have

$$F_x = mg - \mu_s Mg\cos\theta - Mg\sin\theta \ ,$$

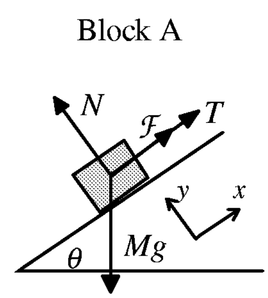
and this is zero when

$$m = M(\mu_s \cos\theta + \sin\theta) = 3.9 \text{ kg} \ ,$$

for $M = 5$ kg and $\theta = 25°$. Any mass B larger than 3.9 kg will thus cause block A to start sliding.

For case (b) the system is moving at constant speed, so again there is no net force, but this time we are dealing with kinetic friction. We simply replace μ_s by μ_k in the above formula to find $m = 3.5$ kg.

Formulate and Solve (c)
For case (c) we need a new force diagram for block A. A's motion is now ***down*** the slope, so the frictional force will act in the opposite direction. The net force on A is now



$$F_x = T + \mathcal{F} - Mg\sin\theta$$
$$F_y = N - Mg\cos\theta \ .$$

In the case where A is just about to start sliding down the slope, we again have no net force and the maximum possible amount of static friction. B's force diagram is unchanged, so T is still mg and we have

$$F_x = mg + \mu_s Mg\cos\theta - Mg\sin\theta \ ,$$

giving

$$m = M(\sin\theta - \mu_s \cos\theta) = 0.3 \text{ kg} \ .$$

Formulate (d)
For the final part, we note that a mass of 2 kg is more than required to prevent A from moving downhill, but not sufficient to move it uphill. Therefore the blocks will be stationary and static friction operates. Block A is not on the point of sliding, so the size of the frictional force is determined by the condition that there is no net force acting. It is not immediately apparent what direction $\mathcal{F}$ acts in: for convenience we shall use the second of our two free-body diagrams (if we are wrong, $\mathcal{F}$ will come out negative).

6E.1, continued:

Solve (d)
Using $F_x = 0$ and $T = mg$ as before, we have

$$F_x = mg + \mathcal{F} - Mg\sin\theta = 0 \ ,$$

i.e.

$$\mathcal{F} = Mg\sin\theta - mg = 1.1 \text{ N} \ .$$

Since this is positive we were in fact correct about the direction of the force: in the absence of friction block A would slide downwards. A slightly more massive block B, say 2.5 kg, would result in a frictional force directed the other way (in the absence of friction, A would then slide upwards). Notice that the frictional force is very much less than $\mu_s N = \mu_s Mg\cos\theta = 18$ N.

Scrutinize
In the absence of friction the mass m needed to give no net force on Block A is $M\sin\theta =$ 2.1 kg. Any greater mass would cause A to accelerate uphill; smaller values would let it accelerate downhill. In the presence of friction we need a greater mass to move A uphill (reasonable, as a greater tension force is needed to overcome the additional force of friction), and less mass to stop it moving downhill (here friction is 'helping' block B).

If $\theta = 90°$, there is no normal force on Block A, and the results should be independent of friction. Our equations satisfy this criterion for parts (a)–(c), as $\cos 90° = 0$. In part (d) we can still calculate a value for $\mathcal{F}$, because we are simply calculating the value required by the condition of zero net force, but in fact the assumption made in part (d)—that neither block moves, because of the effects of static friction—does not hold for $\theta = 90°$, and so the value we obtain is not valid.

Learn
This example demonstrates most of the ways in which friction can enter a force problem. Note that the techniques we apply to solve the problem are exactly the same as those we used in earlier chapters—we just have to decide when drawing the free-body diagram where friction is operating, and when calculating the net force whether we need to apply static friction at its maximum value, static friction at some smaller value, or kinetic friction. We must also consider the direction in which the frictional force acts, as we saw in drawing the free-body diagrams for parts (a) and (c) above: to determine this, we apply the rule that friction between two surfaces always acts to oppose the ***relative*** motion of the surfaces in question.

6E.6 *A curve on a freeway has radius* 300 m, *and has been banked for a design speed of* 80 km/h *(i.e. the inward component of the normal force provides the necessary centripetal acceleration at this speed). The freeway is presently occupied by the getaway car from a bank robbery, with the police in hot pursuit. In dry conditions, how fast can the crooks safely take the bend? What if they had chosen to commit their robbery on the proverbial dark and stormy night? [The coefficients of friction for rubber on dry concrete are* $\mu_s = 1.0, \mu_k = 0.8$*; for wet concrete they are 0.30 and 0.25 respectively.]*

6E.6, continued:

Conceptualize

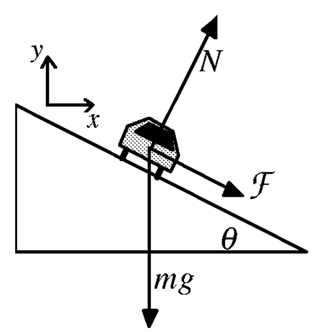
While the car is on the bend, it is engaged in circular motion, and if it is moving with speed v its acceleration must be v^2/r, where r is the radius of the bend. In the absence of friction, this acceleration could only be produced by the horizontal component of the normal force from the banked bend, and there would be only one speed at which the corner could be negotiated successfully: the question states that this speed is 80 km/h. With friction acting, we have an additional horizontal force which will allow the car to take the bend at speeds higher than 80 km/h.

At these higher speeds, the inward component of the normal force will not provide a large enough centripetal acceleration. In the absence of frictional forces, the car will turn less sharply than required (r will increase) and will run off the outside of the bend. The frictional force acts to resist this motion, and is therefore directed inward, as shown on the diagram. Since the car is ***not*** skidding, the part of the tire in contact with the road is ***not*** sliding on the road surface, and therefore we want the ***static*** friction. Further, since we are interested in the maximum possible speed, the static friction must be at its maximum possible value, i.e. $\mu_s N$. We thus have three independent unknowns, θ, v, and N. Our strategy will be to find θ from the known design speed of the bend, and then use the two component equations of the net force to find v and N.

Formulate

Putting $\mathcal{F} = \mu_s N$, the components of the net force are

$$\begin{aligned} F_y &= N\cos\theta - mg - \mu_s N\sin\theta &= 0 \\ F_x &= N\sin\theta + \mu_s N\cos\theta &= mv^2/r\,. \end{aligned} \qquad (1)$$



To solve these for v and N we need the value of θ, which we obtain by considering the case where $\mathcal{F} = 0$ and the bend is taken at its design speed of $v_0 = 80$ km/h. Our equations then become:

$$N\cos\theta - mg = 0$$

$$N\sin\theta = \frac{mv_0^2}{r}\,,$$

from which

$$\tan\theta = \frac{v_0^2}{rg}\,.$$

Solve

From Eq. 1 we have

$$N\left(\cos\theta - \mu_s\sin\theta\right) = mg$$

$$N\left(\sin\theta + \mu_s\cos\theta\right) = \frac{mv^2}{r}\,,$$

6E.6, continued:

giving

$$v^2 = rg\frac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\,.$$

By dividing numerator and denominator by $\cos\theta$ we could, if we wished, express this equation in terms of $\tan\theta$, and hence in terms of our given parameters v_0, r and g, but there is no real need to do this—we can simply substitute the numerical values into the equation for θ, giving, for $g = 9.8\ \mathrm{m/s^2}$, $\tan\theta = 0.168$, or $\theta = 9.5°$. Under dry conditions, $v = 64\ \mathrm{m/s} = 230\ \mathrm{km/h}$ (which isn't very likely to cause the crooks any problems), while for wet conditions we get $v = 38\ \mathrm{m/s} = 140\ \mathrm{km/h}$, or about 80 mph (which might lead to disaster).

Scrutinize

Our equation for v^2 clearly reduces to the frictionless case for $\mu_s = 0$, as it should. The dimensions are correct: $[rg] = [\mathrm{length}] \times [\mathrm{length}]/[\mathrm{time}]^2$, which is the same as v^2. The units require some care: 80 km/h must be converted to 22.2 m/s before using the formula for $\tan\theta$.

Looking at the equation, we see that a larger coefficient of friction would increase the numerator and decrease the denominator, increasing v (as we see in comparing wet and dry conditions). If we express v^2 in terms of $\tan\theta$, giving $rg(\tan\theta + \mu_s)/(1 - \mu_s\tan\theta)$, we see that an increase in θ also increases v: this is sensible, as a larger θ corresponds to a curve with a higher design speed.

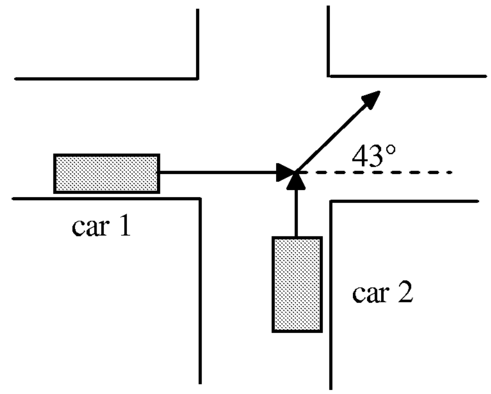
Learn

In problems involving friction we must always ask ourselves two questions: are we dealing with static or kinetic friction, and in which direction does the frictional force act? The answers may seem obvious, but there are pitfalls—in this example we have a ***moving*** car, but must apply ***static*** friction, and the horizontal component of the frictional force acts in the ***same*** direction as the horizontal component of the normal force, not opposed to it. Frictional forces invariably act to resist relative ***motion*** of the two surfaces: generally this means opposing an applied force, but not always!

For ***static*** friction problems, we must also ask whether the size of the frictional force is determined by μ_s or by the condition that the object is not moving. In this example, μ_s is the same for the car doing 80 km/h as it is for the one doing 230 km/h, but in the first case the frictional force is actually zero, because there is no force tending to slide the surfaces of the tires on the surface of the road. For intermediate speeds, there is a frictional force, but its size is determined by the value of v^2/r, not by μ_s. (What happens to the frictional force if the car is doing ***less*** than 80 km/h?)

For rubber on concrete, as for most pairs of surfaces, $\mu_k < \mu_s$. If the car does start to skid, its sideways motion will accelerate as static friction is replaced by kinetic friction. This is why it is so easy to lose control of a skidding auto.

6E.7 *Two cars collide at an intersection. They remain locked together after the collision, and by measuring the skid marks the police conclude that the wreckage traveled* 4.4 m *at an angle of 43° to car 1's original direction. Car 1 had mass* 1000 kg *and car 2* 1300 kg. *If the accident happened in dry conditions when the coefficient of kinetic friction between rubber and concrete is 0.8, calculate the speeds of the two cars immediately before the collision.*



Conceptualize

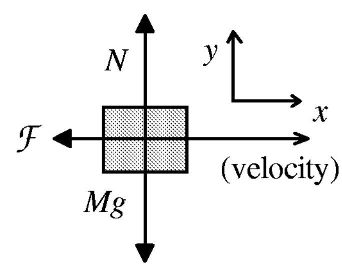
We have two distinct problems here: the collision itself, and the subsequent deceleration of the wreckage due to friction with the road. Our information relates most directly to the second part, so we shall deal with that before tackling the collision.

The only horizontal force acting on the wreckage during its deceleration is friction. The work-energy theorem implies that the work done by friction must therefore equal the change in kinetic energy of the wreck, so by calculating the magnitude of $\mathcal{F}$ we can deduce the initial speed of the wreck. Our collision problem will then have two equations (conservation of momentum in two dimensions) and two unknowns (the magnitudes of the velocities of car 1 and car 2—we know their directions), and we should therefore be able to solve it.

$\Sigma\int$

Formulate

The force diagram for the wreck involves three forces: gravity and the normal force from the road, which must balance each other, and kinetic friction. We conclude that the vertical and horizontal components of the net force are respectively



$$F_{\text{v}} = N - (m_1 + m_2)g = 0$$

$$F_{\text{h}} = -\mathcal{F} = -\mu_k N = -\mu_k (m_1 + m_2)g \ .$$

The work done by friction is $\vec{\mathcal{F}} \cdot \vec{\mathbf{r}}$, where $\vec{\mathbf{r}}$ is the displacement of the wreck. As the wreck moves horizontally, the vertical forces do no work, so by the work-energy theorem the work done by friction must be equal to the change in kinetic energy of the wreck.

Solve

As $\vec{\mathcal{F}}$ is antiparallel to $\vec{\mathbf{r}}$, the work done is

$$\vec{\mathcal{F}} \cdot \vec{\mathbf{r}} = -\mathcal{F} r = -\mu_k (m_1 + m_2) g r.$$

The change in kinetic energy is $-\frac{1}{2}(m_1 + m_2)v_0^2$, where v_0 is the speed of the wreck immediately after the collision (as the wreck comes to a halt, clearly $K_{\text{final}} = 0$). The

6E.7, continued:

work-energy theorem then gives

$$v_0 = \sqrt{2\mu_k\, gr} = \sqrt{2 \times 0.8 \times 9.8\ \text{m/s}^2 \times 4.4\ \text{m}} = 8.3\ \text{m/s, or } 30\ \text{km/h}.$$

Scrutinize

We can also do this using kinematics. If the speed of the wreckage immediately after the collision is v_0, then when it stops $v^2 = 0 = v_0^2 + 2ar$, where r is the distance traveled and $a = -\mu_k\, g$ is the acceleration. Therefore $v_0 = \sqrt{2\mu_k\, gr}$, in agreement with our value from energy conservation.

Conceptualize

Our next task is to deal with the collision. Although friction is obviously present, the collision takes place very quickly over a very short distance, and thus both the impulse applied and the work done by friction during the collision itself are small compared to the contact forces between the cars. We shall therefore neglect frictional effects and treat the collision as an isolated two-body system. This is an inelastic collision (the two cars stick together), so only momentum is conserved.

Formulate and Solve

We define a coordinate system so that x points in the direction of car 1's incoming velocity, and y in the direction of car 2. Momentum conservation then gives us

$$(m_1 + m_2)v_0 \cos 43° = m_1 v_1$$
$$(m_1 + m_2)v_0 \sin 43° = m_2 v_2$$

where v_1 and v_2 are the speeds of car 1 and car 2 immediately before the collision. Putting in the masses and our calculated value of v_0 gives

$$v_1 = 14\ \text{m/s} = 50\ \text{km/h}\ ,$$
$$v_2 = 10\ \text{m/s} = 36\ \text{km/h}\ .$$

Scrutinize

Is it fair to neglect friction during the collision? We may argue as follows: the frictional force on the wreck is $\mu_k(m_1 + m_2)g = 18$ kN. The loss of kinetic energy as a result of the collision is 84 kJ. Even if the two cars crumpled by as much as one meter during the collision (unlikely at such comparatively low speeds), the average contact force exerted during the collision must therefore be 84 kN (equating the work done by the dissipative forces involved in the collision to the loss of kinetic energy), which is some five times the frictional force. (Notice that the more the cars crumple, the smaller the average force exerted during the collision—this is why auto engineers design cars with 'crumple zones' to protect passengers.)

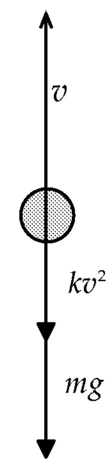
6E.10 *A particle of mass m is launched vertically upward at time $t = 0$ with initial speed v_0. If the air drag is $\vec{f} = -kv^2\hat{\mathbf{v}}$, show that the particle reaches its maximum height at time*

$$t = \sqrt{\frac{m}{kg}}\tan^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right) .$$

What is the maximum height reached? How does this compare with the height reached in the absence of air resistance, for a projectile with $k/m = 0.35\ \mathrm{m}^{-1}$ and $v_0 = 10$ m/s vertically upwards? Take $g = 10\ \mathrm{m/s}^2$.

Conceptualize

The force diagram for this problem looks very simple. The difficulty is that the air drag force is dependent on the velocity, which in turn depends on the past history of the acceleration, which depends on the air drag force. So we are going to end up with a differential equation relating the acceleration, $\mathrm{d}v/\mathrm{d}t$, to the speed v.



Formulate

Taking up to be positive, the acceleration is

$$a = -g - \frac{k}{m}v^2 ,$$

so our differential equation for v is

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g\left(1 + \frac{k}{mg}v^2\right) .$$

Solve

The maximum height is reached when $v = 0$, so we need to integrate our equation from $v_i = v_0$ to $v_f = 0$. We have

$$\int_{v_0}^{0}\frac{\mathrm{d}v}{1 + A^2v^2} = -g\int_0^{t_f}\mathrm{d}t ,$$

where $A^2 = k/mg$. To solve this we write

$$Av = \tan\theta$$
$$A\mathrm{d}v = \sec^2\theta\,\mathrm{d}\theta ,$$

and note that $\sec^2\theta = 1 + \tan^2\theta$ for any θ. The integral then reduces to

$$\int_{\theta_i}^{0}\frac{\mathrm{d}\theta}{A} = -\frac{\theta_i}{A} = -gt_f ,$$

6E.10, continued:

where $\theta_i = \tan^{-1}(Av_0)$. Solving for t_f,

$$t_f = \frac{1}{gA}\theta_i = \sqrt{\frac{m}{kg}}\tan^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right) \ ,$$

as required.

To find the corresponding height $y_{\rm max}$, we first write the solution of our integration in a more general form to get an equation for v at any time t:

$$\theta(t) - \theta_i = \tan^{-1}\left(v\sqrt{\frac{k}{mg}}\right) - \theta_i = -\sqrt{\frac{kg}{m}}\,t \ ,$$

i.e.

$$v(t) = \sqrt{\frac{mg}{k}}\tan\left(\theta_i - \sqrt{\frac{kg}{m}}\,t\right) \ ,$$

and then integrate this from $t = 0$ to $t = t_f$. To do this we need the integral of $\tan\theta$. This is most easily done by putting $u = \cos\theta$, $du = -\sin\theta d\theta$, from which

$$\int \tan\theta\, d\theta = \int \frac{\sin\theta}{\cos\theta} d\theta = -\int \frac{du}{u} = -\ln u = \ln(\sec\theta) \ .$$

Our integral can be cast into this form by putting

$$s = \theta_i - \sqrt{\frac{kg}{m}}\,t$$

$$ds = -\sqrt{\frac{kg}{m}}\,dt \ ,$$

from which

$$\int_0^{t_{\rm max}} v(t)\, dt = y_{\rm max} = -\frac{m}{k}\int_{s_0}^{s_f} \tan s\, ds = -\frac{m}{k}\left(\ln(\sec s_f) - \ln(\sec s_0)\right) \ .$$

Now

$$s_0 = \tan^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)$$

and

$$s_f = \tan^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right) - \sqrt{\frac{kg}{m}}\sqrt{\frac{m}{kg}}\tan^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right) = 0 \ ,$$

so

$$y_{\rm max} = \frac{m}{k}\ln\sqrt{1 + \frac{kv_0^2}{mg}} \ ,$$

6E.10, continued:

using the identity $\sec^2\theta = 1 + \tan^2\theta$.

Putting in the numbers gives

$$y_{\max} = \frac{1}{0.35\ \mathrm{m}^{-1}} \ln\sqrt{1 + (10\ \mathrm{m/s})^2 \frac{0.35\ \mathrm{m}^{-1}}{10\ \mathrm{m/s}^2}}$$

$$= (2.86\ \mathrm{m}) \times \ln(2.12) = 2.2\ \mathrm{m}\ .$$

Scrutinize

This is rather higher than the ball in Problem 6E.9, which started off with an identical v_y and had identical k/m. The reason is that the air drag on that projectile is larger because its ***overall*** speed is larger. The differential equations for its motion mix up x- and y-components, and are very awkward to solve—problems of this sort tend to be done numerically, by computer.

We can check that the hideous expressions we have derived do in fact reduce to the right forms when k/m is small by using various small-number approximations: for small ϵ

$$\sin\epsilon \approx \epsilon \qquad \sec\epsilon \approx 1 + \tfrac{1}{2}\epsilon^2$$
$$\tan\epsilon \approx \epsilon \qquad \ln(1+\epsilon) \approx \epsilon$$

and so

$$t_f \approx \sqrt{\frac{m}{gk}}\left(v_0\sqrt{\frac{k}{mg}}\right) = \frac{v_0}{g}$$

$$y_{\max} \approx \frac{m}{k}\ln\left(1 + \tfrac{1}{2}t_f^2\frac{kg}{m}\right) \approx \tfrac{1}{2}gt_f^2\,.$$

These are indeed the equations we expect.

Learn

Note that although the mathematics of this realistic problem was much more complicated than our idealized systems, the concepts and techniques involved were just the same. This is why we use idealized systems—they illustrate the physics without requiring the additional mathematical baggage.

HINTS FOR PROBLEMS WITH AN (H)

The number of the hint refers to the number of the problem

6A.2 What is the weight's motion if it is ***not*** sliding? What is its acceleration? What force produces this acceleration?

6C.2 Draw a force diagram for the block. What is the net force acting if it moves at constant speed?

What is the definition of work done by a force? What is the direction of the frictional force?

Still stuck? Study the solution to problem 6C.1.

6E.2 Draw a force diagram for the stone. What determines the frictional force? If you're still stuck, review the solution to problem 6E.1.

6E.5 What forces are acting on the block (a) before release; (b) after release? Draw a force diagram for the block at the instant of release. What force makes it possible that the block will remain stationary? What condition must hold if it does?

Draw another force diagram for the moving block. How does the net force depend on its position? What is the net force at the point where the block reaches its maximum speed?

6E.8 What are the horizontal (x) and vertical (y) components of the acceleration at time t?

To integrate the y equation you may find it useful to change variables from v_y to $w = g - \frac{k}{m}v_y$.

6F.1 (a) In an inertial frame of reference, what force is acting on the mass? How does the acceleration of the mass compare with the acceleration of the space probe?

(b) In the space probe's frame of reference, what is the net force on the mass M? What is the value of the fictitious force which must be introduced to achieve this?

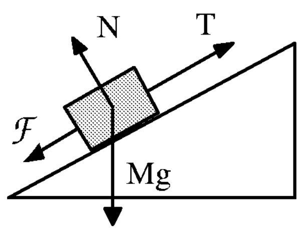
6F.2 (a) What is the fictitious force acting on the plumb bob? Draw a force diagram for the bob including the fictitious force.

(b) Define a coordinate system so that the y-axis is directed parallel to the string of the plumb bob in its equilibrium position. If the bob is displaced from equilibrium by a small angle $\Delta\theta$, what is the x-component of the force on the bob?

ANSWERS TO HINTS

6A.2 Circular motion at 33.3 rpm; v_2/r; static friction.

6C.2 Force diagram:



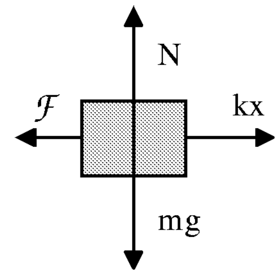
Zero.

$\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$;

opposite to direction of motion of block.

6E.2 Force diagram as for 6C.2. Size of static friction force is determined by $\mu_s N$ or size of opposing force, whichever is less; kinetic friction is $\mu_k N$. Here we want the point at which the block just starts to move, so static friction is $\mu_s N$.

6E.5 Vertically, gravity and normal force from table; horizontally, spring force and force applied by experimenter. After release static friction replaces the last of these.



Static friction; zero net force, i.e. $kx = \mathcal{F}_s$.

Diagram is the same for moving block, but kinetic friction replaces static friction; $F(x) = \mu_k mg - kx$ where x is measured from the point at which the spring is not stretched; zero.

6E.8 $\dfrac{dv_x}{dt} = -\dfrac{k}{m}v_x$; $\dfrac{dv_y}{dt} = -\left(g - \dfrac{k}{m}v_y\right)$

6F.1 (a) $|\vec{\mathbf{F}}| = k(\ell - \ell_0)$, directed along spring toward the fixed end; accelerations are equal.

(b) Net force = zero; $\vec{\mathbf{F}}_{\text{fict}}$ must oppose force of spring, so must have magnitude $k(\ell - \ell_0)$ and must point along spring, away from fixed end.

6F.2 (a) $\vec{\mathbf{F}}_{\text{fict}} = -M\vec{\mathbf{a}}$, where $\vec{\mathbf{a}}$ is the train's acceleration.

(b) $F_x \simeq -T\Delta\theta = -M\Delta\theta\sqrt{g^2 + a^2}$.

ANSWERS TO ALL PROBLEMS

6A.1 b; a

6A.2 0.12

6B.1 c

6B.2 See complete solution.

6B.3 An acceptable answer would be:

"Kinetic friction acts to oppose the relative motion of two surfaces which are already moving. It has a fixed magnitude of $\mu_k N$. Static friction acts to prevent relative motion of two surfaces at rest. It has whatever magnitude is necessary to do this, up to a maximum of $\mu_s N$."

6C.1 See complete solution.

6C.2 (i) $Mg(\sin\theta + \mu_k \cos\theta)$;

(ii) $Mgh(1 + \mu_k \cot\theta)$.

For a given μ_k you need minimal force for a slope with $\tan\theta = \mu_k$; but you do least work simply lifting the object vertically—the shallower the slope, the more work you have to do.

6D.1 (a) Initially, as long as $kv^2 \ll mg$, the effect of the air resistance is small, so the motion is close to a trajectory of uniform acceleration g downward. As kv^2 becomes comparable to mg, the magnitude of the downward acceleration decreases, and the projectile approaches a constant speed, called the ***terminal speed***.

(b) $v_{\text{terminal}} = \sqrt{\dfrac{mg}{k}}$

6D.2 (a) A parachute is basically a device to increase the surface area of a freely falling object (human being, crate of supplies, descending space-probe, etc.) while not greatly increasing its mass. Thus $\sqrt{mg/k}$ is greatly reduced, and the object has a much lower terminal speed (and hence a very much decreased risk of suffering damage on landing).

(b) Cats and humans have similar silhouettes and are made of similar materials, One would therefore expect that $k_{\text{cat}}/k_{\text{human}} \simeq A_{\text{cat}}/A_{\text{human}}$, where A is the surface area of the being in question. Similarly, the density of a cat should be similar to that of a human, so $m_{\text{cat}}/m_{\text{human}} \simeq V_{\text{cat}}/V_{\text{human}}$, where V is the volume. Surface area is basically ℓw, where ℓ is the length of the object and w is its width; volume is $\ell w t$, where t is the thickness. Therefore $(k/m)_{\text{cat}}/(k/m)_{\text{human}} \propto t_{\text{cat}}/t_{\text{human}}$, which is significantly less than one, and so the terminal speed of a human is higher than that of a cat. (This effect, namely that volume increases with increasing size of animal more rapidly than surface area does, is important in a wide range of biological issues: small animals lose heat more quickly (and therefore, if warm-blooded, need faster matabolisms), absorb gas more easily (insects don't need lungs), and are much more affected by surface forces (insects can walk on water; insects and small lizards can walk up walls); also, since muscle strength is proportional to cross-sectional area, small animals seem astonishingly strong, able to leap many times their own body length and so forth.)

(c) Galileo's cannonballs would presumably have had similar size and shape, and therefore similar k, but the lead ball would clearly be much heavier than the wooden ball. Hence the lead ball's terminal speed would be higher, and it would hit the ground first. This effect is probably what caused the ancient Greeks and other pre-Renaissance scientists/philosophers to believe that heavy objects fell faster than light ones.

6E.1 See complete solution.

6E.2 302; 62.

6E.3 If you lock the wheels of your car, the tire surfaces are sliding on the road surface, and therefore the force that is slowing you down is kinetic friction, $\mu_k Mg$, where M is the mass of your car. On wet or icy roads, μ_k may be rather small, and you will not stop very quickly; also, while the wheels are sliding, you have no control over either the rate of deceleration (pushing the brake down harder will not affect the forces acting at all), nor the direction of motion (turning the wheels will not affect the direction of the force, which simply opposes the relative motion of car and road). The goal, then, is to press on the brake as hard as one can without letting the car skid, so one can take advantage of the maximum force of static friction.

6E.4 $\tan\theta = \mu_s$; $\dfrac{g(\mu_s - \mu_k)}{\sqrt{1+\mu_s^2}}$

6E.5 (i) 1.75 cm; (ii) The block will initially accelerate, since $\mu_k < \mu_s$. As the spring extension reduces, the spring force will become smaller, while kinetic friction remains the same: thus at some point the net force will change sign, and the block will decelerate and eventually stop. At this point the spring compression will be less than the starting extension (since the frictional force has done negative work, and reduced the kinetic energy of the block more than the spring acting alone), and therefore the spring force will be insufficient to overcome static friction: once the block stops, it will remain stationary.

1 m/s^2 towards wall; 0.5 cm towards wall.

6E.6 See complete solution.

6E.7 See complete solution.

6E.8 $v_x = u\exp\left(-\dfrac{k}{m}t\right)\ ; \quad v_y = \dfrac{mg}{k}\left(1-\exp\left(-\dfrac{k}{m}t\right)\right)$

The ball's trajectory will tend towards a situation where it is traveling vertically downwards with speed mg/k.

6E.9 The trajectory is asymmetric, with steeper descent, not a parabola, because the horizontal component of velocity is no longer constant. In the absence of air resistance we would expect a range of 20 m (4 m in diagram) and maximum height of 5 m (1.7 m). Air resistance has had a dramatic effect on both, though range has been more affected than height.

In the absence of air resistance, the optimum 45° angle is the result of a tradeoff between the benefits of vertical velocity, which increases flight time, and horizontal velocity, which measures the rate of progress toward the destination. With air resistance, one can see from the diagram that the horizontal velocity is rapidly reduced (recall that the circles are evenly spaced in time), so most of the forward progress is made during the earlier part of the flight. The benefit of increased flight time is therefore reduced, so the tradeoff is skewed in favor of increased horizontal velocity. We conclude that the range is likely to be maximized by aiming the projectile at ***less than 45°*** to the horizontal—option (b). (In fact, numerical simulation indicates that the range is maximal for $\theta \simeq 35^\circ$. It should be pointed out, however, that the projectile described here has an unusually high air drag. Its terminal speed (see Problem 6D.1) is 5.3 m/s, slightly slower than the descent rate of a standard military parachute.)

6E.10 See complete solution.

6F.1 $|\vec{\mathbf{a}}| = \dfrac{k}{M}(\ell - \ell_0)$, directed along spring toward the fixed end.

See the hints to follow the two alternative arguments.

6F.2 (a) $\theta = \tan^{-1}\left(\dfrac{a}{g}\right)$.

(b) $T = 2\pi\sqrt{\dfrac{\ell}{\sqrt{g^2 + a^2}}}$.