ESSENTIALS OF INTRODUCTORY CLASSICAL MECHANICS

Sixth Edition

CHAPTER 7

ENERGY, MOMENTUM AND MACROSCOPIC FORCES

OVERVIEW

This chapter contains no new ideas. Instead, we take the material covered in Chapters 4, 5 and 6 and use it to solve more complicated problems. Notice that in this chapter we will not divide the problems up into sections dealing with specific topics; instead you will have to decide for yourself which of the physical principles you have learned are relevant to a given problem.

When you have completed this chapter you should:

- ✔ be able to extract the essential features of a problem and express them in mathematical equations;
- ✔ be capable of manipulating the equations relevant to a problem to obtain an expression for the required quantity, either symbolically or numerically;
- ✔ be able to analyze a hypothetical situation in terms of the physics presented in previous units, and explain that analysis clearly in non-mathematical terms.

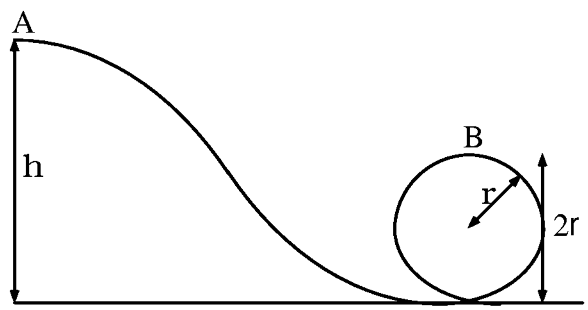
PROBLEMS AND QUESTIONS

By the end of this chapter you should be able to answer or solve the types of questions or problems stated below.

At the end of the chapter there are answers to all the problems. In addition, for problems with an (H) or (S) after the number, there are respectively hints on how to solve the problems or completely worked-out solutions.

7.1 (H) The cars on a rollercoaster ride at a fairground start from rest a height h above ground level, descend to ground level and then execute an essentially circular loop of radius r.

(a) At point B, the top of the loop, passengers feel a sensation of "weight" acting upwards, i.e. towards the outside of the loop. Explain, in terms of the forces acting, the origin of this feeling.



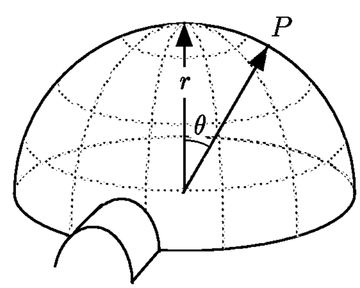
(b) If the "weight" sensation felt by a passenger of mass m is half her normal weight, calculate, in terms of g and r, (i) the speed of the car at point B and (ii) the height h of the starting point A.

7.2 What is the potential energy of the Earth due to its position in the gravitational field of the Sun, if we define the potential energy such that it would be zero if the Earth were infinitely far from the Sun? What is the Earth's kinetic energy if we treat it as a point mass (i.e. neglect all effects of its rotation)? With the same assumption, what is the Earth's total energy? Comment on your result.

[The mass of the Sun is 2×10^{30} kg, that of the Earth is 6×10^{24} kg, and the distance between them is 150 million kilometers. The gravitational constant $G = 6.67 \times 10^{-11}$ N$\cdot$m^2/kg^2.]

7.3 (H) An Eskimo child is using her parents' hemispherical igloo as a slide. She starts off from rest at the top and slides down under the influence of gravity. The surface of the igloo is effectively frictionless.

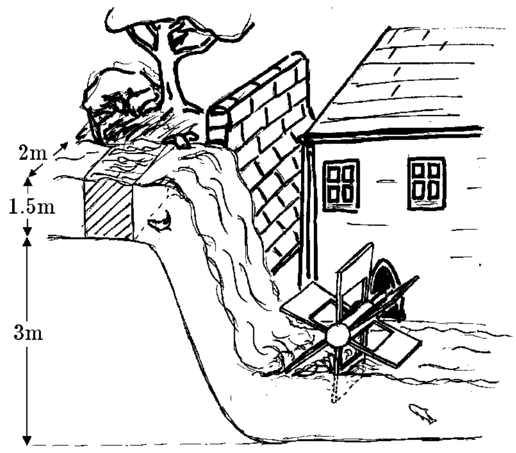
(a) What is her potential energy at point P (see diagram)? Define the potential energy so that it is zero when the child is on the ground.



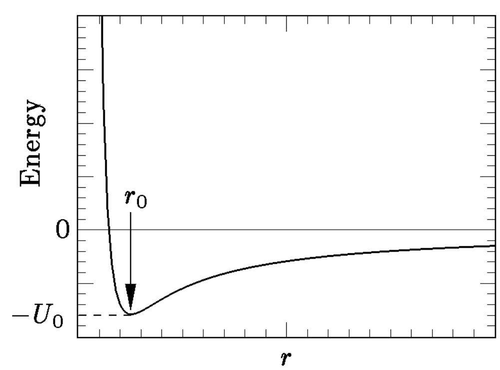
(b) Draw a force diagram for her at point P.

(c) Does she remain in contact with the igloo all the way to the ground? If not, at what angle θ does she lose contact?

7.4 (S) Prior to the Industrial Revolution, waterwheels were commonly used to power machines (particularly mills for grinding flour). Suppose that a waterwheel is driven by a stream which is 2 m wide and 1.5 m deep and which flows at 1 m/s. The stream is made to flow over a weir with a vertical fall of 3 m immediately before striking the wheel. After the weir, the stream is observed to continue with the same width and depth as before the weir. Explain what is happening to the energy of the water and the wheel, and calculate how much energy per second is available to power the mill. The density of water is 1000 kg/m^3.



7.5 (S) The potential energy of an atom bound in a molecule or crystalline solid is given by a function with the general shape shown in the diagram. The zero is chosen so that an atom infinitely distant from the other atoms making up the compound would have zero potential energy.



(a) What force acts on an atom located (i) closer than r_0; (ii) at r_0; (iii) beyond r_0?

(b) What would happen to an atom initially at r_0 if it had a small kinetic energy K? What if it had a large kinetic energy $K > U_0$?

7.6 (H) A man tows his daughter on a sled on level ice, and she in turn tows her teddy-bear behind her on a toy sled. The girl and her sled have a combined mass M; Teddy and its sled have a mass m. Dad's tow-rope is inclined at an angle of θ to the horizontal, while the rope joining the two sleds is horizontal. Friction between the sled runners and the ice is negligible, as is the mass of each rope.

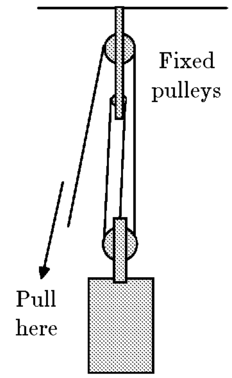
(a) Draw free-body diagrams for each sled. Which forces are Third Law pairs?

(b) Derive expressions for the tension in each rope when the acceleration of the sleds is a (obviously directed forwards). If $M = 30$ kg, $m = 8$ kg, $\theta = 30°$ and the sleds accelerate at 1 m/s^2, what is the magnitude of each tension?

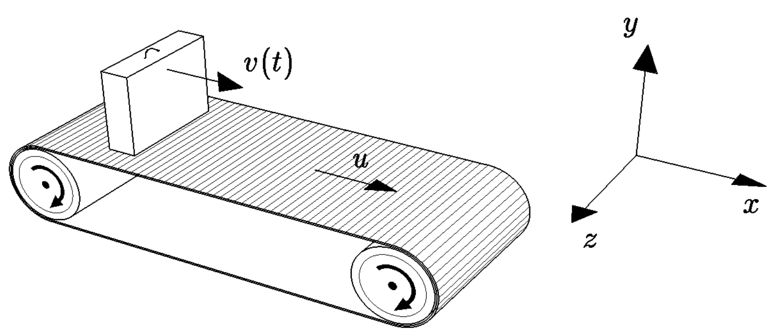
(c) Suppose that there is in fact some non-zero coefficient of kinetic friction μ_k between the sleds and the ice. In terms of m, M, g, $\cos\theta$ and μ_k, what tension does Dad have to exert to keep the sleds moving at constant velocity?

7.7 (S) A common piece of lifting gear is the ***block and tackle***, consisting of a system of pulleys arranged as in the schematic diagram on the right. Assuming that the pulleys are frictionless and that the angle the rope makes with the vertical is always negligible, what force must you apply, and what work must you do, to lift a load of mass $M = 30$ kg a vertical distance of 1 m? Take $g = 10$ m/s^2, and treat the rope as massless.

Why is it easier to lift a heavy object using this device?



7.8 A suitcase of mass M is placed on a level conveyor belt at an airport. The coefficient of static friction between the suitcase and the conveyor belt is μ_s, and the coefficient of kinetic friction is μ_k, with $\mu_k < \mu_s$.



The conveyor belt moves with constant speed u, and at time $t = 0$ the suitcase is placed on the conveyor with speed $v = 0$. At $t = 0$, what is the total force $\vec{\mathbf{F}}$ acting on the suitcase? How long does the suitcase take to reach the speed of the conveyor belt (i.e. at what time t does $v(t) = u$)? What is the work done by friction on the suitcase during this time? Comment briefly on the direction of the frictional force and the sign (positive or negative) of the work done. After the suitcase reaches the speed of the conveyor belt, what is the force of friction that acts on it?

7.9 The level conveyor belt of Problem 7.8 is part of an airport baggage delivery system. The bags are delivered onto the level belt by an inclined conveyor belt which lifts them from the tarmac to the baggage collection region. This 'delivery' conveyor is inclined at an angle θ to the horizontal, and moves with constant speed $u_0 < u$.

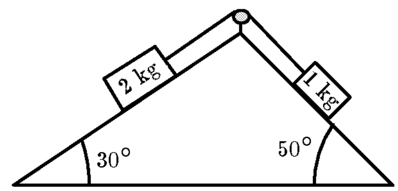
(a) What is the minimum coefficient of static friction between bags and delivery conveyor needed to ensure that bags do not slip as they are delivered to the collection hall? In which direction does the frictional force act?

(b) The delivery conveyor is oriented at right angles to the level conveyor, so our suitcase of mass M lands on the level conveyor with velocity $\vec{\mathbf{v}}(0) = [0, 0, u_0 \cos\theta]$. What is the frictional force $\vec{\mathbf{F}}$ acting on the suitcase just after it lands. At what time t does $v_x(t) = u$, and how far across the conveyor (in the z direction) has the suitcase traveled at that point?

(c) Some way along the level conveyor, maintenance men working on an overhead light fixture have erected a scaffold which passes over the top of the belt. The strap of a hiker's backpack has caught on the scaffold, causing it to get firmly stuck. Our suitcase collides with the backpack and gets jammed behind it, so that both are now stationary with respect to the airport. What contact force is being exerted by the backpack on the suitcase? If the backpack has mass M_2, what is the magnitude of the horizontal component of the tension in the strap that is caught on the overhead scaffold?

7.10 (S) Two masses m and M are connected by a massless rope which passes over a frictionless pulley. If $M > m$ and the rope does not stretch, what is the acceleration of the mass M?

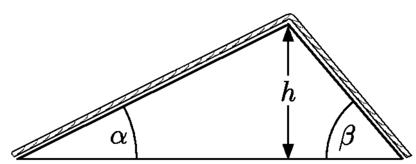
7.11 (S) (a) Two blocks of mass 1 kg and 2 kg are connected by a light string passing over a pulley as shown. Assuming that there is no friction anywhere in the system, what is the acceleration of the blocks? Take $g = 10\ \mathrm{m/s^2}$.



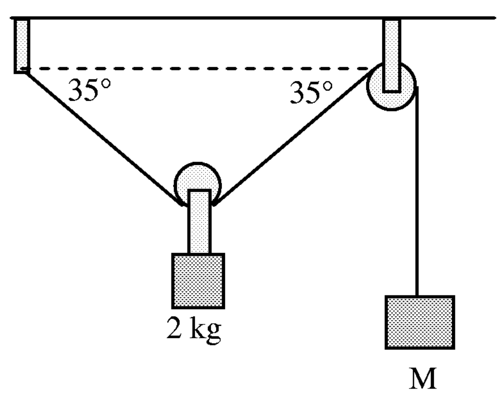
(b) Assuming that the pulley remains frictionless, what is the minimum coefficient of static friction between blocks and slope required to allow the blocks in part (a) to remain motionless? Suppose that the coefficient of static friction has this value, but the coefficient of kinetic friction is smaller. What would happen if the blocks were started in motion sliding towards the left (i.e., downhill for the block on the left, uphill for the block on the right)?

(c) Suppose that the coefficient of static friction between blocks and slope is 0.12. Is it possible for the blocks described in part (b) to be stationary? If so, what is the range of possible values for the tension in the string?

(d) Now suppose that a heavy uniform rope (i.e. a rope with constant mass per length) is laid over a frictionless triangular block as shown. Assume that the peak of the triangle behaves as a frictionless pulley. Will the rope slip?



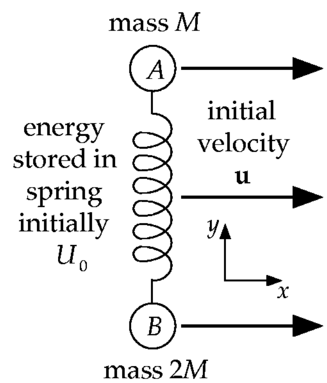
7.12 (H) The setup in the diagram uses massless, frictionless pulleys and a rope of negligible mass. What mass M is required to balance the 2 kg mass, so that if the masses are initially stationary they will remain so? Describe qualitatively what will happen if the mass M is then given a small downward impulse.



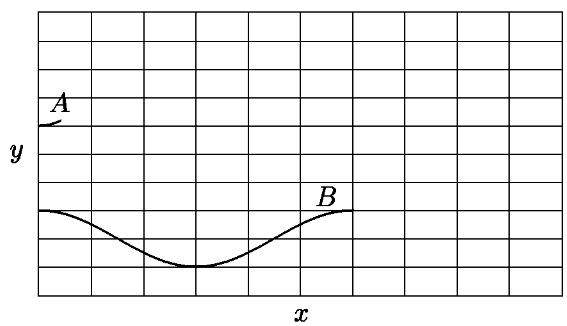
7.13 (H) A rifle of mass 10 kg fires a 10 g bullet into a 3 kg block of wood which is suspended by a thin wire, forming a ***ballistic pendulum***. The bullet remains embedded in the block of wood, which is observed to swing so that it reaches a vertical height of 20 cm above its starting point. What was the speed of the bullet, and with what speed did the rifle recoil?

Ignore air resistance, friction in the suspension, and the vertical motion of the bullet, and assume that the block of wood is small relative to the length of the wire (we will see why this is necessary in Chapter 8).

7.14 Two masses A and B are connected by a spring. A has mass M, B has mass $2M$, and the spring has negligible mass. The spring is compressed so that the potential energy stored in it is U_0. The system is placed on a level horizontal air table (which provides a frictionless surface), given a velocity $\vec{\mathbf{u}}$ in the x-direction as shown, and the spring is released. In the subsequent motion the line BA always points in the positive y-direction.



(a) The diagram shows, to scale, the initial position of A and the path followed by B over a certain length of time. Make a similar drawing which also includes the path followed by A. Mark on your drawing the initial position and subsequent motion of the center of mass of the system.

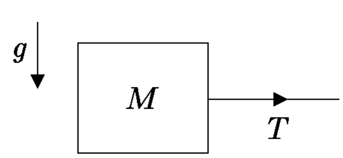


(b) In terms of the given quantities, what is the initial total energy of the system (A, B and the spring)?

(c) How much potential energy is stored in the spring when A and B have kinetic energies K_A and K_B respectively?

(d) At some point in the motion, the y-component of the velocity of mass B is v_y. Write down the velocity vector $\vec{\mathbf{v}}_A$ of A at that instant, expressing your answer in terms of v_y and the quantities given above.

7.15 (H) A block of mass M rests on a horizontal surface. The coefficient of kinetic friction between the block and the surface is μ_k, and the coefficient of static friction is μ_s, with $\mu_s > \mu_k$. The block is pulled horizontally by a massless inextensible rope, with a tension T that is gradually increased until the block starts to slide.

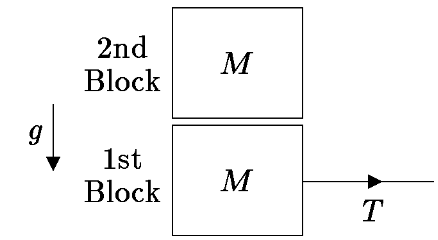


(a) What is the value of the tension T_1 at which the block begins to slide?

(b) When the tension was only $\frac{1}{2}T_1$, before the block began to slide, what was the magnitude and direction of the force of friction?

(c) If the tension is maintained at the value T_1, what is the acceleration of the block?

(d) A second block, identical to the first, is placed directly on top of the first block while both are at rest. The coefficients of friction between the two blocks are $\bar{\mu}_k$ for kinetic friction and $\bar{\mu}_s$ for static friction. As before, a horizontal rope is attached to the first (lower) block, and the tension in the rope is increased gradually from zero. At some value of the tension the two blocks begin to move, but there is initially no relative velocity between the two. As the steady increase in the tension is maintained, they accelerate faster and faster. At what value of the tension will the second block begin to slip relative to the first block? In what direction will it slip, relative to the block below?



7.16 (S) A rocket is a 'Newton's Third Law machine'—it operates by ejecting a high velocity exhaust at one end. Suppose that a rocket with initial mass M_i burns its fuel at a constant rate dm/dt and expels the combustion products with speed u relative to itself. By how much has the rocket's speed increased when its mass has decreased to M_f (the difference having been ejected as exhaust)? (Assume the rocket is in interstellar space, with no other forces acting.)

A deep space probe will be placed in low Earth orbit by a shuttle launch, and will then fire its own booster rocket to leave orbit. You are in charge of designing the booster. If the probe's mass is 500 kg and it needs a velocity change of 5 km/s, what minimum mass of fuel will you need to specify if the exhaust velocity is to be 2500 m/s?

7.17 A spaceship is stationary in outer space, far away from any matter. It is facing a very distant star. At some instant it starts its rocket engines. The hot gases are ejected from the engines with speed v, relative to the spaceship. After some time the spaceship attains a speed greater than v. From that time on, in which direction is the hot gas ejected from the engines moving? In which direction is the overall center of mass of the spaceship and all ejected gas moving? Explain.

Explain also why you would expect the speed v of the ejected gases to be approximately constant relative to the engines, and not relative to the fixed frame of reference.

7.18 (H) A train moving in the x-direction is uniformly accelerating with acceleration $\vec{\mathbf{a}} = [a, 0, 0]$. In one of the train carriages, Alice and Betty stand opposite one another, a distance w apart: in Alice's frame of reference they have coordinates $[0, 0, 0]$ and $[0, w, 0]$ respectively (the z axis points vertically up). Alice throws a ball across the carriage to Betty.

(a) If Alice throws the ball at an angle θ to the horizontal with initial speed v, what must the initial velocity vector $\vec{\mathbf{v}}$ be, in Alice's frame of reference, if the ball is to land precisely in Betty's hands (i.e. the initial and final positions of the ball are the coordinates of Alice and Betty respectively)?

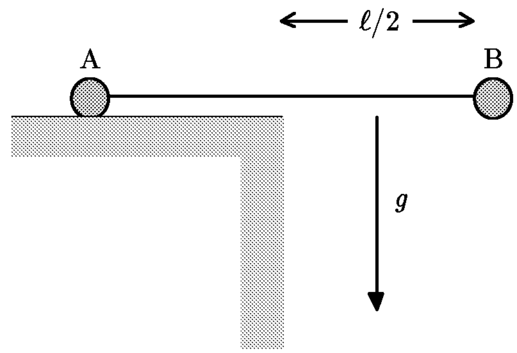
(b) If $\theta = 45°$, at what speed v should Alice throw the ball, and how long will its flight last? Express your answers in terms of g, a and w.

7.19 (H) Two identical small balls, A and B, are connected by a string of length ℓ and negligible mass. Ball A is placed on a frictionless table (with a frictionless edge), and B is held a distance $\ell/2$ from the edge of the table so that the string is horizontal and just taut.

If mass B is now released, will it hit the side of the table before A falls off the edge, or vice versa?

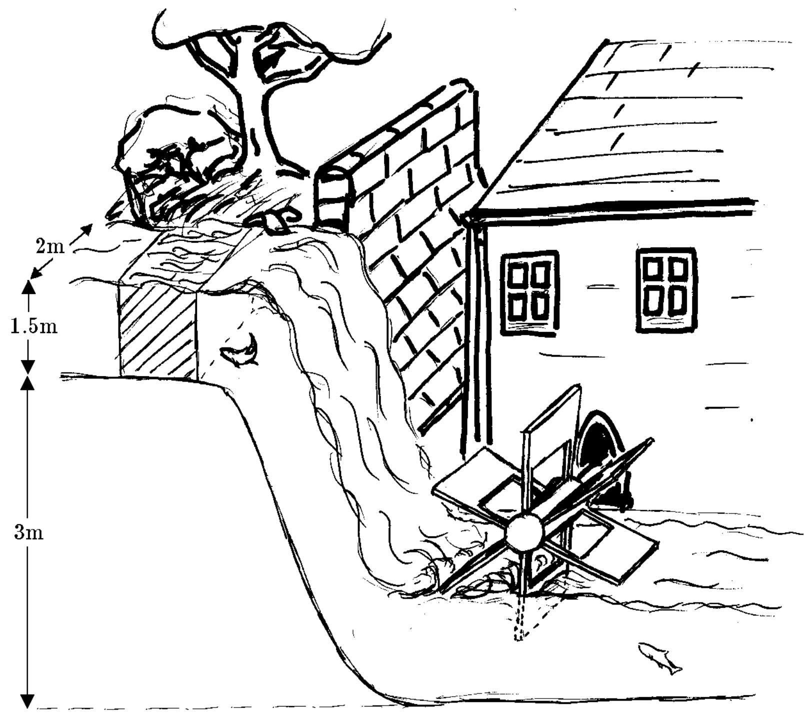
Explain your reasoning clearly!

Note: This problem can be solved by reasoning. It is ***not*** necessary to solve the equations of motion explicitly.



COMPLETE SOLUTIONS TO PROBLEMS WITH AN (S)

7.4 *Prior to the Industrial Revolution, waterwheels were commonly used to power machines (particularly mills for grinding flour). Suppose that a waterwheel is driven by a stream which is* 2 m *wide,* 1.5 m *deep and flows at* 1 m/s. *The stream is made to flow over a weir with a vertical fall of* 3 m *immediately before striking the wheel. After the weir, the stream is observed to continue with the same width and depth as before the weir. Explain what is happening to the energy of the water and the wheel, and calculate how much energy per second is available to power the mill. The density of water is* 1000 kg/m^3.



Conceptualize

We can look at this problem in two ways: by considering the work done on and by the water as it flows over the weir and past the waterwheel, or in terms of energy conservation in the system consisting of water, wheel and Earth. In terms of energy conservation, the water above the waterfall has greater gravitational potential energy in the water-Earth system than does the same amount of water in the pool at the bottom of the waterfall. However, in steady-state the water has to flow out of the millpool at the same rate as it flows in, since the millpool can neither produce water nor cause it to disappear. Since the dimensions of the steam after the weir are the same as before, the water must flow at the same speed. Therefore the loss in potential energy is not balanced by a gain in kinetic energy ***of the water***, so it must be balanced by a gain in energy of the waterwheel. In terms of work done on and by the water, the same series of events can be described as follows:

- Work is done by gravity on the water as it descends the waterfall. Its kinetic energy at the bottom of the fall has increased by an amount corresponding to the work done.

7.4, continued:

- When the water leaves the pool, its kinetic energy has decreased back to its initial value. Therefore work must have been done by the water on some intervening object. The only obvious candidate object is the waterwheel.

These two pictures are entirely equivalent, since the work done by gravity on the water is, by definition, equal to the loss of gravitational potential energy from the water-Earth system. This amount of energy is transferred from the water-Earth to the waterwheel, and within the mill the kinetic energy of the wheel is used to drive the mill mechanism by processes too complicated for us to deal with them at this point.

Formulate and Solve

The amount of water passing over the weir in one second is $(2\,\text{m})\times(1.5\,\text{m})\times(1\,\text{m}) = 3\,\text{m}^3$ (for example, the shaded area in the figure). Its mass is 3000 kg, so its kinetic energy at this point is $\frac{1}{2}mv^2$, or 1.5 kJ. If we treat this water as a point mass located at the position of its center of mass, it also has potential energy $mgh = 88.2$ kJ, compared to a similar mass of water—also considered as a point mass—in the millpond at the bottom of the weir. As the water descends the weir this potential energy is converted to kinetic energy, so at the bottom of the weir its kinetic energy is 89.7 kJ. When the water leaves the millpond it carries with it its original 1.5 kJ of kinetic energy, so the energy available to power the mill, rounded to two significant figures, is 88 kJ/s, or 88 kW.

Scrutinize

If we consider the work done on the water by gravity instead of the potential energy lost, we of course obtain the same result: $(mg)\cdot(h) = mgh$.

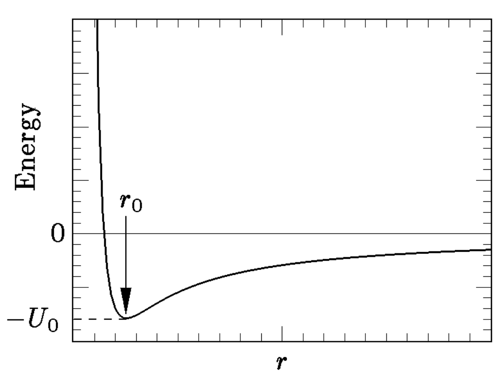
Learn

In practice the mill will not be able to use all the 88 kJ of energy lost by the water. Much of this will be 'wasted' by being transformed into forms such as heat which do not contribute to the turning of the mill.

7.5 *The potential energy of an atom bound in a molecule or crystalline solid is given by a function with the general shape shown in the diagram on the next page. The zero is chosen so that an atom infinitely far away from the other atoms making up the compound would have zero potential energy.*

(a) *What force acts on an atom located (i) closer than r_0; (ii) at r_0; (iii) beyond r_0?*

(b) *What would happen to an atom initially at r_0 if it had a small kinetic energy K? What if it had a large kinetic energy $K > U_0$?*



Conceptualize

We have been given a potential energy and asked for (a) a force and (b) the motion of a particle of a given kinetic energy. We can deal with part (a) by recalling that the force is simply $(-1)\times$ the derivative of the corresponding potential energy with respect to r. We do not have the exact functional form of the potential, so we cannot solve this in a quantitative fashion, but we can deduce the

7.5, continued:

direction of the force by considering the ***sign*** of the slope of U, and estimate its ***magnitude*** by looking at the ***steepness*** of the slope.

Part (b) is an energy conservation problem: we solve it by using the fact that the particle's total energy must remain constant to deduce how its kinetic energy varies with r.

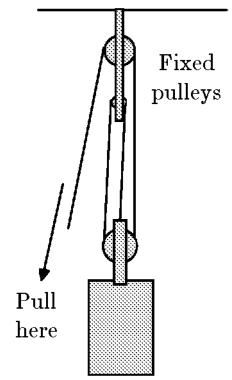
Solve (a)
The slope of $U(r)$ is negative for $r < r_0$, positive for $r > r_0$, and zero at r_0. The corresponding force is large and positive for $r < r_0$, smaller and negative for $r > r_0$ where the slope is less steep, and zero at r_0. In each case, therefore, the force tends to accelerate the atom towards r_0, which is a position of stable equilibrium.

Solve (b)
An atom with a small kinetic energy K would move outwards or inwards from r_0, gaining potential energy and losing kinetic energy as it did so. Its total energy remains constant at $K - U_0$, and it will therefore have zero kinetic energy when its potential energy is equal to $K - U_0$. At this point there will be a force acting on it which will cause it to accelerate back towards r_0. As a result the atom oscillates about the equilibrium position. If the kinetic energy is small enough that the maximum displacement from r_0 is a small fraction of r_0, the atom will perform simple harmonic motion (it will be possible to approximate the shape of the potential energy curve by a quadratic); if it is larger, the motion will still be oscillatory, but its mathematical form will be more complicated.

If the atom's kinetic energy is larger than U_0, and it is moving outwards (towards larger r) it will never reach a point where its potential energy is $K - U_0$, and therefore it will never stop. If it is initially moving inwards, it will reach a point where it has zero kinetic energy, and will then turn round under the influence of the force acting on it and move outwards. Since its total energy is unchanged, it will then be able to move outwards to infinite r. We see that regardless of its initial direction, the atom will eventually wind up infinitely far from the rest of its molecule or solid—it is no longer part of the compound.

Scrutinize and Learn
This is a simplified (we have ignored quantum mechanical effects and reduced the problem to one dimension) but essentially correct analysis of the behavior of bound atoms. We will see later that kinetic energy can be supplied to atoms by heating the material: the escape of our atom when it gains enough kinetic energy thus corresponds to the melting of a solid or the thermal breaking up of a molecule. We could have performed a similar analysis for gravitationally bound systems: a deep-space probe like Pioneer or Voyager has enough kinetic energy to escape from the solar system, whereas a communications satellite does not. (Since the gravitational force does not change sign, there is no equivalent of r_0 in this system.) In general, a particle moving in a particular potential energy is bound (restricted to a limited range of position) if its total energy is less than the maximum value of the potential energy in all directions.

7.7 *A common piece of lifting gear is the* **block and tackle**, *consisting of a system of pulleys arranged as in the schematic diagram on the right. Assuming that the pulleys are frictionless and that the angle the rope makes with the vertical is always negligible, what force must you apply, and what work must you do, to lift a load of mass* $M = 30$ kg *a vertical distance of* 1 m*? Take* $g = 10$ m/s^2*, and treat the rope as massless.*



Why is it easier to lift a heavy object using this device?

Conceptualize

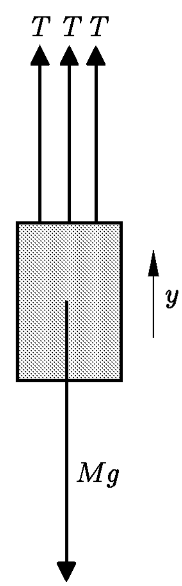
From the schematic, we have three segments of rope exerting an upward force on the block—two on the pulley and one on the pulley frame. You apply a force to a fourth segment of rope, that leading to the upper pulley. To solve this problem we need to relate the force you exert on the rope to the forces the various bits of rope exert on the block.

We stated in Chapter 2, and proved in Problem 5A.5, that the tension along a massless rope is constant. By Newton's third law, the tension T in the segment of rope you pull on is equal to the force you exert on the rope. The tension in the rope at the upper pulley is therefore T. As the rope goes round the pulley, the pulley exerts a normal force perpendicular to the surface of the pulley, and therefore perpendicular to the line of the rope. Since it is a ***frictionless*** pulley, it does not exert any force ***along*** the line of the rope, and so it does not change the magnitude of T (the tension is always directed along the line of the rope). Therefore the magnitude of the tension at the other side of the pulley is still T, although the direction has changed The same logic applies to the other two pulleys of the block and tackle, so we conclude that all three tension forces applied to the load M are equal to T.

Formulate

If the dimensions of the pulleys are small compared to the distance between the upper and lower parts of the block and tackle, all the rope segments will be essentially vertical, and so the problem reduces to one dimension. The force diagram for the load is as shown. The minimum force required to lift the mass just balances the gravitational force on the mass:

$$Mg = 3T.$$



So the force that you must apply is $T = \frac{1}{3}Mg$.

The work done on the block by the rope in raising it a distance s is the dot product of the force exerted by the rope, $[0, 3T, 0]$, and the displacement of the block, $[0, s, 0]$. Since these are parallel, the work done is simply the product of their magnitudes, $3Ts = Mgs$.

Alternatively, one can directly calculate the work that you do on the rope. In raising the block a distance s, each of the three rope segments leading to the block are shortened by a distance s, so the rope end that you are holding moves a distance $3s$. Since you are applying a force of magnitude $\frac{1}{3}Mg$ in the same direction, the work that you do is Mgs.

7.7, continued:

Solve

Putting in the numbers, to lift a 30 kg load through 1 m (at constant speed) you would need to exert a force of 100 N and do 300 J of work.

If you had simply lifted the mass, you would have had to exert a force $Mg = 300$ N, and you would have done work $Mgs = (300\,\mathrm{N}) \times (1\,\mathrm{m}) = 300$ J. The block and tackle does not decrease the work you have to do (as physicists, we would be extremely surprised if it did, since this would violate energy conservation!), but it does decrease the force you must exert. It is physiologically easier to exert a small force over a long distance than a large force over a short distance, so you find it easier to lift the load using the block and tackle.

Scrutinize

In terms of work and energy, you have done $3Ts$ of work on the rope, and the rope has done $3Ts$ of work on the mass. The mass's kinetic energy has not changed, because no net work has been done on it (gravity, acting downwards, has done work $-3Ts$). Looking at the mass-Earth system, $3Ts$ of work has been done by an external force (you), and the mass's gravitational potential energy has increased by $Mgs = 3Ts$. Energy conservation is thus satisfied in this problem.

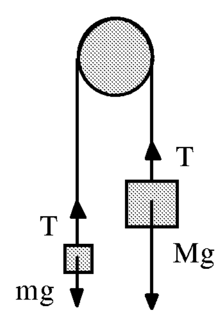
Learn

As long as there is no friction in the system, the magnitude of the tension along a massless rope is constant. The direction of the tension is always along the direction of the rope.

7.10 *Two masses m and M are connected by a massless rope which passes over a frictionless pulley. If $M > m$ and the rope does not stretch, what is the acceleration of the mass M?*

Conceptualize

The force diagram for this problem is shown on the right. As this is another massless rope and frictionless pulley, the two tension forces are equal in magnitude (and in this case also in direction). Since all the forces act vertically, this is essentially a one-dimensional problem.



Formulate

We treat the two masses as separate problems, linked by the facts that (i) the tension in the rope is the same for both, because the rope is massless, and (ii) the magnitude of the acceleration is the same for both, because the rope does not stretch. Taking up to be positive, the net force F on the large mass M is

$$F = T - Mg,$$

and by Newton's second law $F = Ma$, so

$$Ma = T - Mg.$$

7.10, continued:

Similarly, for the net force f on the small mass m

$$f = T - mg = ma'.$$

Solve

Since the rope is of fixed length, the acceleration of the small mass must be equal in size to the acceleration of the large mass, but opposite in direction (if one goes up, the other goes down), i.e. $a' = -a$. Hence we can eliminate T by subtracting our two equations:

$$F - f = (m + M)a = (m - M)g.$$

The acceleration is

$$a = -\frac{M - m}{M + m}g,$$

where the minus sign indicates (since we took up to be positive) that the large mass is accelerating downwards.

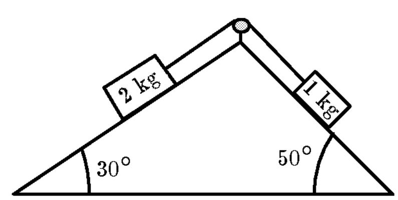
Scrutinize

Does our result make sense? We can conduct a mental check by looking at some special values. If $m = 0$, then $a = -g$, which is obviously right (the large mass is then a freely falling body). If $m = M$, there is no acceleration, which is sensible enough, and if $m > M$, the acceleration is positive, which means M is going up and m is going down, as we would expect.

Learn

Notice that if we make m and M nearly equal, a will be very small. Provided we can really make the effects of friction unimportant, this would be a good way to measure the value of g without needing accurate timing equipment—for instance, if we choose $M = 1$ kg and $m = 0.99$ kg, the acceleration will be only about 5 cm/s^2, and the mass will take more than 6 s to fall one meter. In fact this type of experimental arrangement, called an Atwood's machine, has indeed been used to make accurate measurements of g.

7.11 (a) *Two blocks of mass* 1 kg *and* 2 kg *are connected by a light string passing over a pulley as shown. Assuming that there is no friction anywhere in the system, what is the acceleration of the blocks? Take* $g = 10$ m/s^2.



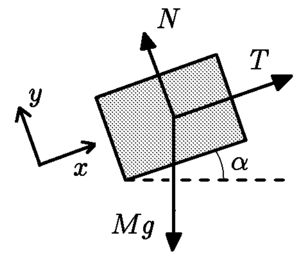
Conceptualize

This is a problem very much like 5A.7. We saw in that problem that the action of the frictionless pulley is to change the direction of the string without changing the magnitude of its tension. Because the string does not stretch, the magnitude of the acceleration is also the same for each block.

7.11, continued:

Formulate

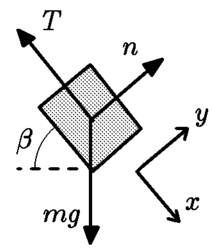
For the $M \equiv 2$ kg block we define a coordinate system with the x-axis pointing up the slope. In these coordinates the x- and y-components of the net force are



$$F_x = T - Mg\sin\alpha \qquad (1)$$

$$F_y = N - Mg\cos\alpha\ , \qquad (2)$$

where $\alpha = 30°$. The y-component must be zero, because we know the block will not accelerate perpendicular to the slope.



For the $m \equiv 1$ kg block we choose a coordinate system with the x-axis pointing down the slope. Notice that since we are treating the blocks as separate entities, there is no requirement to choose the same coordinate system for both blocks; for convenience, however, we have chosen the x directions so that a positive acceleration for one block corresponds to a positive acceleration for the other. With this choice of coordinates, the components of the net force are

$$f_x = mg\sin\beta - T \qquad (3)$$

$$f_y = n - mg\cos\beta\ , \qquad (4)$$

where $\beta = 50°$. Again, the y-component is zero. Applying Newton's second law for each block, we can write the equations of motion as

$$Ma = T - Mg\sin\alpha \qquad (1')$$

$$0 = N - Mg\cos\alpha \qquad (2')$$

$$ma = mg\sin\beta - T \qquad (3')$$

$$0 = n - mg\cos\beta\ , \qquad (4')$$

where a denotes the x-component of the acceleration, which must be the same for both blocks, since the string has constant length. Thus we have four equations for the four unknowns, a, T, N, and n. Because we don't need to know N or n, we can forget the y equations $(2')$ and $(4')$, leaving two equations in two unknowns.

Solve

Adding the x-component equations $(1')$ and $(3')$, one finds

$$(M + m)\,a = mg\sin\beta - Mg\sin\alpha\ . \qquad (5)$$

The acceleration is therefore

$$a = \frac{mg\sin\beta - Mg\sin\alpha}{M+m} = \frac{(1\ \text{kg})\sin 50° - (2\ \text{kg})\sin 30°}{(3\ \text{kg})} \times (10\ \text{m/s}^2) = -0.78\ \text{m/s}^2.$$

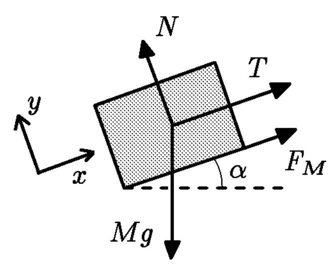
7.11, continued:

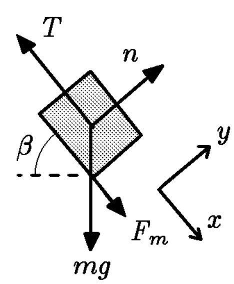
This is negative, so the acceleration is directed in the negative x-direction: downhill for M, uphill for m.

Scrutinize
The dimensions of our final expression for acceleration are correct, since the numerator has the dimensions of mass times acceleration, and the denominator has the dimension of mass. The answer can be checked for special cases: if we put either mass equal to zero, the other has acceleration $g \sin\beta$ (or $-g\sin\alpha$), as we would expect for a free body sliding down an inclined plane.

(b) *Assuming that the pulley remains frictionless, what is the minimum coefficient of static friction between blocks and slope required to allow the blocks in part (a) to remain motionless? Suppose that the coefficient of static friction has this value, but the coefficient of kinetic friction is smaller. What would happen if the blocks were started in motion sliding towards the left (i.e., downhill for the block on the left, uphill for the block on the right)?*

Conceptualize
The situation is essentially as in part (a), but we must add the frictional forces to our free-body diagrams. In the absence of friction, we saw that the blocks move in the negative x-direction, i.e. the 2 kg block moves downhill. The frictional forces will both act to oppose this motion, pointing uphill for the 2 kg block and downhill for the 1 kg block. Because we are dealing with static friction, we do not immediately know the magnitudes of the frictional forces. However, since we are looking for the minimum value of μ_s, we can expect that at least one frictional force will have its maximum magnitude of $\mu_s N$.





Formulate
The x-component equations (1′) and (3′) become

$$Ma = T - Mg\sin\alpha + F_M \tag{6}$$

$$ma = mg\sin\beta - T + F_m\ , \tag{7}$$

where F_M and F_m are the x-components of the frictional forces on the two blocks. Since we seek a solution for which the blocks are not sliding, we can set $a = 0$. We can then add the two equations to obtain

$$F_M + F_m = Mg\sin\alpha - mg\sin\beta\ . \tag{8}$$

Solve
In general the force of static friction obeys the inequality $|\vec{\mathbf{F}}_s| \le \mu_s|\vec{\mathbf{N}}|$, so in this case

$$|F_M| \le \mu_s N = \mu_s Mg\cos\alpha \tag{9}$$

$$|F_m| \le \mu_s n = \mu_s mg\cos\beta\ , \tag{10}$$

7.11, continued:

where we have obtained the values of N and n from the y-component equations $(2')$ and $(4')$. If we insert these inequalities for F_M and F_m into Eq. (8), we obtain

$$|Mg\sin\alpha - mg\sin\beta| = |F_M + F_m| \le |F_M| + |F_m| \le \mu_s(Mg\cos\alpha + mg\cos\beta)\ ,$$

so

$$\mu_s \ge \frac{|M\sin\alpha - m\sin\beta|}{M\cos\alpha + m\cos\beta}\ .$$

We were asked to find the ***minimal*** value for μ_s which allows a static configuration, so that question is answered by

$$\mu_s\big|_{\min} = \frac{|M\sin\alpha - m\sin\beta|}{M\cos\alpha + m\cos\beta}\ . \tag{11}$$

Putting in the numbers,

$$\mu_s\big|_{\min} = \frac{|(2\text{ kg})\sin 30^\circ - (1\text{ kg})\sin 50^\circ|}{(2\text{ kg})\cos 30^\circ + (1\text{ kg})\cos 50^\circ} = 0.099\,.$$

We were also asked what would happen if $\mu_s = \mu_s\big|_{\min}$, with $\mu_k < \mu_s$, and the blocks were started in motion sliding towards the left. Recall that this is the same direction in which the blocks would slide if there were no friction, as we found in part (a). Once the blocks are in motion the frictional force on the two blocks will have magnitudes $\mu_k N$ and $\mu_k n$, respectively. These forces are smaller than $\mu_s N$ and $\mu_s n$, which together are just barely enough to prevent the blocks from sliding. Thus the blocks will continue to slide, accelerating in the negative x-direction, until the 2 kg mass reaches the bottom of the incline.

Scrutinize

The expression (11) for $\mu_s\big|_{\min}$ is dimensionless, as it must be since μ_s is a pure number (being a ratio of two forces). If we set $m = 0$, the required coefficient of friction is simply $\tan\alpha$, as in Problem 6E.4; if we put $M = 0$, we get $\tan\beta$, which is also in agreement with Problem 6E.4.

Learn

Signs can be very tricky in problems involving friction. The direction of both static and kinetic friction forces generally depends on the values of other quantities in the problem, so one frequently uses formulas that are valid only for a restricted range of values. In the above derivation we have taken pains to make no assumptions about the signs of the expressions, so Eq. (11) is valid for all values of α and β, from 0 to 90°, and for all values of M and m. However, we knew that $M\sin\alpha - m\sin\beta$ is positive for the numbers given in the problem, so we could have concluded that F_M and F_m are positive. If we had used this knowledge, we could have obtained a formula just like Eq. (11), but without the absolute value signs. In that case, however, we would have found a negative value for $\mu_s\big|_{\min}$ when we applied the equation to the special case $M = 0$. Coefficients of friction are never negative, so such an answer would have been

7.11, continued:

a mistake. Thus, it is important to keep track of any assumptions that you make about the signs of the quantities in your equations, and you should expect your equations to fail when they are applied to cases that violate those assumptions.

(c) *Suppose that the coefficient of static friction between blocks and slope is 0.12. Is it possible for the blocks described in part (b) to be stationary? If so, what is the range of possible values for the tension in the string?*

Conceptualize

Since we have learned from part (b) that a coefficient of static friction of 0.099 is sufficient to allow a static configuration of the blocks, the value of 0.12 given in this part is clearly sufficient. So the nontrivial part of the problem is to find the range of possible values of the tension. This problem is unusual, as most of the problems we encounter have unique answers, while in this problem we are asked to find a range of possible answers.

Formulate

Although the question is different, the physical situation is identical to that of part (b). Thus, Eqs. (6), (7), (9), and (10) are all still valid. The first two of these equations allow us to relate the tension to the frictional forces, while the latter two give us the range of possibilities for the frictional forces.

Solve

We are describing a static situation, so $a = 0$. From Eqs. (6) and (7), it follows that

$$T = Mg\sin\alpha - F_M$$
$$T = mg\sin\beta + F_m \; .$$

Eqs. (9) and (10) can be rewritten as

$$-\mu_s Mg\cos\alpha \le F_M \le \mu_s Mg\cos\alpha$$
$$-\mu_s mg\cos\beta \le F_m \le \mu_s mg\cos\beta \; .$$

Combining with the previous equations, we have the inequalities

$$Mg\sin\alpha - \mu_s Mg\cos\alpha \le T \le Mg\sin\alpha + \mu_s Mg\cos\alpha$$
$$mg\sin\beta - \mu_s mg\cos\beta \le T \le mg\sin\beta + \mu_s mg\cos\beta \; .$$

Numerically,

$$7.9 \text{ N} \le T \le 12.1 \text{ N}$$
$$6.9 \text{ N} \le T \le 8.4 \text{ N} \; .$$

Since both of these must hold, the final answer is the intersection of the ranges given by the first and second inequalities:

$$7.9 \text{ N} \le T \le 8.4 \text{ N} \; .$$

7.11, continued:

Scrutinize

The dimensions are clearly correct, since each term is a mass times an accleration. The dependence on μ_s can also be seen to be at least qualitatively correct: as μ_s is increased the lower limits get lower and the upper limits get higher, so the range is monotonically increased. This behavior agrees with our intuition, because a larger amount of friction should increase the range of parameters for which a stationary solution exists. Finally, we can check that the range of values disappears at precisely the value of μ_s that was calculated in (b). For the numerical values in this problem, the most restrictive of the above inequalities give

$$Mg\sin\alpha - \mu_s Mg\cos\alpha \le T \le mg\sin\beta + \mu_s mg\cos\beta \ .$$

The upper and lower limits are equal, and hence the range disappears, precisely when

$$\mu_s = \frac{Mg\sin\alpha - mg\sin\beta}{mg\sin\beta + mg\cos\beta} \ ,$$

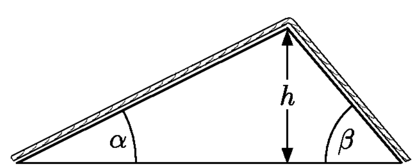
which is exactly the value of $\mu_s\big|_{\rm min}$ found in part (b).

Learn

It is unusual to have a problem with a range of answers, but this situation can arise when there are forces—such as static friction, normal forces, or the tension in an inextensible string—which have magnitudes that adjust to the other forces that are acting. A classic example of such a problem, which we will learn how to analyze in Chapter 9, is the distribution of weight among the four legs of a rigid table. For a three-legged table, on the other hand, the answer is unique.

If one were to physically construct the system described in this question, one would of course expect that the tension in the string would have some definite value. But how can the physical system pick out a definite value, when our calculation could not? The answer is that our calculation was overly idealized. In reality there is no such thing as an inextensible string, but any real string has some elasticity. The string stretches slightly, and the amount of stretching increases with the tension. So the tension in the string would depend on the precise distance between the blocks.

(d) *Now suppose that a heavy uniform rope (i.e. a rope with constant mass per length) is laid over a frictionless triangular block as shown. Assume that the peak of the triangle behaves as a frictionless pulley. Will the rope slip?*



Conceptualize

We know that a system of particles accelerates under the influence of external forces as if all its mass were concentrated at the center of mass. By treating the pieces of string on either side of the peak as two separate systems, we can reduce this problem to something equivalent to part (a).

7.11, continued:

Formulate

The length of the left-hand segment of rope is

$$\ell_1 = \frac{h}{\sin\alpha} ,$$

while the length of the right-hand segment is

$$\ell_2 = \frac{h}{\sin\beta} .$$

If we let σ denote the mass per length of the system, then the masses of the left- and right-hand segments are

$$m_1 = \frac{\sigma h}{\sin\alpha}$$
$$m_2 = \frac{\sigma h}{\sin\beta} .$$

If we model the peak of the triangular block as a frictionless pulley, then we can apply Eqs. (1′) and (3′) from part (a), using these masses:

$$m_1 a = T - m_1 g \sin\alpha = T - \sigma h g$$
$$m_2 a = m_2 g \sin\beta - T = \sigma h g - T ,$$

where T is the tension of the rope at the peak, which determines the magnitude of the tension force acting on each segment of the rope. Adding the equations gives

$$(m_1 + m_2) a = 0 ,$$

so the rope will remain stationary if it starts stationary. On the other hand, if it is given a small nudge, say to the right, it will not only continue to move, but will actually accelerate. This is because the mass of rope on the right will increase while the mass on the left decreases, so a net force to the right will develop. Our rope is in a position of unstable equilibrium.

Scrutinize

The crucial point is that the length of rope on each side is inversely proportional to $\sin\alpha$ (or $\sin\beta$), whereas the component of $\vec{g}$ down the slope is directly proportional to $\sin\alpha$ (or $\sin\beta$). Hence the component of the gravitational force down the slope is independent of the angle.

Learn

The rope problem gives an interesting illustration of how one can take advantage of the freedom to call any selection of particles a system. No matter what system we choose, we always know that the acceleration of the center of mass is given by the total force acting on it divided by the its total mass. Here we started with one rope, but found it

7.11, continued:

useful to treat it as *two* systems, separately considering the segments of rope on either side of the peak.

7.16 *A rocket is a 'Newton's Third Law machine'—it operates by ejecting a high velocity exhaust at one end. Suppose that a rocket with initial mass M_i burns its fuel at a constant rate* dm/dt *and expels the combustion products with speed u relative to itself. By how much has the rocket's speed increased when its mass has decreased to M_f (the difference having been ejected as exhaust)? (Assume the rocket is in interstellar space, with no other forces acting.)*

A deep space probe will be placed in low Earth orbit by a shuttle launch, and will then fire its own booster rocket to leave orbit. You are in charge of designing the booster. If the probe's mass is 500 kg *and it needs a velocity change of* 5 km/s, *what minimum mass of fuel will you need to specify if the exhaust velocity is to be* 2500 m/s*?*

Conceptualize

This is an application of conservation of momentum. The rocket and its exhaust together form an isolated system, with no external force acting, so their combined momentum must be conserved. The difficulty is that the masses of rocket and exhaust are not constant: the rocket is losing mass continuously, and the total mass of combustion products expelled is increasing. Because this is a continuous process, it is necessary to use calculus to solve this problem.

Σ∫

Formulate

Consider a time t at which the rocket's mass is m and its speed is v. A short time interval Δt later its mass is $m + \Delta m$ and its speed is $v + \Delta v$; its momentum has changed by an amount $v\Delta m + m\Delta v$. (Notice that Δm will be a negative number, because the rocket's mass is decreasing—we do it this way because we want the differential dm/dt (i.e. $\lim_{\Delta t \to 0}(\Delta m/\Delta t)$) to be the rate of change of the rocket's mass, which is what interests us, and not the rate of change of the amount of mass ejected as exhaust. Also note that the term $\Delta m\Delta v$ in the momentum change is neglected: if Δt is small, $|\Delta m| \ll m$ and $|\Delta v| \ll v$, so $\Delta m\Delta v$ is much smaller than either of the other two terms.)

The rocket's momentum change is balanced by the momentum $(v-u)(-\Delta m)$ of the exhaust ejected during this time interval, $v - u$ being the speed of the exhaust in a stationary reference frame. (A change of Δm in the mass of the rocket clearly implies ejection of $-\Delta m$ of fuel.) This gives us

$$v\Delta m + m\Delta v - v\Delta m + u\Delta m = 0,$$
$$\Rightarrow m\Delta v = -u\Delta m.$$

If we divide this by the time interval Δt and take the limit as $\Delta t \to 0$ it tells us that the net force on the rocket (i.e. its mass times its acceleration dv/dt) is its exhaust velocity times the rate at which it burns and expels fuel, $u\,dm/dt$. This is called the ***thrust*** of the rocket.

7.16, continued:

Solve

If we divide the equation above by m, we obtain the equation for Δv:

$$\Delta v = -\frac{u}{m}\Delta m \ .$$

The equation means that for any small interval during which the mass of the rocket changes by Δm, the speed changes by $-(u/m)\Delta m$. If we are interested in calculating the change in the speed of the rocket over time, we could imagine dividing the time interval into a huge number of small increments, and then applying the above formula to each increment in sequence. For each increment we would have a different value of m, since the mass decreases as exhaust is ejected. The total change in speed would be the sum of the small increments, which is exactly how one defines an integral:

$$v_f - v_i = -u \int_{M_i}^{M_f} \frac{\mathrm{d}m}{m} \ .$$

Doing the integration gives $v_f - v_i = -u \ \ln\left(M_f/M_i\right) = u \ \ln\left(M_i/M_f\right)$. Note that if the rocket carries enough fuel, we can have $v > u$, since $\ln(M_i/M_f)$ can be greater than 1. Assuming we are not dealing with velocities near the speed of light, the limiting factor is the ratio of payload mass to fuel mass. This ratio is improved if fuel tanks etc. can be discarded when they are empty, hence the use of multi-stage rockets in satellite launches.

Applying this formula to our probe booster, we have $u = 2500$ m/s, $M_f = 500$ kg, and $v_f - v_i = 5000$ m/s. To extract M_i, we take the antilog of our formula:

$$\frac{M_i}{M_f} = \exp\left(\frac{v_f - v_i}{u}\right) .$$

This gives $M_i = (500 \text{ kg}) \times \exp(2) = 3700$ kg. The minimum mass of fuel is thus 3200 kg, assuming that this is a single-stage rocket and that the original 500 kg already included the mass of the necessary fuel tanks.

Scrutinize

The impulse applied to the 500 kg payload is 2.5 MN·s. If we had done this by ejecting the whole of the 3200 kg of fuel in one instantaneous pulse, the required exhaust velocity would be $(2.5 \times 10^6 \text{ kg} \cdot \text{m/s})/(3200 \text{ kg}) = 780$ m/s, which is considerably less than the actual exhaust velocity. This makes sense, because such an instantaneous ejection is more efficient than our continuous boost—in the early stages of acceleration, what we are accelerating is mainly fuel rather than payload. (The disadvantage of the instantaneous-boost approach is that the instantaneous ***force*** applied to the payload approaches infinity, so what actually gets launched is a squashed heap of wreckage rather than a delicate scientific instrument!)

Although the overall ***momentum*** change of the system of rocket plus exhaust is zero, the overall change in ***kinetic energy*** is large. The source of this energy is the chemical potential energy liberated by burning the fuel.

7.16, continued:

Learn

The rocket problem is a classic example of a problem involving the continuous transfer of material. A similar problem is that of a railroad flatcar onto which is poured a continuous stream of water or sand. In all such problems, the key idea is to think incrementally. Don't try to see through the problem in one step, but instead begin by considering an arbitrarily short interval of time Δt. If you can figure out how the conditions at the end of Δt are related to what they were at the start, then the mechanical manipulations of calculus will usually do the rest of the work for you.

Another fascinating feature of the rocket problem is the fact that the velocity of the center of mass of the whole system—rocket plus exhaust—can never change. If the rocket begins at rest in empty space, then the center of mass will always remain at its initial location, no matter how far the payload might travel.

HINTS FOR PROBLEMS WITH AN (H)

The number of the hint refers to the number of the problem

7.1 What is the acceleration of the car at point B? What forces could be acting to cause this acceleration? Do any of these forces have Third Law partners which are relevant to the problem?

7.3 (c) What quantity is conserved throughout the child's motion? If at point P she is sliding with speed v, what is v in terms of g, r and θ?

What is her centripetal acceleration at point P? What is the net centripetal force on her?

What would the normal force on the child be if she were not in contact with the igloo? What do you think the normal force would be at the exact point where she loses contact with the igloo?

7.6 For both part (a) and part (c), ask yourself the following questions.

Which quantities are the same for the two sleds?

What is the net horizontal force on each sled? The net vertical force?

If you're still stuck, try reviewing the solutions to problems 7.10 and 7.11.

7.12 Draw free-body diagrams for each mass. What is the condition for balance?

If the mass M descends slightly, what happens to (a) the position of the 2 kg mass; (b) the forces acting on the 2 kg mass?

7.13 There are three stages to this problem: (i) firing the rifle; (ii) the impact of the bullet; (iii) the motion of the block (and bullet) after impact. In each case, which quantity is conserved? Which is not conserved?

7.15 For part (d), what is the horizontal force acting on the upper block? Before the block starts to slip, how is the value of this force related to (i) the acceleration, (ii) the tension?

7.18 In Alice's frame of reference, what is the acceleration of the ball? What is the position of the ball at time t?

7.19 It may help to draw a force diagram for the system after mass B has fallen some distance. Study the directions of all the forces involved (do not try to calculate their magnitudes, but do consider which forces must be equal in magnitude). You may also find it useful to list the external forces acting on the system of two balls and a string, and to visualize their directions.

ANSWERS TO HINTS

7.1 v^2/r; gravity and a normal force from the track; yes, the Third Law pair of the normal force exerted by the track on the car is a force exerted by the car on the track.

7.3 Mechanical energy (kinetic plus potential); $\sqrt{2rg(1-\cos\theta)}$

v^2/r; $mg\cos\theta - N$, where N is the normal force from the igloo.

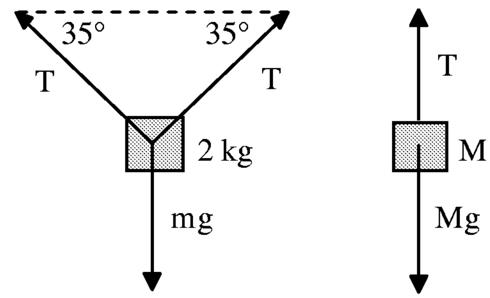
Zero; zero.

7.6 The acceleration, and the magnitude of the tension in the rope between them.

For parts (a) and (b), $ma = 30$ N for the girl's sled and 8 N for Teddy's. The vertical components are both zero.

For part (c), net force is zero in both horizontal and vertical directions. Horizontal force is $T\cos\theta - t - \mu_k N$ for girl's sled, and $t - \mu_k n$ for Teddy, where T and t are the tensions in the first and second ropes and N and n are the respective normal forces.

7.12



$2T\sin 35° - mg = 0$ (2 kg mass); $T - Mg = 0$ (mass M).

2 kg mass moves upwards; downward force unchanged, upward forces decrease (because angle decreases).

7.13 Conserved: (i), (ii) momentum; (iii) mechanical (kinetic plus potential) energy.

Not conserved: (i), (ii) kinetic energy; (iii) momentum.

7.15 Static friction, F_s; $F_s = Ma$; $F_s = \frac{1}{2}(T - 2\mu_k Mg)$.

7.18 Acceleration $= [-a, 0, -g]$; position given by $x = vt\cos\theta\sin\alpha - \frac{1}{2}at^2$, $y = vt\cos\theta\cos\alpha$, $z = vt\sin\theta - \frac{1}{2}gt^2$, where α is the angle between the ball's direction and the y-axis.

7.19 Forces you should have on your diagram: gravity; normal force from surface of table; normal force from edge of table; tension.

ANSWERS TO ALL PROBLEMS

7.1 (a) The downward force acting on each passenger must be mv^2/r, to maintain the circular motion. If $v^2/r > g$, some of this downward force must come from a normal force exerted by the track on the car (and by the car seats on the passengers). The Third Law pair to this normal force, which is exerted by the car on the track (and by the passengers on the seats), acts outward, and is responsible for the feeling of weight.

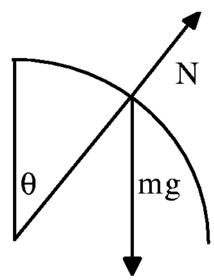
(b) $v = \sqrt{\frac{3}{2}gr}$; $h = \frac{11}{4}r$

7.2 -5.3×10^{33} J; 2.7×10^{33} J; -2.7×10^{33} J. Total energy is negative, as expected for a bound state—cf. problem 7.5. [In fact, in this case the kinetic energy is exactly half the magnitude of the potential energy: $K = -\frac{1}{2}U$.]

7.3 (a) $mgr\cos\theta$.

(b) see diagram at right.

(c) No; at 48.2° ($\cos\theta = \frac{2}{3}$)



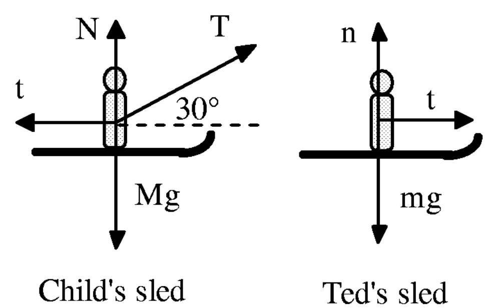
7.4 See complete solution.

7.5 See complete solution.

7.6 Tension t in the rope between the two sleds forms a Third Law pair with the force exerted by the child's sled on the rope, and another pair with the force exerted by Teddy's sled on the rope.

$T = 44$ N; $t = 8$ N.

$$T = \frac{\mu_k(m+M)g}{\cos\theta + \mu_k \sin\theta}.$$



7.7 See complete solution.

7.8 $\vec{\mathbf{F}} = [\mu_k Mg, 0, 0]$; $\dfrac{u}{\mu_k g}$; $\frac{1}{2}Mu^2$. Because friction always acts to oppose ***relative*** motion, the frictional force in this case acts to ***accelerate*** the suitcase, and does positive work. After the suitcase reaches the speed of the conveyor belt, the force of friction acting on it is zero.

7.9 (a) $\mu_s = \tan\theta$; in the direction of motion of the conveyor.

(b) $\vec{\mathbf{F}} = \mu_k Mg\left[\dfrac{u}{\sqrt{u^2+u_0^2\cos^2\theta}}, 0, -\dfrac{u_0\cos\theta}{\sqrt{u^2+u_0^2\cos^2\theta}}\right]$; $t = \dfrac{\sqrt{u^2+u_0^2\cos^2\theta}}{\mu_k g}$; $z(t) = \dfrac{u_0\cos\theta\sqrt{u^2+u_0^2\cos^2\theta}}{2\mu_k g}$.

(c) $\vec{\mathbf{F}} = [-\mu_k Mg, 0, 0]$; $\mu_k(M+M_2)g$.

7.10 See complete solution.

7.11 See complete solution.

7.12 1.7 kg. The system will oscillate.

7.13 600 m/s; 0.6 m/s (taking $g = 9.8$ m/s^2).

7.14 (b) $U_0 + \frac{3}{2}Mu^2$; (c) $U_0 + \frac{3}{2}Mu^2 - K_A - K_B$ (d) $[u, -2v_y, 0]$.

7.15 (a) $\mu_s Mg$.

(b) $\frac{1}{2}\mu_s Mg$, in direction opposite the tension force.

(c) $(\mu_s - \mu_k)g$, in direction of tension force.

(d) $2(\bar{\mu}_s + \mu_k)Mg$; backwards (opposite to direction of motion).

7.16 See complete solution.

7.17 towards the star; stationary.

If we let $\hat{\boldsymbol{u}}$ denote a unit vector pointing toward the point A, then the velocity vector of the spaceship can be written as $V\hat{\boldsymbol{u}}$, where V is the spaceship's speed. The exhaust gas moves with velocity $-v\hat{\boldsymbol{u}}$ relative to the spaceship, and therefore with velocity $(V - v)\hat{\boldsymbol{u}}$ relative to a stationary observer. If $V > v$, the exhaust velocity vector therefore points towards A.

There is no external force acting on the system, so the system center of mass has no acceleration. Since the center of mass was at rest before the spaceship fired its rocket engines, it must remain at rest. (Note that the internal kinetic energy of the system certainly increases, but this is not a violation of energy conservation: the increase in kinetic energy is balanced by a decrease in the chemical potential energy of the fuel.)

The speed v of the exhaust gases is determined by the chemical potential energy of the fuel and the design of the engines. If we knew the parameters, we could calculate v for the case when the spaceship is at rest. If we now consider the instant when the rocket has attained velocity $\vec{\mathbf{V}}$, and repeat this calculation in a reference frame moving at the same velocity (so that the spacecraft is instantaneously at rest in this frame*), we will be doing essentially the same calculation as the first one. That is, since everything relevant to the calculation (e.g., the fuel tanks, the pumps, and the combustion chamber) is moving with the spaceship, everything will be at rest in the new frame of reference. As long as nothing has happened in the meantime to affect the behavior of the engine, then the calculation will be unchanged, and we will get the same answer for the exhaust speed. Therefore the exhaust speed must be approximately constant relative to the rocket engine.

There is, however, one further complication that is worth exploring. The above argument shows that if nothing relevant to the mechanics of the rocket changes as the rocket accelerates, then the change in the speed of the rocket will have no effect on the speed of the exhaust relative to the rocket. There is, however, one property of a rocket that typically changes dramatically over the course of a flight: its mass. Most of the mass of a typical

---

* The spacecraft itself defines an accelerating, non-inertial, reference frame, but at any given moment it is at rest relative to an inertial frame moving with the same instantaneous velocity $\vec{\mathbf{V}}$.

rocket is fuel, so the mass of the rocket decreases very significantly with time. The above argument does not tell us whether the change in the rocket mass will change the exhaust speed, so to check that we need to do a more quantitative calculation by using conservation of energy and momentum. Suppose that at time t the rocket plus its remaining fuel has mass $m+\Delta m$. In the short time interval Δt it burns Δm of fuel, converting ΔE of chemical potential energy to kinetic energy and ejecting Δm of exhaust gas at speed v relative to the rest frame of the rocket at time t. By conservation of momentum, the rest of the rocket must move in the opposite direction with speed V, where $mV = \Delta m v$. By conservation of energy, $\Delta E = \frac{1}{2}mV^2 + \frac{1}{2}\Delta m v^2$. Combining these two equations,

$$\begin{aligned} \Delta E &= \frac{1}{2}m\left(\frac{\Delta m}{m}v\right)^2 + \frac{1}{2}\Delta m v^2 \\ &= \frac{1}{2}\Delta m v^2\left(\frac{\Delta m}{m}+1\right) \\ \implies \quad v &= \sqrt{\frac{2E}{\Delta m\left(1+\frac{\Delta m}{m}\right)}}. \end{aligned}$$

As Δm was the fuel burned in a short time interval, we can safely assume that Δm is small compared to m, and so $1+\Delta m/m \simeq 1$. Hence the exhaust speed is approximately constant in the rest frame of the rocket (it depends only on the amount of gas expelled in time Δt and the amount of energy liberated by burning that amount of fuel).

7.18 (a) $\vec{\mathbf{v}} = v\left[\frac{a}{g}\sin\theta, \sqrt{\cos^2\theta - \frac{a^2}{g^2}\sin^2\theta}, \sin\theta\right]$.

(b) $v = \sqrt{\frac{gw}{\cos\alpha}}$, $t = \sqrt{\frac{2w}{g\cos\alpha}}$, where $\sin\alpha = \frac{a}{g}$.

7.19 Ball A falls off the edge before ball B hits it. There are at least two conclusive arguments: (i) the only horizontal force on the two-ball-plus-string system is the force of the table edge on the string, which has a component to the right; therefore the center of mass must move to the right; (ii) thinking of the balls one at a time, the only horizontal force is from the tension in the string; the tension force has equal magnitude on the two balls, but only for the left ball is the force purely horizontal.