ESSENTIALS OF INTRODUCTORY CLASSICAL MECHANICS

Sixth Edition

CHAPTER 8

ROTATION IN TWO DIMENSIONS

OVERVIEW

We have seen in earlier chapters that dealing explicitly with a system of many bodies becomes increasingly difficult as the number of individual bodies increases. In Chapter 6 we looked at one way of handling this problem — approximating the individual interactions of the surface atoms of two bodies by a single macroscopic force, i.e. friction. This chapter introduces the idea of ***rigid bodies***, where the configuration of the system (the shape of the body) remains fixed at all times. Rigid bodies can move as point particles; they can also ***rotate*** about some axis.

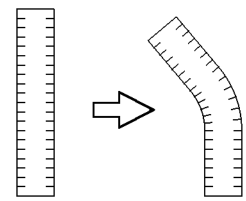
In this chapter we shall deal with rigid bodies moving in a two-dimensional plane and rotating about an axis perpendicular to this plane. This is the rotational analogue of one-dimensional linear motion — it simplifies the algebra but still allows us to develop all the new physical concepts we need. Since the direction of the axis of rotation is fixed, we will not have to describe the direction in the course of our calculations. This means that we will be able to use scalars to represent quantities — such as ***angular velocity***, ***torque***, and ***angular momentum***, which under more general circumstances would require vectors. In the following chapter we will extend our discussion to cases in which the axis is not fixed, and then we will introduce the full vector formalism needed to describe the rotations of rigid bodies in three dimensions.

When you have completed this chapter you should:

- ✔ be able to explain the concept of a rigid body and recognize its similarities to and differences from a real physical object;
- ✔ know what is meant by the terms angular velocity, angular acceleration, rotational kinetic energy, moment of inertia, torque, and angular momentum;
- ✔ be able to calculate the moment of inertia of a simple solid object about a fixed axis;
- ✔ be aware that two-dimensional motion of a rigid body can always be regarded as a combination of translation and rotation about an axis through the center of mass and perpendicular to the plane of motion;
- ✔ know how to use the rotational equivalents of Newton's laws to predict the outcome of motion involving translation and rotation about an axis with fixed orientation.

ESSENTIALS

Many solid bodies, such as blocks of wood or bars of steel, assume a shape that is maintained under a wide variety of conditions. The idealized solid object for which this is exactly true is called a ***rigid body***. In an ideal rigid body, the distance between any two constituent parts remains exactly constant over time. Rigid bodies cannot bend, twist, expand, or contract.

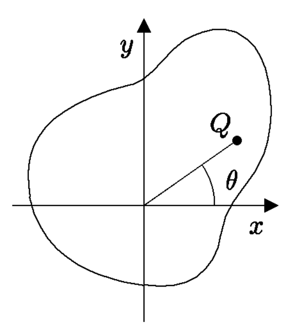


Bending
In a bent bar, the distances between constituent particles are stretched on one side and shortened on the other. In a rigid body such distances cannot change.

A rigid body can move in space while retaining the same orientation, a form of motion called ***translation***. A translation can be described in the same way that we describe the motion of point particles. A rigid body can also change its orientation, a form of motion called ***rotation***. The most general motion of a rigid body involves simultaneous translation and rotation, but for simplicity we begin our discussion by considering rotations alone, without any translation. We therefore assume for now that one point on the rigid body, which we call P, is held fixed. The motion of the rigid body is then called a ***rotation about the point P***.

Although the definition of a rotation requires only that a single point P be held fixed, it can be shown that for any such motion there is an entire line of points that remains fixed. This line of fixed points is called the ***axis*** of the rotation. The fixed point P and perhaps even the entire axis will sometimes lie outside the physical body, but one can always imagine extending the rigid body to avoid this complication.

In studying translational motion we started by confining ourselves to motion in one dimension. For similar reasons, we begin our discussion of rotations by considering cases in which the axis is fixed. That is, we assume that the physical device is constructed so that the only possible motion is rotation about a specified axis. An example would be a wheel, rotating about an axle rigidly attached to a stationary table. In this case the motion of the wheel can be described by a single coordinate, giving the angle at which the wheel is oriented at any given time. For definiteness, we can introduce a Cartesian coordinate system with the z-axis along the axis of rotation, and we can choose a reference point Q which is on the rigid body and in the xy-plane. We then define the coordinate θ as the angle between the x-axis and the line joining the rotation axis to the point Q. By convention the angle is defined to increase as the reference point Q moves in the counterclockwise direction, from the x-axis towards the y-axis.



Following the analogy with one-dimensional motion, we refer to the first and second time-derivatives of the coordinate as a velocity and an acceleration, respectively. Specifically, for rotational motion we define the ***angular velocity*** (measured in ***radians per second***)

$$\boxed{\omega \equiv \frac{d\theta}{dt}\ .}$$

We also define the ***angular acceleration*** (measured in ***radians per second***^{2})

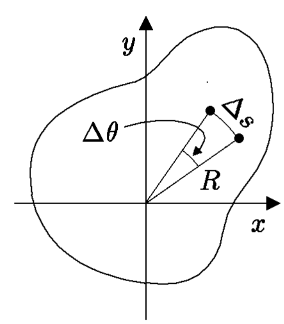
$$\boxed{\alpha \equiv \frac{d\omega}{dt}\ .}$$

Problem 8A.1

For clarity we emphasize that angular velocity and angular acceleration are not the same quantities as the (linear) velocity and acceleration that we have been discussing since Chapter 1; the connection is only by analogy.

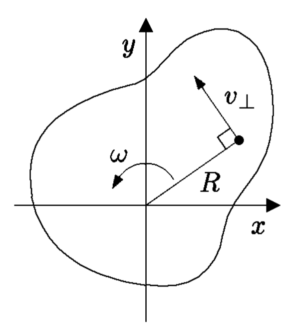
While a rigid body rotates about a fixed axis, each point particle that makes up the rigid body is moving on a circular trajectory. Since each point particle moves according to Newton's laws of motion, we can deduce the laws that govern rigid body motion from the laws that we already know for point particles. To do this, we begin by relating the velocity and acceleration of the point particles to the angular velocity and angular acceleration of the rigid body.

Consider a point particle on the rigid body, located a distance R from the fixed axis. (Whenever we speak of the distance between a point and a line, we mean the distance measured perpendicularly to the line.) In a small time interval Δt, the point will rotate through a small angle $\Delta\theta = \omega\Delta t$. The particle moves along the arc of a circle, moving tangentially (i.e., perpendicular to the radial direction) through an arc length $\Delta s = R\,\Delta\theta$. Therefore, if $\omega > 0$, the particle's velocity $\vec{\mathbf{v}}$ has magnitude



$$v = |\vec{\mathbf{v}}| = \frac{\Delta s}{\Delta t} = R\frac{\Delta\theta}{\Delta t} = R\omega\ .$$

Denoting the outward radial component of a vector by a subscript r, and the counterclockwise tangential component of a vector by a subscript $\perp$, the velocity of the particle can be specified by

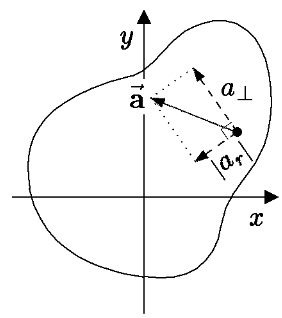


$$\boxed{v_r = 0\ ; \qquad v_\perp = R\omega\ .}$$

The equation above holds for either sign of ω.

The acceleration is a bit more complicated. If ω is constant then the particle is undergoing uniform circular motion, so we already know that the acceleration is directed towards the origin, with magnitude v^2/R. When ω is not constant, it is shown in Problem 8A.2 that the centripetal acceleration is unchanged, but there is also a tangential component to the acceleration given by $\mathrm{d}v/\mathrm{d}t = R\alpha$, so the full acceleration vector is described by

Problem 8A.2



$$\boxed{a_r = -\frac{v^2}{R} = -R\omega^2 \; ; \qquad a_\perp = R\alpha \; .}$$

We now want to relate the motion of the rigid body to the applied forces. The most straightforward approach is to study the kinetic energy of the rotating rigid body, which is an example of a system of particles, as discussed in Chapter 5. The kinetic energy of such a system is given by

$$K = \sum_i \frac{1}{2} m_i v_i^2 \; ,$$

the sum of the kinetic energies of the individual particles. For a rigid body rotating about a fixed axis, we can replace each v_i by $R_i\omega$ to obtain

$$K = \frac{1}{2}\left(\sum_i m_i R_i^2\right)\omega^2 \; .$$

We call this the rotational kinetic energy of the body, and define the quantity in parentheses to be the ***moment of inertia*** of the rigid body about the specified axis:

$$\boxed{I = \sum_i m_i R_i^2 \; .}$$

Problems 8C.1 and 8C.2

(For a solid body, we replace the sum over R_i by an integral over volume.)

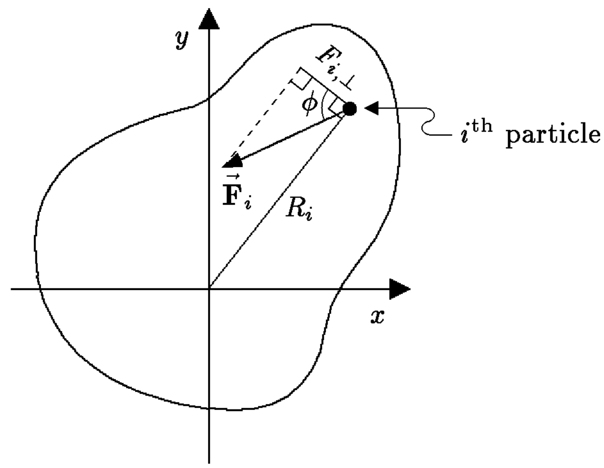
The rotational kinetic energy is therefore given by

$$\boxed{K = \frac{1}{2} I \omega^2 \; ,}$$

Problem 8D.1

which is closely analogous to the formula $\frac{1}{2}Mv^2$ for ordinary translational kinetic energy. While angular velocity and angular acceleration are concepts that are ***analogous*** to the concepts of velocity and acceleration discussed earlier in the book, rotational kinetic energy is not analogous to kinetic energy—it really ***is*** kinetic energy, as defined in Chapter 4.

Having obtained a simple expression for the kinetic energy, we can now use the work-energy theorem of Chapter 4 to relate the change in kinetic energy of the rotating body to the total work done by external forces. If the rigid body moves through a small angle $\Delta\theta$, the work done on the i^{th} point particle is given by



$$\Delta W_i = \vec{\mathbf{F}}_i \cdot \overrightarrow{\boldsymbol{\Delta R_i}} = |\vec{\mathbf{F}}_i||\overrightarrow{\boldsymbol{\Delta R_i}}|\cos\phi \ ,$$

where $\vec{\mathbf{F}}_i$ is the total force (both internal and external) applied to the i^{th} particle, $|\overrightarrow{\boldsymbol{\Delta R_i}}| = R_i\Delta\theta$ is the distance the particle moves, and ϕ is the angle between the force and the displacement vector $\overrightarrow{\boldsymbol{\Delta R_i}}$. Since $\overrightarrow{\boldsymbol{\Delta R_i}}$ is tangential, $|\vec{\mathbf{F}}_i|\cos\phi$ is just the tangential component of $\vec{\mathbf{F}}_i$, which we call $F_{i,\perp}$. (Note that although the particle motion in the rotating body is purely tangential, the acceleration of a particle, and hence the force acting on it, can have a radial component as well. Only the tangential component, however, can do work on the body.) Summing over all the particles, the total work done on the rigid body is

$$\Delta W = \sum_i F_{i,\perp} R_i \, \Delta\theta \ .$$

The quantity $F_{i,\perp} R_i$ will play an important role in rotational motion, so it is given a name, the ***torque*** τ:

$$\boxed{\tau_i \equiv F_{i,\perp} R_i \ ; \qquad \tau_{\text{tot}} = \sum_i \tau_i = \sum_i F_{i,\perp} R_i \ .}$$

Problems 8B

Recalling the definition of $\perp$ on page 267, we see that torque, like angle, is positive when counterclockwise.

In terms of the torque, the work done on the rigid body as it rotates through a small angle $\Delta\theta$ can be written as

$$\boxed{\Delta W = \tau_{\text{tot}} \, \Delta\theta \ .}$$

By the work-energy theorem, this work must equal the change in the kinetic energy of rotation. We can obtain an expression for the power applied to the rotating body by dividing the above equation by Δt and then taking the limit as $\Delta t \to 0$:

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \tau_{\mathrm{tot}} \frac{\mathrm{d}\theta}{\mathrm{d}t} = \tau_{\mathrm{tot}}\omega \ .$$

Equating this expression to the rate of increase of the kinetic energy of the body, we find

$$\tau_{\mathrm{tot}}\omega = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}I\omega^2\right) = I\omega\frac{\mathrm{d}\omega}{\mathrm{d}t} = I\omega\alpha \ ,$$

from which we can see that

$$\boxed{\tau_{\mathrm{tot}} = I\alpha \ .}$$

With this formula we have achieved our goal of determining the angular acceleration of the rigid body from the forces acting on it. In the process, we have discovered the torque, a quantity which for rotational motion is the analogue of the force.

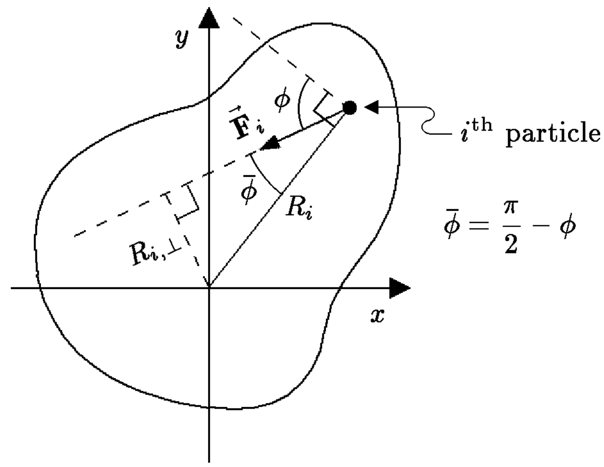
The angular acceleration of a rotating rigid body is proportional to the total torque acting on it, just as the linear acceleration of an object is proportional to the total force acting on it. Similarly, the angular acceleration is inversely proportional to the moment of inertia, just as the linear acceleration is inversely proportional to the mass. The unit of torque is the newton-meter, abbreviated $\mathrm{N \cdot m}$. (This is actually the same as the unit of energy, but to emphasize that torque is analogous to force we always use $\mathrm{N \cdot m}$ for its unit, never J.) Both the moment of inertia and the torque depend upon the choice of the rotation axis, since the R_i appearing in the definitions are measured from the axis.

While we derived the above equation from the work-energy theorem, it is easy to verify that it also follows as a consequence of Newton's second law. Replacing $\vec{\mathbf{F}}_i$ by $m_i\vec{\mathbf{a}}$ in the definition of the torque, we find

$$\tau_{\mathrm{tot}} \equiv \sum_i F_{i,\perp}R_i = \sum_i m_i R_i a_{i,\perp} = \sum_i m_i R_i^2 \alpha = I\alpha \ ,$$

where in the last step we used $a_\perp = R\alpha$ and the definition of I.

In calculating the torque, it is sometimes useful to express it in a slightly different way. Since $F_{i,\perp} R_i = |\vec{\mathbf{F}}_i| R_i \cos\phi$, where ϕ is the angle between $\vec{\mathbf{F}}_i$ and the tangential direction, one can instead write the contribution to the torque from the force on the i^{th} particle as

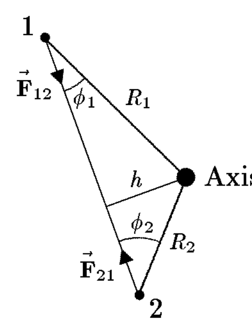


$$\tau_i = \pm |\vec{\mathbf{F}}_i| R_{i,\perp} ,$$

where $R_{i,\perp} \equiv R_i \cos\phi = R_i \sin\bar{\phi}$ is the component of the displacement vector of the i^{th} particle in the direction perpendicular to the force $\vec{\mathbf{F}}_i$. (The $\pm$ sign is included because the factors on the right are by definition nonnegative, while the torque τ_i is negative when the torque is clockwise.)

The formula $\tau_{\text{tot}} = I\alpha$ allows us to calculate the angular acceleration if we know the total torque, but this is still complicated, since the total torque includes contributions from both internal and external forces. This situation is reminiscent of Chapter 5, where we discussed the role of internal and external forces in the translational motion of a system of particles. We found that Newton's third law of motion implied that the internal forces always canceled, so the acceleration of the center of mass of the system could be calculated in terms of the external forces alone. If this were not the case, then a block could accelerate with no external forces acting on it, and conservation of energy would not hold. Now we want to address the analogous question for rotational motion: does Newton's third law imply that the internal torques always cancel? If not, then a rigid body could undergo accelerated rotation with no external forces acting on it, and conservation of energy would have to be abandoned as a physical principle.

To show that internal torques always cancel, it is necessary to assume a form of Newton's third law that is stronger than the version that was postulated by Newton and discussed in Chapter 5. Specifically, we must assume not only that the forces between any two particles are equal in magnitude and opposite in direction, but also that ***they are directed along the line joining the two particles***. As can be seen from the diagram on the right, this added assumption implies that the forces between two particles share the same value of $R_{i,\perp}$: $R_{1,\perp} = R_{2,\perp} = h$. Then



$$|\tau_{12}| = |\vec{\mathbf{F}}_{12}| R_{1,\perp} = |\vec{\mathbf{F}}_{21}| R_{2,\perp} = |\tau_{21}| ,$$

and one can see from the diagram that the two torques have opposite signs. Thus, the strengthened form of Newton's third law implies that the internal torques all cancel, just as the internal forces do for translational motion. Conservation of energy therefore remains valid, and angular accelerations can be calculated by including only the external torques:

$$\boxed{\tau^{\text{ext}} = I\alpha \ .}$$

Problems 8E.1, 8E.2, and 8E.6

This result is crucially important, since it allows us to calculate the rotation of a rigid body without any reference to its internal forces.

In Chapter 5 we learned that the total external force applied to a system of particles is equal to the rate of change of its total momentum, so we might ask whether the total external torque could be involved in an analogous relationship. The answer is yes. Since I is constant for a rigid body rotating about a fixed axis, the equation above can be rewritten by defining the ***angular momentum*** L of the rotating object about the axis by

$$\boxed{L \equiv I\omega \ ,}$$

giving

$$\boxed{\tau^{\text{ext}} = \frac{\mathrm{d}L}{\mathrm{d}t} \ .}$$

L is the rotational analogue of the linear momentum, Mv. If there are no external torques, the above formula implies that angular momentum is conserved.

So far, however, we have demonstrated this conservation law only for a rigid body rotating about a fixed axis, in which case it reduces to the statement that $\omega =$ constant. We will see in the next chapter, however, that the principle is much more general: ***angular momentum is conserved for any system for which there are no external torques***. The classic example is the (non-rigid) spinning skater who pulls in her arms, thereby reducing her moment of inertia I and causing her angular velocity ω to increase, so that the product $L = I\omega$ is conserved. Like the conservation of energy and momentum, the conservation of angular momentum is, so far as we know, an exact law of nature. Within our experience it has never been seen to fail under any circumstances.*

So far we have considered rotation about a ***fixed*** axis, such as the rotation of a wheel about a stationary axle. In this case the axle—the supporting shaft about which the wheel is pivoted—coincides with the axis of rotation—the line of points which have zero instantaneous velocity. In many applications, however, one is interested in a rolling wheel, for which the axle is in motion. Such problems combine rotational motion with translational motion. Nonetheless, as long as the motion is two-dimensional in the sense that all particle velocities lie in the xy-plane, these problems can be treated by a minor extension of the methods already developed.

To describe the most general possible two-dimensional motion of a rigid body, we begin by introducing the "center-of-mass axis," the line parallel to the z-axis which passes through the center of mass. In the rolling wheel problem, for example, the center-of-mass axis would lie along the axle of the wheel (assuming that the wheel is symmetric about its axle). We then change our frame of reference by making a time-dependent translation in the x- and y- directions to a frame that follows the center of mass, so that the center-of-mass axis is at all times the z-axis of the new frame of reference. Even though the new frame will in many cases be non-inertial (i.e., it may accelerate), it is still a convenient system to describe the motion. In the new frame the z-axis is by construction a line of fixed points, and hence the z-axis is the axis of rotation. In this frame the problem reduces to the rotation of the rigid body about the z-axis. Returning to the original "laboratory" frame of reference, ***the full motion of the rigid body is described by specifying the translational motion of the center of mass and the rotational motion about the center-of-mass axis.***

Beware, however, the requirement that the motion remain two-dimensional is not trivial. For reasons that will be discussed in the next chapter, asymmetric objects will usually wobble, rather than rotate about an axis of fixed orientation. One condition for which such

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* The strengthened form of Newton's third law, however, which we used to demonstrate the conservation of angular momentum, is more problematic. If interpreted literally as a statement about the forces between two particles, the strengthened form of Newton's third law is not always valid. It holds, however, for all forces between two particles discussed in this book, such as electrostatic and gravitational forces. For contact forces, such as normal forces and friction, the two particles are idealized as being at precisely the same point. In this case the extension to Newton's third law is ill-defined, since no line is defined by two coincident points, but the original form of the third law is sufficient to guarantee that the torques on the two particles are equal and opposite. If one goes beyond the level of this book, however, to consider the forces felt and exerted by electrically charged particles in motion, then the issue becomes more complicated. As discussed in the Supplementary Notes at the end of Chapter 5, in this case even the original form of Newton's third law fails, if one considers only the particles. If one takes into account the momentum and angular momentum carried by the electromagnetic field, however, then both conservation laws are exact, to within the accuracy of the best measurements that have been performed.

stable rotation is possible, however, is when the object is symmetric about an axis parallel to the z-axis—as an idealized wheel usually is.

The equations describing combined translation and rotation in two dimensions are easier to state than they are to demonstrate, so we will begin by stating them. The translational motion of the rigid body is described fully by the methods introduced in Chapter 5, where we learned that the acceleration of the center of mass $\vec{\mathbf{a}}_{\mathrm{cm}}$ of any system of particles can be determined from the total external force $\sum \vec{\mathbf{F}}^{\mathrm{ext}}$ acting on the system:

$$\boxed{\sum \vec{\mathbf{F}}^{\mathrm{ext}} = M\vec{\mathbf{a}}_{\mathrm{cm}} = \frac{\mathrm{d}\vec{\mathbf{p}}}{\mathrm{d}t} \qquad \text{(translational)}\ .}$$

Here M is the mass of the rigid body, and $\vec{\mathbf{p}}$ is its momentum. The rotational motion is described by the equation

$$\boxed{\sum \tau^{\mathrm{ext}} = I_{\mathrm{cm}}\alpha = \frac{\mathrm{d}L}{\mathrm{d}t} \qquad \text{(rotational)}\ ,}$$

Problems 8E.8, 9D.6, 10.10

where $\sum \tau^{\mathrm{ext}}$ is the total external torque calculated about the center-of-mass axis, I_{cm} is the moment of inertia about the center-of-mass axis, and L describes the angular momentum about this axis. Note that these equations hold even though the axis through the center of mass might be both moving and accelerating.

Warning — *Difficult Material*

The easiest way to derive the above equation for the rotational motion is to use a coordinate system that follows the center of mass of the body, so that the center-of-mass axis is the z-axis of the coordinate system. Even though this center-of-mass coordinate system will in many cases be accelerating, it is still the simplest way to describe the rotational motion, which reduces to the problem of rotation about the z-axis. (Note that the axis of rotation is frame-dependent. In the original frame of reference the center of mass of the rigid body is generally moving, so the center-of-mass axis cannot be the axis of rotation.)

To relate the behavior in the center-of-mass frame to that in the original "laboratory" frame, we must be able to convert angles, angular velocities, and angular accelerations between the two frames. The two coordinate frames are related to each other by a time-dependent translation, but happily the orientation of an object does not change under a translation. Thus the quantity θ describing the orientation of the object, and the quantities ω and α describing the angular velocity and acceleration of the object, all have the same value in the center-of-mass and laboratory frames:

$$\theta_{\rm cm} = \theta_{\rm lab}\ ; \quad \omega_{\rm cm} = \omega_{\rm lab}\ ; \quad \alpha_{\rm cm} = \alpha_{\rm lab}\ .$$

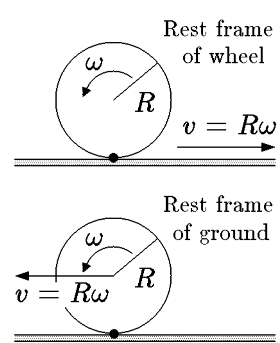
Warning — Difficult Material

The torque is more intricate, since in the center-of-mass frame we must include fictitious forces to compensate for the noninertial nature of the frame, as discussed in Chapter 6. Specifically, to each particle of mass m_i we must assign a fictitious force $\vec{\mathbf{F}}_{\rm fict}(t) = -m_i\vec{\mathbf{a}}_{\rm cm}(t)$, where $\vec{\mathbf{a}}_{\rm cm}(t)$ is the acceleration of the center of mass as measured in the inertial laboratory frame. These fictitious forces are equivalent to a uniform gravitational field with acceleration vector $\vec{\mathbf{g}}(t) = -\vec{\mathbf{a}}_{\rm cm}(t)$. It is shown in Problem 8E.7 that the total torque about a horizontal axis caused by a uniform gravitational field can be calculated as if all of the force were applied directly to the center of mass. Hence, in the center-of-mass frame the fictitious forces produce no torque about an axis that goes through the center of mass, and hence no torque about the axis of rotation, the z-axis. That is the reason why we chose a coordinate system centered on the center of mass. The torque which appears in the rotational equation of motion can therefore be calculated directly from the real physical forces, such as the force of friction or forces applied by ropes, ignoring the fictitious forces.

To describe a wheel that is rolling without slipping, it is also necessary to understand the relation that such rolling imposes between the angular and linear velocities. In the center-of-mass frame of the wheel, the axle of the wheel is at rest and the edges are moving with a speed $v = R|\omega|$, where R is the radius of the wheel and ω is its angular velocity. (The absolute value sign is necessary because the speed v is by definition positive, but ω may not be.) If the ground is in contact with the wheel and there is no slippage at the interface, then the ground must move at the same speed v. In the rest frame of the ground, therefore, the axle of the wheel must be moving at speed

$$\boxed{v = \pm R|\omega|.}$$

For $\omega > 0$:



As we saw in Chapter 5, the kinetic energy of an arbitrary system of particles can be conveniently decomposed into contributions from the center-of-mass motion and from the motion about the center of mass:

$$K_{\rm tot} = \frac{1}{2}M_{\rm tot}v_{\rm cm}^2 + \sum_i \frac{1}{2}m_i\left(\vec{\mathbf{v}}_i - \vec{\mathbf{v}}_{\rm cm}\right)^2\ .$$

Specializing this relation to the motion of a rigid body, it becomes

$$\boxed{K_{\text{tot}} = \frac{1}{2}M_{\text{tot}}v_{\text{cm}}^2 + \frac{1}{2}I_{\text{cm}}\omega^2 \ ,}$$

Problems 8E.5, 8E.8, 10.10

rigid body KE rotational kinetic energy i.e. the sum of the ***translational*** kinetic energy of the body taken as a point mass and the ***rotational*** kinetic energy about an axis through the center of mass.

Apart from rotational kinetic energy, the new concepts in this chapter (i.e., angular velocity, angular acceleration, torque, and angular momentum) are analogous to ***vector*** quantities in linear motion, and we shall see in the next chapter that they are indeed vectors. We can treat them as scalars for the special case of rotation about an axis of fixed orientation, just as we can treat force and velocity as scalars when dealing with motion in a straight line. To solve more complicated problems in three dimensions, we will need to treat all of these quantities as vectors, as will be discussed in the next chapter.

To apply the techniques of this chapter, we frequently need to calculate I for the body and axis of interest. In many cases we might know or be able to calculate the moment of inertia about some other axis, so it is useful to know how the moments of inertia about different axes are related. One useful relation is the ***parallel-axis theorem***, which relates the moment of inertia I_{cm} about any axis through the center of mass to the moment of inertia $I_{\parallel}$ about any axis parallel to the first:

$$I_{\parallel} = I_{\text{cm}} + Md^2 \ ,$$

Problems 8D.2, 8E.3, and 8E.4

where M is the mass of the object and d is the distance between the two axes. Md^2 is simply the moment of inertia about the $I_{\parallel}$ axis of a single point particle at the center of mass of the body, so this formula is very similar to the one for the kinetic energy of a system of particles.

Another useful relation is the ***perpendicular-axis theorem***, which applies to bodies in the shape of a flat sheet, such as a sheet of paper or a compact disc. If we define a Cartesian coordinate system for which the object lies in the xy-plane, then the moments of inertia about the three axes are related by

$$I_z = I_x + I_y \ ,$$

Problems 8C.3, 8C.4, and 8C.5

where I_z is the moment of inertia about the z-axis, etc.

TABLE OF STANDARD MOMENTS OF INERTIA

The moment of inertia of any arbitrary rigid body or system of rigid bodies about any axis can always be calculated from the definition, $I = \sum_i m_i R_i^2$, generalizing the sum to an integral if necessary. The integration, however, can be quite complicated, so for convenience the following table contains the moments of inertia of some simple objects (each with mass m) about various axes. More complicated shapes, or moments around different axes, can often be constructed from these using the parallel and perpendicular-axis theorems, along with the principle that the moment of inertia of a complicated system is equal to the sum of the moments of inertia of its parts (all taken about the same axis). In the following table the right-hand column is left as an exercise for you—the examples listed can be constructed from the values in the left-hand column.

Slender uniform rod of length ℓ, axis through center and perpendicular to axis of rod	$\frac{1}{12}m\ell^2$	Slender uniform rod of length ℓ, axis through one end and perpendicular to axis of rod	
Rectangular plate with dimensions $a \times b$, axis along one of the b edges	$\frac{1}{3}ma^2$	Rectangular plate with dimensions $a \times b$, axis through center and perpendicular to plate	
Thin-walled hollow cylinder of radius R, axis along axis of cylinder	mR^2	Thick-walled hollow cylinder of inner radius R_1 and outer R_2, axis along axis of cylinder	
Uniform solid cylinder of radius R, axis along axis of cylinder	$\frac{1}{2}mR^2$		
Thin-walled hollow sphere of radius R, axis through center	$\frac{2}{3}mR^2$		
Solid uniform sphere of radius R, axis through center	$\frac{2}{5}mR^2$		

SUMMARY

* A rigid body is an idealized version of a physical solid object in which the configuration of the body (i.e. the relative positions of its constituent atoms or molecules) is completely fixed. Such a body has only two possible forms of motion: translational motion (like that of a point particle) and rotation.

* Rotational motion in two dimensions is governed by a system of equations closely analogous to those we have already encountered for point-particle motion (Newton's laws and the work-energy theorem). The concepts of velocity, acceleration, mass, force, and momentum are replaced by analogous concepts of angular velocity, angular acceleration, moment of inertia, torque, and angular momentum.

* Angular momentum—defined for two-dimensional motion as the product of the angular velocity and the moment of inertia of the object about the rotation axis—is conserved for any closed system. Like the conservation of energy and momentum, the conservation of angular momentum is believed to be an exact principle of nature, with a validity extending beyond that of classical mechanics.

* Combined translational and rotational motion of a rigid body can also be described straightforwardly, provided that (1) the translation is confined to a plane, (2) the axis of rotation is perpendicular to that plane, and (3) the rotating body is symmetrical about an axis parallel to the axis of rotation. If all three conditions are met, then the motion is two-dimensional. (Two-dimensional motion is actually possible under more general conditions, but their description is beyond the scope of this book.) The acceleration of the center of mass is equal to the total external force divided by the total mass; the angular acceleration is equal to the total external torque about the axis through the center of mass and parallel to the rotation axis, divided by the moment of inertia about this center-of-mass axis.

* Physical concepts introduced in this chapter: rigid body; angular velocity, angular acceleration, rotational kinetic energy, moment of inertia, torque, angular momentum.

* Mathematical concepts introduced in this chapter: none (but you should be sure that you understand the use of radians in measuring angles).

* Equations introduced in this chapter:

  Most of the equations in this chapter are most easily remembered in the context of the analogous equations for linear motion in one dimension. These are tabulated on the following page.

  Other equations introduced in this chapter:

$$v_r = 0\ ; \quad v_\perp = R\omega \qquad \text{(velocity of point on rotating body);}$$

$$a_r = -\frac{v^2}{R} = -R\omega^2\ ; \quad a_\perp = R\alpha \qquad \text{(acceleration of point on rotating body);}$$

$$v = \pm R|\omega| \qquad \text{(rolling without slipping);}$$

$$\left.\begin{aligned} \sum \vec{\mathbf{F}}^{\text{ext}} &= M\vec{\mathbf{a}}_{\text{cm}} = \frac{\mathrm{d}\vec{\mathbf{p}}}{\mathrm{d}t} \\ \sum \tau^{\text{ext}} &= I_{\text{cm}}\alpha = \frac{\mathrm{d}L}{\mathrm{d}t} \end{aligned}\right\}$$ (combined translational and rotational motion);

$K_{\text{tot}} = \frac{1}{2}Mv_{\text{cm}}^2 + \frac{1}{2}I_{\text{cm}}\omega^2$ (kinetic energy for combined translational and rotational motion);

$I_{\|} = I_{\text{cm}} + Md^2$ (parallel-axis theorem);

$I_z = I_x + I_y$ (perpendicular-axis theorem).

TRANSLATION (one dimension)		ROTATION (about fixed axis)	
Name	Symbol	Name	Symbol
Position	x	Orientation	θ
Velocity	$v = \dfrac{\mathrm{d}x}{\mathrm{d}t}$	Angular velocity	$\omega = \dfrac{\mathrm{d}\theta}{\mathrm{d}t}$
Acceleration	$a = \dfrac{\mathrm{d}v}{\mathrm{d}t}$	Angular acceleration	$\alpha = \dfrac{\mathrm{d}\omega}{\mathrm{d}t}$
Mass	$M = \sum_i m_i$	Moment of inertia	$I = \sum_i m_i R_i^2$
Force	F	Torque	$\tau = F_\perp R$ $= \pm\lvert\vec{\mathbf{F}}\rvert R_\perp$
Force equation	$\sum_i \vec{\mathbf{F}}^{\text{ext}} = M\vec{\mathbf{a}}_{\text{cm}}$	Torque equation	$\sum_i \tau^{\text{ext}} = I\alpha$
Momentum	$p = Mv$	Angular momentum	$L = I\omega$
Kinetic energy	$\frac{1}{2}Mv^2$	Kinetic energy	$\frac{1}{2}I\omega^2$
Work done	$\vec{\mathbf{F}} \cdot \overrightarrow{\Delta r}$	Work done	$\tau\,\Delta\theta$

PROBLEMS AND QUESTIONS

By the end of this chapter you should be able to answer or solve the types of questions or problems stated below.

Note: throughout this book, in multiple-choice problems, the answers have been rounded off to 2 significant figures, unless otherwise stated.

At the end of the chapter there are answers to all the problems. In addition, for problems with an (H) or (S) after the number, there are respectively hints on how to solve the problems or completely worked-out solutions.

8A ANGULAR MOTION

8A.1 At the start of play, a compact disc is spinning at about 700 revolutions per minute; by the end of the disc, one hour later, it has slowed to 200 revolutions per minute. What is its average angular acceleration?

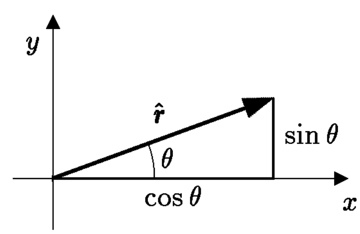
(a) $-8.3\ \mathrm{rad/s^2}$; (b) $-0.0023\ \mathrm{rad/s^2}$; (c) $-0.87\ \mathrm{rad/s^2}$; (d) none of these.

If the angular acceleration is *constant*, what is the total angle through which any point on the CD has turned during the hour? .

(a) 2.7×10^4 rad; (b) 1.7×10^5 rad; (c) 9.7×10^6 rad; (d) none of these.

8A.2 (S) (a) Since the concepts of radial and tangential directions are used so frequently in describing rotational motion, it is sometimes useful to define unit vectors in these directions. These unit vectors are peculiar, however, because their directions depend upon the position of the particle under discussion. For a particle located in the xy-plane at an angle θ counterclockwise from the x-axis, a unit vector in the radial direction can be written as

$$\hat{\boldsymbol{r}}(\theta) = [\cos\theta, \sin\theta, 0]\ .$$



Find the corresponding expression for the unit (counterclockwise) tangential vector $\hat{\boldsymbol{u}}_\perp(\theta)$.

(b) Show that the derivatives of these unit vectors are given by

$$\frac{d\hat{\boldsymbol{r}}(\theta)}{d\theta} = \hat{\boldsymbol{u}}_\perp(\theta)$$
$$\frac{d\hat{\boldsymbol{u}}_\perp(\theta)}{d\theta} = -\hat{\boldsymbol{r}}(\theta)\ .$$

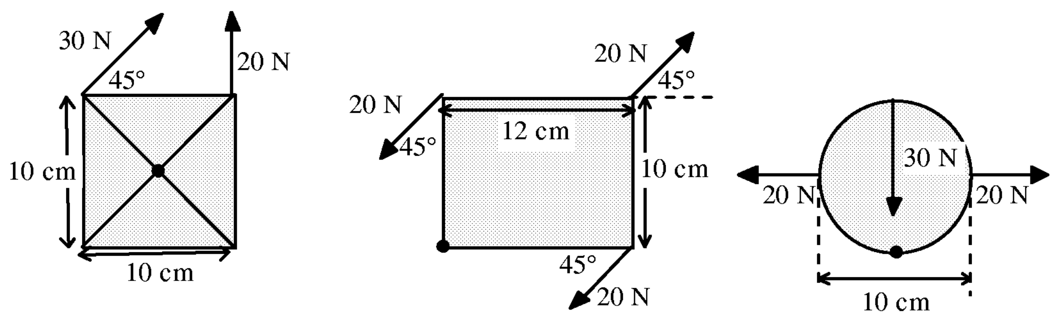
(c) Use these results to prove the formula given in the *Essentials* for the acceleration of a particle on a rigid body rotating about a fixed axis. Specifically, show that the radial and tangential components of the acceleration are given, respectively, by $a_r = -v^2/R = -R\omega^2$ and $a_\perp = R\alpha$, where ω is the angular velocity, α is the angular acceleration, and R is the distance of the particle from the axis of rotation.

8B TORQUE ABOUT AN AXIS

8B.1 The handle of a door is 1 m from its hinged side. You apply a force of 20 N to the door handle, pulling at right angles to the door. What is the torque about the axis through the door hinges?

(a) 20 N; (b) 20 N · m; (c) no net torque; (d) none of these.

8B.2 (H) Calculate the net torque in each of the following cases about a fixed axis perpendicular to the page and passing through the indicated black dot. (Assume you are looking down on the xy-plane, so counterclockwise torques are positive.)

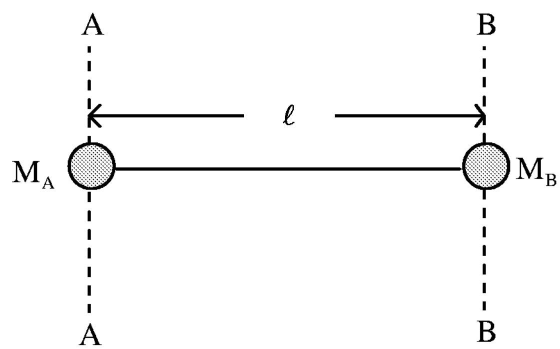


8C MOMENT OF INERTIA

8C.1 (S) A binary star system consists of two stars, one of mass 2×10^{30} kg and one of mass 3×10^{30} kg, separated by 1.5×10^{14} m. Find the moment of inertia of this system about an axis through its center of mass and perpendicular to the line joining the two stars.

8C.2 A point mass M_A is connected to a point mass M_B by a rod of length ℓ and negligible mass. It is observed that the ratio of the moments of inertia of the system about the two axes AA and BB, which are parallel to each other and perpendicular to the rod, is

$$\frac{I_{\mathrm{BB}}}{I_{\mathrm{AA}}} = 3 \, .$$



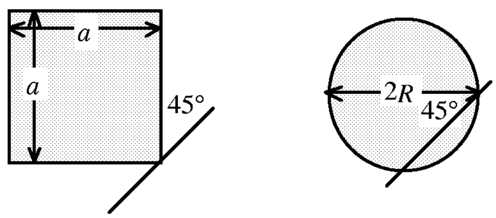
The distance of the center of mass of the system from mass M_A is:

(a) $3\ell/4$; (b) $2\ell/3$; (c) $\ell/2$; (d) $\ell/3$; (e) $\ell/4$; (f) $\ell/9$.

8C.3 (S) Prove the parallel-axis and perpendicular-axis theorems—i.e. (a) prove that the moment of inertia of a rigid body of mass M about an axis through its center of mass is related to the moment of inertia $I_{\|}$ about any axis parallel to the first by the formula $I_{\|} = I_{\mathrm{cm}} + Md^2$, where d is the distance between the two axes; and (b) prove that for a thin flat object in the xy-plane, the moment of inertia about the z-axis is equal to the sum of the moments of inertia about the x- and y-axes.

8C.4 (S) Calculate the moments of inertia of (a) a thin rod about an axis through its center and perpendicular to the rod; (b) a thin flat disk about an axis through its center and perpendicular to its plane; (c) a solid sphere about any diameter. Then deduce the moments of inertia of (d) a thin rod about an axis at one end; (e) a thin flat disk about any diameter; (f) a hollow spherical shell. Assume all the objects are of uniform density and each has mass M.

8C.5 (H) Calculate the moment of inertia of the following shapes about the specified axes (shown by a thick line). In each case assume that the object in question is a thin sheet of mass M. Note that you do *not* need to do any integration to solve this problem (but you do need the table at the end of the *Essentials)*.



If I apply a torque τ with respect to the specified axis to each of these objects, what will be the angular acceleration in each case?

8D ROTATIONAL KINETIC ENERGY AND ANGULAR MOMENTUM ABOUT AN AXIS

8D.1 The Earth's moment of inertia about its axis of rotation is known to be 8.1×10^{37} kg $\cdot$ m^2. What is the Earth's rotational kinetic energy as a result of its spinning around its axis? (Neglect the effects of the Earth's orbit around the Sun.)

(a) 2.1×10^{29} J; (b) 4.2×10^{29} J; (c) 3.4×10^{28} J; (d) 4.2×10^{33} J.

8D.2 (H) A ruler of mass m, length ℓ and width w has a hole drilled in it a short distance x from one end and equidistant from both sides. It is anchored on a frictionless air table by a nail driven through the hole and is then set rotating about the nail such that its far end (a distance $\ell - x$ from the hole) is moving with (linear) speed v. Obtain expressions for the rotational kinetic energy K and angular momentum L of the ruler with respect to the axis through the nail. Calculate the values of K and L if the ruler is 35 cm long, 2 cm wide and has a mass of 50 g, the hole is 1 cm from one end, and the other end is moving at 0.2 m/s.

8D.3 Suppose that a long thin rod, of mass M and length ℓ, is suspended horizontally from a stiff wire, with the wire passing through the midpoint of the rod, and the rod is then rotated by an angle θ about the axis formed by the wire. It is found experimentally that a wire twisted in this way exerts a restoring torque $\tau = -K\theta$, where K is a constant, so the rod will undergo simple harmonic motion, forming a ***torsion pendulum***. Let θ_m denote the amplitude of the oscillations.

(a) Find the period of the oscillations.

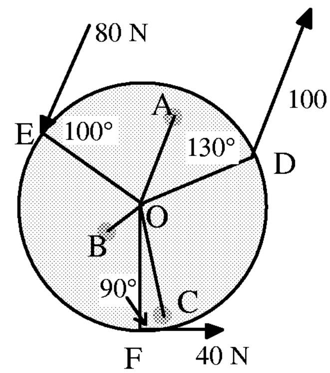
(b) What is (i) the rotational kinetic energy, and (ii) the angular momentum, of the rod at the point when it reaches its maximum angular speed?

8D.3, continued:

(c) If I cut the rod down to three-quarters of its original length and re-hang it, what will the new period be? (Assume that the wire still passes through the center of the reduced rod, and express your answer in terms of the original values of K, M and ℓ.)

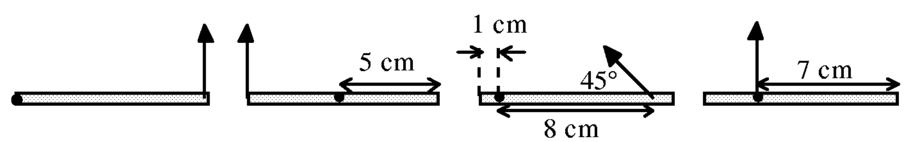
8E ROTATION OF A RIGID BODY ABOUT AN AXIS

8E.1 (S) A children's playground contains a circular merry-go-round of radius 2 m. One afternoon, three children are sitting on the merry-go-round while three of their friends push its rim to make it revolve. The positions of the children and the forces they apply are shown on the diagram. What is the net torque on the merry-go-round, and what would its acceleration be in the absence of friction, given that the merry-go-round itself is a uniform disk of mass 200 kg and Alfredo, Betty and Chris have masses 30, 40 and 25 kg respectively?

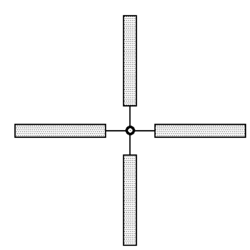


OA = 1.5 m, OB = 0.7 m, OC = 1.8 m

8E.2 Find the torque about the specified axis in each of the following situations, given that the force is 20 N in each case, the rod is 10 cm long, and the fixed axis is perpendicular to the page and passes through the black dot. If the mass of the rod is 0.5 kg, also find the angular acceleration.



8E.3 (S) A ceiling fan consists of four blades arranged in a cross. Each blade has mass 2 kg and is one meter long and 10 cm wide, and is connected to the central spindle of the fan by a rod of length 10 cm and negligible mass. The fan rotates at a steady rate of 30 revolutions per minute. What is its kinetic energy and its angular momentum? If it takes 10 s to reach this speed when switched on, what is the average net torque delivered by the motor during this period, and how much work is done?



8E.4 (H) The ruler of problem 8D.2, with length ℓ, width w and mass m, is suspended by a rod passing through the hole x from one end, which thus forms a fixed axis perpendicular to the plane of the ruler. If the ruler is held so that it makes an angle of θ to the vertical and then released, what is the torque acting on it, and what is its angular acceleration? Could such an arrangement be used as a pendulum (i.e. will it undergo simple harmonic motion)?

8E.5 (S) You are on a camping trip in a rural district. Your water supply comes from a well, and is obtained by hauling up a large bucket using a windlass, i.e. a cylindrical spindle turned by a handle. You have just hauled a full bucket of water, mass 15 kg, the 10

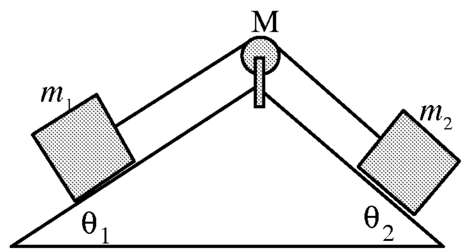
8E.5, continued:

m from the water surface to ground level (at negligible speed) when your hand slips off the windlass handle.

(a) If the windlass is a solid cylinder of mass 20 kg and radius 12 cm, how fast is it spinning when the bucket hits the water? Neglect the mass of the rope and the friction in the bearings of the windlass, take $g = 9.8\ \mathrm{m/s^2}$, and assume the rope does not slip.

(b) What was the tension in the rope while the bucket was falling, and how long did the fall take?

8E.6 (H) Masses m_1 and m_2 are connected by a light string passing over a pulley as shown. The pulley is a solid uniform disk of mass M and radius R, and the friction between it and the rope is such that when the blocks move the rope turns the pulley without slipping. The friction between the blocks and the slopes is negligible and the rope does not stretch. Find the accelerations of both masses and the angular acceleration of the pulley.



8E.7 (S) (a) A rigid body of unspecified shape is allowed to pivot freely about a horizontal axis through its center of mass. The only force acting is gravity. Show that there is no net torque.

(b) Hence show that the net force of gravity on a rigid body can always be regarded as acting through the center of mass—i.e., the torque it exerts about any horizontal axis is the same as if the total force acted at the center of mass.

8E.8 (S) A rigid body of mass M and having a circular cross-section of radius R rolls without slipping down a slope making an angle θ to the horizontal. The moment of inertia of the body about its central axis of symmetry is kMR^2, where k is a numerical constant.

(a) What is its speed when it has descended through a vertical distance h?

(b) What is the minimum coefficient of static friction required to ensure that it rolls, rather than slides, down the slope?

8E.9 A group of children are playing a game involving rolling an assortment of objects down a slope. The objects include a solid uniform rubber ball, a thin hoop, a solid uniform cylindrical log, a spherical leather soccer ball, essentially all of whose mass is in its leather cover rather than its interior, and a discarded wheel from a toy cart, which has the form of a solid disk of radius R with a central hole of radius $R/2$. All the objects roll without slipping down the slope.

(a) The slope has length ℓ and is at an angle of θ to the horizontal. How long does each object take to reach the bottom? Assume all the objects are small compared to ℓ.

(b) Your little sister is desperate to win this game. Suggest an object that you could make for her which would complete the course faster than any of those currently in play.

8E.5, continued:

8E.10 You are helping out at a local auto mechanic's shop, and he asks you to fetch a truck tire from the store. The tire has a mass of 30 kg, divided equally between its outer tread (which has a diameter of 90 cm) and its sidewalls (which have an inner radius of 30 cm). You roll the tire across the level shop floor at a speed of 1.5 m/s.

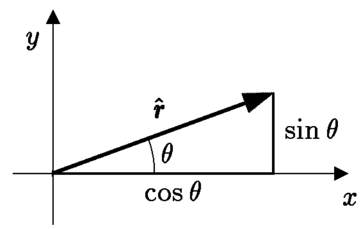
(a) From the description, we can model the tire as a thin cylinder of diameter 90 cm and mass 15 kg, and two disks each with outer diameter 90 cm, inner diameter 60 cm, and mass 7.5 kg. Using this model, calculate the work that you did in accelerating the tire from rest to 1.5 m/s.

(b) What impulse did you supply to the tire? If it took you 4 s to reach your final speed, what average torque did you exert?

COMPLETE SOLUTIONS TO PROBLEMS WITH AN (S)

8A.2 (a) *Since the concepts of radial and tangential directions are used so frequently in describing rotational motion, it is sometimes useful to define unit vectors in these directions. These unit vectors are peculiar, however, because their directions depend upon the position of the particle under discussion. For a particle located in the xy-plane at an angle θ counterclockwise from the x-axis, a unit vector in the radial direction can be written as*

$$\hat{\boldsymbol{r}}(\theta) = [\cos\theta, \sin\theta, 0] \ .$$



Find the corresponding expression for the unit (counterclockwise) tangential vector $\hat{\boldsymbol{u}}_\perp(\theta)$.

(b) *Show that the derivatives of these unit vectors are given by*

$$\frac{\mathrm{d}\hat{\boldsymbol{r}}(\theta)}{\mathrm{d}\theta} = \hat{\boldsymbol{u}}_\perp(\theta)$$
$$\frac{\mathrm{d}\hat{\boldsymbol{u}}_\perp(\theta)}{\mathrm{d}\theta} = -\hat{\boldsymbol{r}}(\theta) \ .$$

(c) *Use these results to prove the formula given in the* **Essentials** *for the acceleration of a particle on a rigid body rotating about a fixed axis. Specifically, show that the radial and tangential components of the acceleration are given, respectively, by $a_r = -v^2/R = -R\omega^2$ and $a_\perp = R\alpha$, where ω is the angular velocity, α is the angular acceleration, and R is the distance of the particle from the axis of rotation.*

Conceptualize

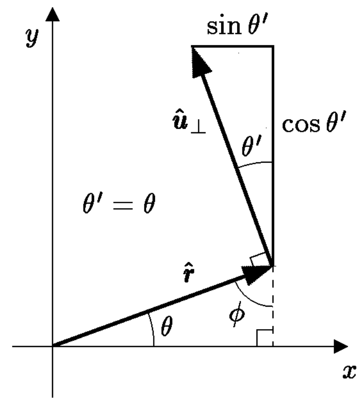
In this case the questioner has been helpful, breaking up the problem into relatively small steps. One can view part (c) as being the main question, in which case parts (a) and (b) can be construed as the "Formulate" steps in answering it.

One can seek an expression for $\hat{\boldsymbol{u}}_\perp$ with either a geometric approach, by drawing a careful diagram, or with an algebraic approach, by seeking a unit vector which has a vanishing dot product with $\hat{\boldsymbol{r}}$. To distinguish counterclockwise from clockwise, however, it seems that at least a crude diagram is necessary. Here we will use the geometric approach, but we will verify that the answer has the right algebraic properties. By the time we reach parts (b) and (c), each result will follow from the previous results by straightforward calculus, so diagrams will no longer be necessary.

8A.2, continued:

Formulate and Solve (a)
From the diagram it seems obvious that $\theta' = \theta$. One can prove it by first noticing that θ and ϕ are two angles of a right triangle, so ϕ is the complement of θ (i.e., $\phi = 90° - \theta$); then notice that ϕ, θ', and a right angle meet to form a straight angle (180°), so θ' is the complement of ϕ, and hence $\theta' = \theta$. One can then see that the horizontal and vertical components of $\hat{\boldsymbol{u}}_\perp$ are $-\sin\theta$ and $\cos\theta$, so

$$\hat{\boldsymbol{u}}_\perp = [-\sin\theta, \cos\theta, 0] \ .$$



(If you are still having difficulty distinguishing the sine from the cosine, a practical technique is to always draw your angles noticeably smaller than 45°, as was done here. Then the short side of the triangle is always proportional to the sine, and the long side is proportional to the cosine.)

Scrutinize (a)
One can easily check that the expression for $\hat{\boldsymbol{u}}_\perp$ has unit length,

$$\left|[-\sin\theta, \cos\theta, 0]\right|^2 = \sin^2\theta + \cos^2\theta = 1,$$

and that it is perpendicular (i.e., has zero dot product) with $\hat{\boldsymbol{r}}$:

$$\hat{\boldsymbol{u}}_\perp \cdot \hat{\boldsymbol{r}} = [-\sin\theta, \cos\theta, 0] \cdot [\cos\theta, \sin\theta, 0] = -\sin\theta\cos\theta + \cos\theta\sin\theta = 0 \ .$$

However, these checks do not test whether $\hat{\boldsymbol{u}}_\perp$ is counterclockwise or clockwise. To get that right we need to think about the diagram. If θ is in the first quadrant (i.e., $0 < \theta < \pi/2$), then a counterclockwise vector should have a horizontal component to the left, which means the negative x-direction. Since the first component of our expression for $\hat{\boldsymbol{u}}_\perp$ is negative, we must have this right.

Formulate and Solve (b)
Since we have an explicit component expression for the vectors, we can proceed by differentiating the components:

$$\begin{aligned}\frac{d\hat{\boldsymbol{r}}(\theta)}{d\theta} &= \frac{d}{d\theta}[\cos\theta, \sin\theta, 0] \\ &= [-\sin\theta, \cos\theta, 0] = \hat{\boldsymbol{u}}_\perp \ .\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{d\hat{\boldsymbol{u}}_\perp(\theta)}{d\theta} &= \frac{d}{d\theta}[-\sin\theta, \cos\theta, 0] \\ &= [-\cos\theta, -\sin\theta, 0] = -\hat{\boldsymbol{r}} \ .\end{aligned}$$

8A.2, continued:

Scrutinize (b)
You may have noticed that in both cases ($\hat{\mathbf{r}}$ and $\hat{\mathbf{u}}_\perp$), the derivative of the unit vector with respect to θ is perpendicular to the original unit vector. In fact, it can be shown that differentiation of any ***unit*** vector, with respect to any parameter, always produces a vector that is perpendicular to the original unit vector. To see this, let $\hat{\mathbf{u}}(\lambda)$ represent an arbitrary unit vector that depends on some parameter called λ. Since $\hat{\mathbf{u}}$ is assumed to be a unit vector for all values of λ, we can write

$$\hat{\mathbf{u}}(\lambda) \cdot \hat{\mathbf{u}}(\lambda) = 1 \ ,$$

which we can differentiate to find

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{\hat{\mathbf{u}}(\lambda) \cdot \hat{\mathbf{u}}(\lambda)\right\} = \frac{\mathrm{d}}{\mathrm{d}\lambda} 1 \ ,$$

which implies

$$2\hat{\mathbf{u}} \cdot \frac{\mathrm{d}\hat{\mathbf{u}}}{\mathrm{d}\lambda} = 0 \ .$$

Thus $\hat{\mathbf{u}}$ is perpendicular to $\mathrm{d}\hat{\mathbf{u}}/\mathrm{d}\lambda$.

Formulate (c)
The particle on the rigid body lies at a distance R from the fixed axis of rotation, and we can call its z-coordinate z. We can then describe its position completely by specifying at any time the angle $\theta(t)$ of its x- and y- coordinates in the xy-plane, measured counterclockwise from the x-axis. Specifically, the position vector $\vec{\mathbf{r}}(t)$ describing the trajectory of the particle can be written as

$$\vec{\mathbf{r}}(t) = [0, 0, z] + R\,\hat{\mathbf{r}}\big(\theta(t)\big) \ .$$

Given this expression, we will be able to find the velocity and acceleration by straightforward differentiation.

Solve (c)
To differentiate the expression above, one notices that the only time dependence appears in $\theta(t)$. Using the chain rule for differentiating a function of a function,

$$\begin{aligned}
\vec{\mathbf{v}}(t) &= \frac{\mathrm{d}\vec{\mathbf{r}}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{[0,0,z] + R\,\hat{\mathbf{r}}\big(\theta(t)\big)\right\} \\
&= R\,\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}\theta}\,\frac{\mathrm{d}\theta}{\mathrm{d}t} = R\,\hat{\mathbf{u}}_\perp(\theta)\,\omega(t) = R\,\omega(t)\,\hat{\mathbf{u}}_\perp(\theta) \ .
\end{aligned}$$

Differentiating again, using the product rule and once more the chain rule,

$$\begin{aligned}
\vec{\mathbf{a}}(t) &= \frac{\mathrm{d}\vec{\mathbf{v}}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{R\,\omega(t)\,\hat{\mathbf{u}}_\perp(\theta)\right\} \\
&= R\,\omega\,\frac{\mathrm{d}\hat{\mathbf{u}}_\perp(\theta)}{\mathrm{d}t} + R\,\frac{\mathrm{d}\omega(t)}{\mathrm{d}t}\hat{\mathbf{u}}_\perp = R\,\omega\,\frac{\mathrm{d}\hat{\mathbf{u}}_\perp}{\mathrm{d}\theta}\,\frac{\mathrm{d}\theta}{\mathrm{d}t} + R\,\alpha\,\hat{\mathbf{u}}_\perp = -R\,\omega^2\,\hat{\mathbf{r}} + R\,\alpha\,\hat{\mathbf{u}}_\perp \ .
\end{aligned}$$

8A.2, continued:

This verifies the final result, except to check that the radial acceleration can also be written as $-v^2/R$. By using $v = |\vec{\mathbf{v}}| = R\omega$ from the intermediate result, however, it is easy to see that $-v^2/R = -(R\omega)^2/R = -R\omega^2$.

Scrutinize (c)
The result for the acceleration is reasonable, since we have known for some time that particles in uniform circular motion have an acceleration toward the origin of magnitude v^2/R. The new piece $R\,\alpha\,\hat{\boldsymbol{u}}_\perp$ arises when ω is not constant, in which case we would expect a contribution to the acceleration related to the rate of change of the tangential velocity.

Learn
While the results of parts (a) and (b) of this question were used as intermediate steps toward the goal of finding the answer to (c), they are also valuable results in themselves. For example, if we wanted to plot the trajectory of a particle on a rolling wheel, we would probably want to use the vectors $\hat{\boldsymbol{r}}(\theta)$ and $\hat{\boldsymbol{u}}_\perp(\theta)$, and the properties derived here.

8C.1 *A binary star system consists of two stars, one of mass* 2×10^{30} kg *and one of mass* 3×10^{30} kg*, separated by* 1.5×10^{14} m. *Find the moment of inertia of this system about an axis through its center of mass and perpendicular to the line joining the two stars.*

Conceptualize
We can treat the stars as point masses, since they are small compared to their separation. (The radius of the Sun, for example, is 7×10^8 m.) To solve the problem we must first find the center of mass, and then find the moment of inertia of each star about this axis. Adding these will give us the total moment of inertia. For convenience we can choose an x-axis which goes through both of the stars, and we can even arrange for the origin ($x = 0$) to be located at the center of mass.

Formulate
The position $x_{\rm cm}$ of the center of mass of two bodies, one with mass m_1 at position x_1 and the other with mass m_2 at position x_2, is given by

$$x_{\rm cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \ .$$

By choosing the origin to lie at the center of mass, we require the condition

$$m_1 x_1 + m_2 x_2 = 0 \ .$$

Using ℓ to denote the separation between the stars, the relation above can be combined with

$$\ell = x_2 - x_1$$

to obtain a pair of equations that can be solved for the two unknowns, x_1 and x_2. Since x_1 and x_2 are the distances of the two stars from the axis of rotation, the moment of inertia is then given by

$$I = \sum_{i=1}^{2} m_i x_i^2 = m_1 x_1^2 + m_2 x_2^2 \ .$$

8C.1, continued:

Solve
Solving the pair of equations for x_1 and x_2, one finds

$$x_1 = -\frac{m_2}{m_1 + m_2}\ell \qquad \textit{and} \qquad x_2 = \frac{m_1}{m_1 + m_2}\ell \ .$$

The next step is to insert these expressions into the equation for I and then simplify, with the result

$$I = \frac{m_1 m_2}{m_1 + m_2}\ell^2 \ .$$

Numerically, this gives

$$\begin{aligned} I &= \frac{(2 \times 10^{30}\ \mathrm{kg})(3 \times 10^{30}\ \mathrm{kg})}{(2+3) \times 10^{30}\ \mathrm{kg}} \times (1.5 \times 10^{14}\ \mathrm{m})^2 \\ &= 2.7 \times 10^{58}\ \mathrm{kg \cdot m^2} \ . \end{aligned}$$

Scrutinize
The dimensions of I are [mass]$\times$[length]^2, as can be seen from its definition. The answer approaches zero as either of the two masses approaches zero, which is a property that we might have foreseen: if one mass vanishes, then the distance of the center of mass from the other mass is zero, and hence the moment of inertia is zero.

Learn
This example is fairly straightforward, but note that *any* calculation of I uses exactly the same principles. The quantity $m_1 m_2/(m_1 + m_2)$ appears so often in problems involving two-body systems that it is given a name: the ***reduced mass***.

8C.3 *Prove the parallel-axis and perpendicular-axis theorems—i.e. (a) prove that the moment of inertia of a rigid body of mass M about an axis through its center of mass is related to the moment of inertia $I_\parallel$ about any axis parallel to the first by the formula $I_\parallel = I_{\mathrm{cm}} + Md^2$, where d is the distance between the two axes; and (b) prove that for a thin flat object in the xy-plane, the moment of inertia about the z-axis is equal to the sum of the moments of inertia about the x- and y-axes.*

Conceptualize
The moment of inertia of a rigid body about a specified axis is defined as $I = \sum\limits_i m_i R_i^2$, where R_i is the distance of the mass element m_i from the axis. To discuss the parallel axis theorem, we will call the axis through the center of mass N_{cm}, and we will call the other axis $N_\parallel$. To prove the theorem, we will need to re-express the distance of the mass element m_i from the $N_\parallel$ axis in terms of its distance from the N_{cm} axis; for the perpendicular-axis theorem, we need the distance from the z-axis in terms of the distances from the x- and y-axes.

Formulate (a)
For the parallel-axis theorem, it is easiest to choose a coordinate system with the origin at the center of mass, and with the z-axis coincident with the axis N_{cm}. The moment of inertia about this axis is then given by

$$I_{\mathrm{cm}} = \sum_i m_i R_i^2 = \sum_i m_i (x_i^2 + y_i^2) \ .$$

8C.3, continued:

The condition that the center of mass lies at $x_{\rm cm} = y_{\rm cm} = 0$ can be expressed as

$$x_{\rm cm} = \frac{1}{M_{\rm tot}} \sum_i m_i x_i = 0$$

$$y_{\rm cm} = \frac{1}{M_{\rm tot}} \sum_i m_i y_i = 0 \ .$$

The axis $N_\parallel$ is parallel to the z-axis, so it can be described by its x- and y-coordinates, $x_\parallel$ and $y_\parallel$. Since $N_\parallel$ is a distance d from the center of mass, it follows that

$$x_\parallel^2 + y_\parallel^2 = d^2 \ .$$

If we let R_i' denote the distance between the mass element m_i and the $N_\parallel$ axis, the moment of inertia about this axis can be written

$$I_\parallel = \sum_i m_i R_i'^2 = \sum_i m_i \left[(x_i - x_\parallel)^2 + (y_i - y_\parallel)^2 \right] \ .$$

Solve (a)
Expanding the expression above for $I_\parallel$,

$$\begin{aligned} I_\parallel &= \sum_i m_i \left[x_i^2 + y_i^2 - 2x_\parallel x_i - 2y_\parallel y_i + \left(x_\parallel^2 + y_\parallel^2 \right) \right] \\ &= \sum_i m_i \left(x_i^2 + y_i^2 \right) - 2x_\parallel \sum_i m_i x_i - 2y_\parallel \sum_i m_i y_i + \left(x_\parallel^2 + y_\parallel^2 \right) \sum_i m_i \ . \end{aligned}$$

The first term on the right-hand side is $I_{\rm cm}$, while the second and third terms vanish due to the center of mass condition written above. The final term can be written as Md^2, so we have the sought-after result:

$$I_\parallel = I_{\rm cm} + Md^2 \ .$$

Formulate (b)
The perpendicular-axis theorem is easier to prove. Since the object lies entirely in the xy-plane, the distance of a mass element m_i from the x-axis is $|y_i|$, and the distance from the y-axis is $|x_i|$. The distance of the mass element from the z-axis is $\sqrt{x_i^2 + y_i^2}$.

Solve (b)
Given the distance relationships discussed in the paragraph above, it is straightforward to see that

$$I_z = \sum_i m_i R_i^2 = \sum_i m_i (x_i^2 + y_i^2) = \sum_i m_i x_i^2 + \sum_i m_i y_i^2 = I_y + I_x \ ,$$

8C.3, continued:

which is the desired result.

Scrutinize

Although these proofs were written in terms of a summation over a set of discrete mass elements m_i, they could easily be extended to describe continuous bodies. The sums would then be replaced by integrations, but the proofs would be otherwise unchanged. Since an integral can be defined as the limit of a summation, all the necessary properties of summations are valid for integrals as well.

Learn

These formulas are useful because one often knows the moment of inertia of an object about one axis, but wants to know it about another. In particular, tables of moments of inertia, such as the one at the end of the *Essentials,* cannot possibly list the values of all possible axes. By using the parallel and perpendicular axis theorems, the applicability of such tables can be dramatically increased.

While the parallel and perpendicular axis theorems are very useful, please bear in mind their limitations:

- The perpendicular-axis theorem is only valid for an object in the form of a thin flat sheet, with two of the axes in the plane of the sheet and the third one perpendicular to it. The origin can be anywhere in the plane of the sheet.
- The parallel-axis theorem is only valid if one of the axes passes through the center of mass. Note, however, that as long as you know where the center of mass is, you can use the parallel axis theorem to relate the moments of inertia about any two axes that are parallel to each other. To do this, simply use the parallel axis theorem twice, relating each of the given axes to the axis which is parallel to both and runs through the center of mass.

8C.4 *Calculate the moments of inertia of (a) a thin rod about an axis through its center and perpendicular to the rod; (b) a thin flat disk about an axis through its center and perpendicular to its plane; (c) a solid sphere about any diameter. Then deduce the moments of inertia of (d) a thin rod about an axis at one end; (e) a thin flat disk about any diameter; (f) a hollow spherical shell. Assume all the objects are of uniform density and each has mass M.*

Conceptualize

These are applications of the basic formulas for calculating a moment of inertia. As solid objects are involved, we will first have to consider how to express the sum over discrete masses as an integral over volume—these are not ***conceptually*** different, but they do require different calculational techniques.

The wording of the question suggests that the last three cases can be solved without an explicit integration. Comparing (d) with (a), we can see that they satisfy the conditions for applying the parallel-axis theorem, while (b) and (e) have the appropriate geometry for the perpendicular-axis theorem. Although it is not immediately clear how to relate (f) and (c), a solid sphere can be built up from a series of concentric hollow shells, so

8C.4, continued:

the moment of inertia of a solid sphere is the sum of the moments of inertia of a series of shells.

Formulate

To convert our sum to an integral, we need to replace the 'mass m_i' at point i by a mass element dm and integrate over the whole mass:

$$I = \int_{\text{body}} R^2 \mathrm{d}m \ ,$$

where R is the distance of mass element dm from the axis of rotation. For a body of density ρ, dm is simply ρdV, where V is a volume element, so we have

$$I = \int_{\text{body}} \rho R^2 \mathrm{d}V \ .$$

The choice of dV depends on the object, as can be seen in the examples.

Solve (a): *Thin rod about perpendicular axis through center:*

This is quite straightforward. We choose the x-axis running down the rod with its origin at the rod's center. Then the volume element we need is just $A\mathrm{d}x$, where A is the (small) cross-sectional area of the rod, and we have

$$I = \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \rho A x^2 \mathrm{d}x \ ,$$

where ℓ is the total length of the rod. The mass of the rod is $\rho A\ell$, and we conclude

$$I = \frac{1}{12} M\ell^2 \ .$$

Solve (b): *Thin flat disk about perpendicular axis through center:*

Choose as a volume element the ring of material at a distance between r and $r + dr$ from the center of the disk. The volume of this ring is d$V = 2\pi r z \mathrm{d}r$, where z is the thickness of the disk. Our integral is therefore

$$I = 2\pi\rho z \int_0^R r^3 \mathrm{d}r \ ,$$

where R is the radius of the disk. This is a straightforward integration giving

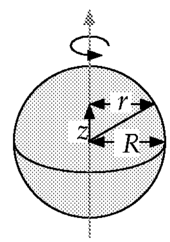
$$I = \frac{1}{2}\pi\rho z R^4 \ .$$

The total mass M of the disk is $\pi R^2 z\rho$, so we can rewrite this as $I = \frac{1}{2}MR^2$. In fact this holds not just for a thin disk, but for any solid cylinder (we never had to use the fact that z is small). Note that in this case we choose our volume element deliberately

8C.4, continued:

so that we only have to integrate over a single variable r; if we had chosen a simpler volume element (the small piece of disk at coordinates (x, y), say), we would have had to do a much more complicated two-dimensional integral.

<u>Solve</u> (c): *Solid sphere about a diameter:*
Since the moment of inertia is basically a sum, we can break it into pieces, just as we did when calculating centers of mass. Thus we can regard our sphere as a stack of thin disks, each with radius r and moment of inertia $\frac{1}{2}\pi\rho r^4 \mathrm{d}z$, where $\mathrm{d}z$ is the thickness of each disk. The moment of inertia of the whole sphere is then



$$I = \int_{-R}^{R} \frac{1}{2}\pi\rho r^4 \mathrm{d}z \ .$$

From the diagram, $r^2 = R^2 - z^2$, and thus the integral becomes

$$I = \frac{1}{2}\pi\rho \int_{-R}^{R} (R^4 - 2R^2 z^2 + z^4)\mathrm{d}z \ ,$$

which comes to $\frac{8}{15}\pi\rho R^5$. The mass of the sphere is $M = \frac{4}{3}\pi\rho R^3$, so we can write this as $I = \frac{2}{5}MR^2$.

<u>Solve</u> (d): *Rod about one end:*
We want the parallel-axis theorem, since the moment of inertia we have already calculated is the one about the center of mass. It follows that

$$\begin{aligned} I_{\text{end}} &= I_{\text{center}} + M(\ell/2)^2 \\ &= \frac{1}{12}M\ell^2 + \frac{1}{4}M\ell^2 \\ &= \frac{1}{3}M\ell^2 \ . \end{aligned}$$

<u>Solve</u> (e): *Thin flat disk about diameter:*
This is an obvious candidate for the perpendicular-axis theorem. If we take a coordinate system with origin at the center of the disk and assume the disk lies in the xy-plane, we have

$$I_z = I_x + I_y \ .$$

We have just calculated I_z, and clearly I_x and I_y are both moments of inertia about a diameter of the disk. They must therefore be equal to each other (there is nothing about our uniform disk to distinguish any diameter from any other diameter) and so

$$I_x = I_y = \frac{1}{2}I_z = \frac{1}{4}MR^2 \ .$$

8C.4, continued:

__Solve__ (f): *Hollow spherical shell about diameter:*
Because moments of inertia add, we can see that the moment of inertia of a sphere of radius $R + \mathrm{d}R$ must be equal to the moment of inertia of a sphere of radius R plus that of a hollow spherical shell of radius R and thickness $\mathrm{d}R$. Therefore

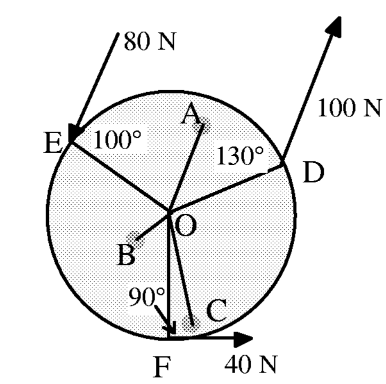
$$\begin{aligned} I_{\text{shell}} &= \frac{8}{15}\pi\rho(R+\mathrm{d}R)^5 - \frac{8}{15}\pi\rho R^5 \\ &= \frac{8}{15}\pi\rho\, 5R^4\mathrm{d}R \ , \end{aligned}$$

where we have ignored terms involving higher powers of $\mathrm{d}R$ (for a thin shell, $\mathrm{d}R/R \ll 1$, so $R^3(\mathrm{d}R)^2 \ll R^4\mathrm{d}R$ and so on). The mass of the shell is $M = 4\pi\rho R^2\mathrm{d}R$, so we can write the moment of inertia as $I_{\text{shell}} = \frac{2}{3}MR^2$. We could of course have obtained this by direct integration, using rings as volume elements, but the method used above is much easier once we have already done the solid sphere.

__*Scrutinize*__
Observe that, as we would expect from dimensional arguments, all the moments of inertia are of the form kMR^2, where k is a dimensionless constant and R is some characteristic length. However, the ***values*** of k vary quite significantly. For a given torque about the specified axis, which of these objects would have the greatest angular acceleration?

__*Learn*__
These examples illustrate the techniques for calculating moments of inertia. See the table at the end of the *Essentials* for a list of moments of inertia for some basic shapes about standard axes. More complicated objects can usually be built up using a combination of these with appropriate application of the parallel-axis and perpendicular-axis theorems.

8E.1 *A children's playground contains a circular merry-go-round of radius* 2 m. *One afternoon, three children are sitting on the merry-go-round while three of their friends push its rim to make it revolve. The positions of the children and the forces they apply are shown on the diagram. What is the net torque on the merry-go-round, and what would its acceleration be in the absence of friction, given that the merry-go-round itself is a uniform disk of mass* 200 kg *and Alfredo, Betty and Chris have masses* 30, 40 *and* 25 kg *respectively?*



OA = 1.5 m, OB = 0.7 m,
OC = 1.8 m

8E.1, continued:

Conceptualize

There are three separate calculations in this problem. We have to determine the torque applied to the object consisting of merry-go-round plus children; calculate the moment of inertia of that object; and finally combine our results to find the angular acceleration.

The torque calculation is straightforward: we know the forces applied and the points of application, and the relevant axis is vertically through the center of the disk of the merry-go-round.

For the moment of inertia, we will have to add the moments of inertia of the merry-go-round (which is a uniform disk) and the children. We are not told the dimensions of the children, but we saw above that moments of inertia around the center of mass take the form kMR^2, where k is usually less than one. Our children's masses are small compared to the mass of the merry-go-round, and their dimensions are at most comparable to its radius, so it is likely that the effect of each child's orientation is small compared to the overall moment of inertia of the system. Therefore, as a first approximation, we can treat them as point particles. We can come back and reconsider this later, when we have a feel for the numerical values involved.

Formulate

The torque exerted by a force about a specified axis is given by $FR\sin\theta$, where R is the distance between the point at which the force acts and the axis, and θ is the angle between $\vec{\mathbf{F}}$ and $\vec{\mathbf{R}}$, assuming that both vectors lie in the plane perpendicular to the axis.

The moment of inertia of a uniform disk about an axis through its center is $\frac{1}{2}MR^2$ (see table, or Problem 8C.4). For a point mass m a distance d from the axis, $I = md^2$.

The relation between torque and angular acceleration is $\tau = I\alpha$.

Solve

The torques exerted by our three forces about the central axis are

$$\tau_{\rm D} = (100\ {\rm N}) \times (2\ {\rm m}) \times \sin 130^\circ = 153\ {\rm N \cdot m}\ ;$$

$$\tau_{\rm E} = (80\ {\rm N}) \times (2\ {\rm m}) \times \sin 100^\circ = 158\ {\rm N \cdot m}\ ;$$

$$\tau_{\rm F} = (40\ {\rm N}) \times (2\ {\rm m}) \times \sin 90^\circ = 80\ {\rm N \cdot m}\ .$$

All of these act to produce a counter-clockwise acceleration, so they all have the same sign and the total torque is $390\ {\rm N \cdot m}$.

The moment of inertia of the merry-go-round is

$$I_M = \frac{1}{2}MR^2 = (100\ {\rm kg}) \times (2\ {\rm m})^2 = 400\ {\rm kg \cdot m^2}\ ,$$

and the children each contribute md^2, i.e. 68, 20 and 81 ${\rm kg \cdot m^2}$ for Alfredo, Betty and Chris respectively. The total moment of inertia is therefore

$$I = 570\ {\rm kg \cdot m^2}\ .$$

8E.1, continued:

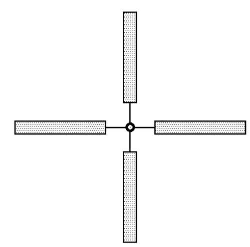
In the absence of friction, the angular acceleration is therefore

$$\alpha = (390\ \mathrm{N \cdot m})/(570\ \mathrm{kg \cdot m^2}) = 0.68\ \mathrm{rad/s^2}\ .$$

Scrutinize

How bad an approximation is it to treat the children as point particles? Let's try a slightly more precise calculation: assume Alfredo is standing upright, and treat him as a uniform cylinder of radius 15 cm. Then his moment of inertia about a vertical axis through his center of mass is $\frac{1}{2}mR^2 = (15\ \mathrm{kg}) \times (0.15\ \mathrm{m})^2 = 0.34\ \mathrm{kg \cdot m^2}$, and by the parallel-axis theorem we add this to md^2 to get his total moment of inertia about the merry-go-round axis. Clearly it makes a negligible difference! If he is 1.2 m tall and he lies down, treating him as a thin rod gives him a moment of inertia about his center of mass of $\frac{1}{12}m\ell^2 = 3.6\ \mathrm{kg \cdot m^2}$, which is still only a 5% correction to md^2. We conclude that it is entirely reasonable to treat the children as point particles in this problem.

8E.3 *A ceiling fan consists of four blades arranged in a cross. Each blade has mass* 2 kg, *is one meter long and* 10 cm *wide, and is connected to the central spindle of the fan by a rod of length* 10 cm *and negligible mass. The fan rotates at a steady rate of 30 revolutions per minute. What is its kinetic energy and its angular momentum? If it takes* 10 s *to reach this speed when switched on, what is the average net torque delivered by the motor during this period, and how much work is done?*



Conceptualize

We know ω, the angular velocity of the fan, and by dividing this by the time the fan takes to spin up we can find the average angular acceleration. To calculate rotational kinetic energy, angular momentum and torque, we also need the moment of inertia of the fan. We have its mass and dimensions, so in principle we can determine I directly by integration, but this looks hard, as the overall shape is quite complicated. Therefore, if possible, we should break the fan down into simpler shapes and use the parallel and/or perpendicular axis theorems to calculate I. The obvious breakdown is to treat each blade as a thin rectangular sheet, rotating about an axis perpendicular to the sheet and located 10 cm from one short side.

Formulate

The moment of inertia of the rectangular plate about an axis through the center of mass, in the plane of the plate, and parallel to the side of length a is $\frac{1}{12}Mb^2$ (both the result and the calculation are identical to problem 8C.4 (a)). The perpendicular-axis theorem says that to get the moment of inertia of the plate about a perpendicular axis through the center we simply add this to the corresponding value for an axis parallel to the b sides, giving $\frac{1}{12}M(a^2 + b^2)$.

This is the basic equation we need to solve this problem. We can apply the parallel-axis theorem to move the axis from the center of mass to the point we want, and then add the four blades together to make the whole fan.

8E.3, continued:

Solve

Applying the parallel-axis theorem:

$$I_{\text{blade}} = \frac{1}{12}M(a^2 + b^2) + Md^2 \ ,$$

and putting in the numbers gives us $I_{\text{blade}} = 0.888 \text{ kg} \cdot \text{m}^2$. This is the moment of inertia for one blade; to find that of the whole fan we just multiply by four to get

$$I = 3.55 \text{ kg} \cdot \text{m}^2 \ .$$

The angular velocity of the fan is $\omega = 2\pi \times (30 \text{ rpm})/(60 \text{ s/min}) = 3.14$ radians per second. Its kinetic energy is therefore $\frac{1}{2}I\omega^2 = 18$ J and its angular momentum $I\omega$ with respect to this axis of rotation is $11 \text{ kg} \cdot \text{m}^2/\text{s}$.

If the fan takes 10 s to reach its final speed, the average torque $\mathrm{d}L/\mathrm{d}t$ must be $(11 \text{ kg} \cdot \text{m}^2/\text{s})/(10 \text{ s})$, or $1.1 \text{ N} \cdot \text{m}$. The work done is converted to the fan's kinetic energy, so (neglecting any frictional effects) 18 J of work (or 1.8 W of power) is required.

Scrutinize and Learn

We have cheated slightly in this problem by implicitly assuming that the fan blades are horizontal, whereas in fact they would probably be somewhat tilted to increase the air flow. However, even with these large extended objects the dominant contribution to the moment of inertia was Md^2, which is four times as big as $\frac{1}{12}M(a^2 + b^2)$, so the effect of neglecting the tilt probably isn't too serious.

Note that the dimensions of ***torque***, ***moment of inertia*** and ***angular momentum*** differ from the corresponding translational quantities of force, mass and momentum, but rotational kinetic energy is genuinely an energy, with the same dimensions as $\frac{1}{2}mv^2$ or mgh, and can be used directly in energy conservation calculations.

8E.5 *You are on a camping trip in a rural district. Your water supply comes from a well, and is obtained by hauling up a large bucket using a windlass, i.e. a cylindrical spindle turned by a handle. You have just hauled a full bucket of water, mass* 15 kg, *the* 10 m *from the water surface to ground level (at negligible speed) when your hand slips off the windlass handle.*

(a) *If the windlass is a solid cylinder of mass* 20 kg *and radius* 12 cm, *how fast is it spinning when the bucket hits the water? Neglect the mass of the rope and the friction in the bearings of the windlass, take* $g = 9.8 \text{ m/s}^2$, *and assume the rope does not slip.*

(b) *What was the tension in the rope while the bucket was falling, and how long did the fall take?*

Conceptualize (a)

The first part of the problem can be solved using energy conservation. Before you let go, the bucket+windlass+rope system had a total energy of mgh—the bucket's gravitational potential energy relative to the level of the water. When the bucket hit the water, it had zero gravitational potential energy and a kinetic energy of $\frac{1}{2}mv^2$, and the spinning

8E.5, continued:

windlass had a kinetic energy of $\frac{1}{2}I\omega^2$. in the absence of friction, the total mechanical energy remains constant, so we can solve this problem if we can relate v and ω.

Formulate (a)
Energy conservation gives

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \ .$$

We already know that for a solid cylinder rotating about its axis $I = \frac{1}{2}Mr^2$. If the bucket's speed is v, and the rope does not stretch, the speed of any piece of rope must also be v, and if it doesn't slip, the speed of any section of the outer surface of the cylinder must be v as well. We conclude that the angular velocity $\omega = v/r$, where r is the radius of the cylinder.

Solve (a)
The rotational kinetic energy of the windlass is

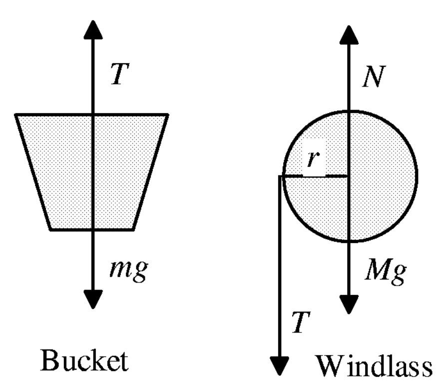
$$\frac{1}{2}I\omega^2 = \frac{1}{2}(\frac{1}{2}Mr^2)(v/r)^2 = \frac{1}{4}Mv^2 \ ,$$

so energy conservation implies

$$mgh = \frac{1}{2}mv^2 + \frac{1}{4}Mv^2 = \frac{1}{4}v^2(2m + M) \ .$$

Now $mgh = (15 \text{ kg}) \times (9.8 \text{ m/s}^2) \times (10m) = 1470$ J, so the velocity of the bucket when it hit the water was $\sqrt{(4 \times 1470 \text{ J})/((2 \times 15 \text{ kg}) + (20 \text{ kg}))} = 11$ m/s, and the angular speed of the windlass was $v/r = 90$ rad/s.

Conceptualize (b)
We can't find the tension in the rope by energy methods—it is an internal force and does no work on the system. Instead we need to consider the forces acting on the bucket and on the windlass. Their force diagrams are shown on the right. There will be three equations in all, two for force and one for torque, and three unknowns: the torque τ, the tension T and the normal force N.



Bucket
Windlass

Formulate (b)
Taking down and counter-clockwise to be positive for motion and rotation respectively, we find

$$F_{\text{B}} = mg - T$$
$$\tau_{\text{W}} = Tr$$
$$F_{\text{W}} = T + Mg - N \ ,$$

8E.5, continued:

where B stands for bucket, W for windlass, and N is the normal force exerted on the windlass by its supporting axle. Neither N nor the weight contributes to the torque because both act through the axis of rotation.

Solve (b)
Because the rope doesn't stretch or slip, the angular acceleration $\alpha = a/r$ by the same logic that we used earlier for v and ω. The equations for F_B and τ_W become

$$ma = mg - T$$
$$\tfrac{1}{2}Mra = Tr\,.$$

From the second of these, $T = \frac{1}{2}Ma$, and from the first

$$a = \frac{mg}{(m + \frac{1}{2}M)}, \qquad \text{giving} \qquad T = \frac{Mmg}{(2m+M)}\ .$$

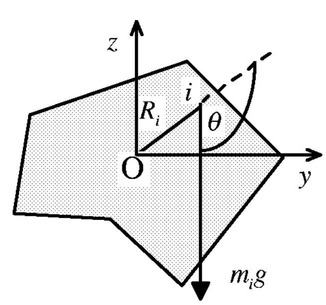
Substituting the numbers gives us $a = 5.9\ \mathrm{m/s^2}$, $T = 59$ N. The time of fall is given by $h = \frac{1}{2}at^2$, which for a 10 m drop gives $t = 1.8$ s.

Scrutinize
We can use the results of part (b) to recheck the speed of our bucket: $v = at = 11$ m/s, in agreement with our earlier result. We can also confirm that our results are of the right general form by looking at extreme cases. If the windlass were very light compared to the bucket ($M \ll m$), the bucket would essentially be in free fall, and sure enough we find an acceleration of g and a very small tension $T = \frac{1}{2}Mg$. If on the other hand the bucket were very light compared to the windlass ($m \ll M$), its weight would not be sufficient to turn the cylinder and it would simply hang there—and indeed this case gives us $a \sim 0$ and $T = mg$.

8E.7 (a) *A rigid body of unspecified shape is allowed to pivot freely about a horizontal axis through its center of mass. The only force acting is gravity. Show that there is no net torque.*

Conceptualize
To find the total torque, we can use the same approach as we did when proving the parallel-axis theorem in Problem 8C.3—find the torque acting on a small piece of the body at (x_i, y_i), with mass m_i, and then sum over all such small pieces (equivalent to an integral over the body, as in 8C.5).

Formulate and Solve
Take the center of mass to be the origin of coordinates, let the z-axis be vertical, and assume the body pivots about the x-axis. Then the torque acting on the element of mass m_i at point i is $m_i g R_i \sin\theta = m_i g y_i$. The total torque on the body is therefore



$$\tau = \sum_i m_i y_i g\ .$$

8E.7, continued:

But $\sum_i m_i y_i = M y_{\rm cm}$, by definition. Since we are considering an axis through the center of mass, $y_{\rm cm}$ is zero, and hence there is no net torque.

Learn

This result explains why you can balance an extended object on a knife-edge if it is arranged so that the knife-edge is located under the center of mass. The term 'center of gravity' is often used as an alternative to 'center of mass' in recognition of this.

(b) *Hence show that the net force of gravity on a rigid body can always be regarded as acting through the center of mass—i.e., the torque it exerts about any horizontal axis is the same as if the total force acted at the center of mass.*

Solve

The conceptualization and formulation of this problem are exactly the same as part (a). The expression for the torque,

$$\tau = \sum_i m_i y_i \, g = M y_{\rm cm} \, g ,$$

does not depend on the location of the origin of coordinates.

If we now consider a point mass M at the location of the center of mass, $[x_{\rm cm}, y_{\rm cm}, z_{\rm cm}]$, then, just as for m_i above,

$$\tau_{\rm cm} = M g R_{\rm cm} \sin\theta = M g y_{\rm cm} \ .$$

Therefore $\tau = \tau_{\rm cm}$: about any horizontal axis, the torque exerted by gravity on a rigid body is the same as the torque on a point particle having the same mass and located at the position of the rigid body's center of mass.

Learn

This is an extremely useful result which basically states that gravity (or indeed any other force which acts uniformly on every part of an object) can always be regarded as acting at the center of mass. We used this implicitly in a couple of earlier examples.

An important application of this result concerns the "fictitious forces" which have to be introduced when we work in non-inertial reference frames. Since these forces are introduced to compensate for the ***frame's*** acceleration, they necessarily act like gravity, giving everything in the frame the same acceleration. This theorem therefore holds for them, and we can treat all such forces as acting through the center of mass of any extended object.

8E.8 *A rigid body of mass M and having a circular cross-section of radius R rolls without slipping down a slope making an angle θ to the horizontal. The moment of inertia of the body about its central axis of symmetry is kMR^2, where k is a numerical constant.*

(a) *What is its speed when it has descended through a vertical distance h?*

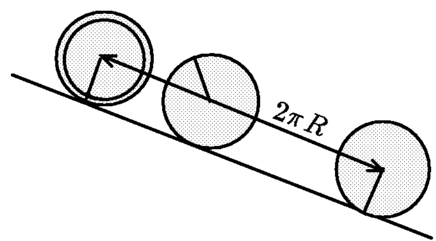
(b) *What is the minimum coefficient of static friction required to ensure that it rolls, rather than slides, down the slope?*

8E.8, continued:

Conceptualize (a)
The only forces acting are friction and gravity. Since the body rolls without slipping, friction does no work in this situation (it is always being applied to the contact point between the body and the surface, and if the body rolls without slipping there is no relative motion between the two surfaces at the contact point). Therefore the change in the body's kinetic energy must come from the work done by gravity, Mgh. To determine the speed that this implies, we can decompose the body's motion into translational motion of the center of mass and rotational motion around the center of mass, and then use the condition of rolling without slipping to relate the angular speed ω to the translational speed v_{cm}.

Formulate (a)
The total kinetic energy of a rigid body of mass M and moment of inertia I is

$$K = \tfrac{1}{2}Mv_{\mathrm{cm}}^2 + \tfrac{1}{2}I\omega^2.$$



If the body rolls without slipping, in one complete rotation (2π radians) it covers a linear distance $2\pi R$ (see diagram), and so $v_{\mathrm{cm}} = R\omega$.

Solve (a)
The work-energy theorem gives

$$Mgh = \tfrac{1}{2}M(R\omega)^2 + \tfrac{1}{2}(kMR^2)\omega^2.$$

Hence

$$Mgh = \tfrac{1}{2}(1+k)MR^2\omega^2,$$

and so

$$\omega = \sqrt{\frac{2gh}{R^2(1+k)}}, \qquad v_{\mathrm{cm}} = \sqrt{\frac{2gh}{1+k}}.$$

Scrutinize (a)
As is usual with inclined planes, the result does not depend on the mass of the body; it is somewhat more surprising to find that the translational speed doesn't depend on the radius of the body either. It does depend on k, which is to say the shape of the object: balls roll down slopes faster than hoops, for example. If we set $k = 0$, treating the object as a point mass, v_{cm} reduces to the value that we have obtained in earlier problems (e.g. in Chapter 4).

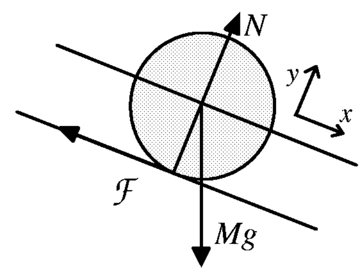
Conceptualize (b)
To find the coefficient of friction, we need the force diagram for the body. We also need to choose a reference point about which to evaluate the torque: as is often the case, it is convenient to choose the center of mass. As in part (a), we are then treating the motion as translation ***of*** the center of mass and rotation ***about*** the center of mass. We

8E.8, continued:

have already determined, in part (a), the relation between these two motions for the condition of rolling without slipping: $v_{\rm cm} = \omega R$.

Formulate (b)
As usual with inclined-plane problems, we define a coordinate system with x pointing down the slope. The components of the net force are

$$F_x = Mg\sin\theta - \mathcal{F}$$
$$F_y = N - Mg\cos\theta$$



and the net torque about the center of mass is

$$\tau = \mathcal{F}R.$$

Since $v_{\rm cm} = \omega R$ and R is constant, it follows that $\mathrm{d}v_{\rm cm}/\mathrm{d}t = R\,\mathrm{d}\omega/\mathrm{d}t$, i.e. $a = \alpha R$. Putting $F_x = Ma$ and $\tau = I\alpha$, where $I = kMR^2$, gives

$$Ma = Mg\sin\theta - \mathcal{F}$$
$$kMRa = \mathcal{F}R.$$

Solve (b)
The torque equation gives $\mathcal{F} = kMa$, and putting this into the force equation we get

$$a = \frac{g\sin\theta}{1+k}, \qquad \mathcal{F} = \frac{kMg\sin\theta}{1+k}.$$

If the object is not slipping, there is no sliding motion between the part of its surface in contact with the slope and the slope itself. Therefore we are dealing with static friction. The minimum value of μ_s for which this is possible is given by

$$\mu_s N = \frac{kMg\sin\theta}{1+k},$$

and since $N = Mg\cos\theta$ (from the fact that $F_y = 0$) we conclude that

$$\mu_s = \frac{k}{1+k}\tan\theta.$$

Scrutinize
The result for μ_s depends, as we would expect, on the shape of the body (and thus the value of k) and the gradient of the slope. Smaller values of k imply that less torque is required to induce the necessary angular acceleration, so we need less friction (since the torque is provided by the frictional force). Steeper gradients increase the net downslope force, hence producing a larger linear acceleration, and thus require a higher angular acceleration to match it. This implies a larger torque, and so more friction.

8E.8, continued:

We can also use our value of a to confirm the energy conservation results for the velocity of the object. Since $v_{\rm cm} = 0$ at $t = 0$, $v_{\rm cm}$ after traveling a distance $x = h/\sin\theta$ must be given by $v_{\rm cm}^2 = 2ax = 2gh/(1+k)$, in agreement with part (a).

Learn

Note that the effect of friction on rolling motion is quite different from its effect on sliding motion! One can define a 'coefficient of rolling friction' in analogy to static and kinetic friction, but its interpretation in physical terms is even more complicated than the interpretation of ordinary sliding friction.

Note that we could also solve part (b) using the point of contact as an instantaneous axis of rotation. Only Mg contributes a torque about this point, namely $\tau = MgR\sin\theta$; the parallel-axis theorem gives $I = (k+1)MR^2$, so $\alpha = g\sin\theta/R(1+k)$, and $a = \alpha R$ for the same result.

HINTS FOR PROBLEMS WITH AN (H)

The number of the hint refers to the number of the problem

8B.2 For the square: what is the *perpendicular* distance between the line of action of each force and the black dot? What is the torque produced by each force? Do both torques act in the same direction?

8C.5 Note that both shapes are said to be thin sheets. Are there any theorems you can apply to such objects which relate moments of inertia about different axes?

For the square: what is the moment of inertia about either diagonal? How can you use this to find I about the given axis? (A similar method is used for the disk.)

Still stuck? Study the solutions to problems 8C.3 and 8C.4.

8D.2 What do you have to know to calculate the kinetic energy and angular momentum of an object rotating about an axis?

What is the moment of inertia of the ruler about the given axis? Is there a way to determine this without having to do an integration?

8E.4 Draw a force diagram for the ruler. What force produces a torque about its axis of rotation?

How does the angular acceleration depend on the angular position (i.e. on θ)? What relationship would you expect for simple harmonic motion?

8E.6 Draw free-body diagrams for the pulley and the two masses. Are the tension forces on both sides of the pulley equal in magnitude?

What is the relationship between the angular acceleration of the pulley and the linear acceleration of the two blocks?

If you are still confused, try reviewing the solution to problem 8E.5.

ANSWERS TO HINTS

8B.2 7 cm for 30 N force; 5 cm for 20 N force; −2.1 N·m and 1 N·m respectively, taking counterclockwise to be positive. No.

8C.5 Can use both parallel-axis and perpendicular-axis theorems (former always, latter for thin sheets).

(From perpendicular-axis theorem) $\frac{1}{12}Ma^2$; use parallel-axis theorem.

8D.2 Angular velocity ω and moment of inertia I.

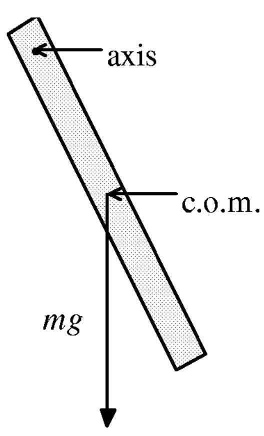
$$\frac{1}{12}m(\ell^2+w^2)+m(\frac{1}{2}\ell-x)^2$$
$$=m(\frac{1}{3}\ell^2+\frac{1}{12}w^2-\ell x+x^2)$$

Use parallel-axis theorem, as above.

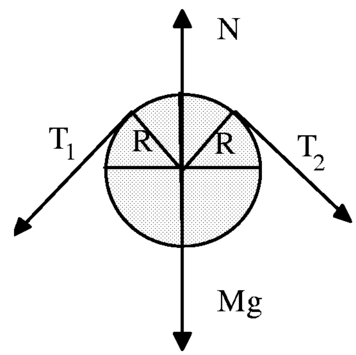
8E.4 Gravity

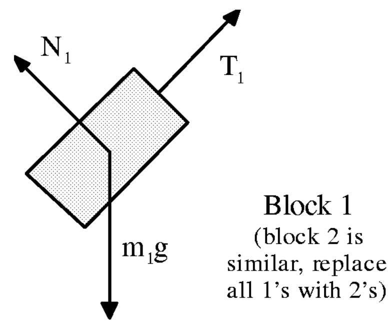
$\alpha=-k\sin\theta$,

$\alpha=-k\theta$, k a positive constant.



8E.6





No: T_1 for block 1 $= T_1$ on pulley, and likewise for T_2, but $T_1 \neq T_2$.

$\alpha = a/R$, where a is the linear acceleration of either block (they are equal), and α is the angular acceleration of the pulley.

ANSWERS TO ALL PROBLEMS

8A.1 d; b.

8A.2 See complete solution.

8B.1 b.

8B.2 $\tau = 1.1$ N·m clockwise; zero; zero.

8C.1 See complete solution.

8C.2 e.

8C.3 See complete solution.

8C.4 See complete solution.

8C.5 $\dfrac{7}{12}Ma^2$; $\dfrac{3}{4}MR^2$.

$\dfrac{12\tau}{7Ma^2}$; $\dfrac{4\tau}{3MR^2}$.

8D.1 a.

8D.2 $K = \frac{1}{2}I\omega^2$, $L = I\omega$, where $\omega = v/(\ell - x)$ and $I = m(\frac{1}{3}\ell^2 + \frac{1}{12}w^2 + x^2 - \ell x)$.

$K = 3.3 \times 10^{-4}$ J; $L = 1.1 \times 10^{-3}$ kg · m^2/s.

8D.3 (a) $\pi\ell\sqrt{\dfrac{M}{3K}}$

(b) Rotational kinetic energy $\frac{1}{2}K\theta_m^2$; angular momentum $\dfrac{1}{2\sqrt{3}}\,\ell\,\theta_m\sqrt{MK}$

(c) $\frac{3}{8}\pi\ell\sqrt{\dfrac{M}{K}}$

8E.1 See complete solution.

8E.2 2 N · m; 1 N · m; 1.1 N · m; 0.

1200 rad/s^2; 2400 rad/s^2; 930 rad/s^2; 0.

8E.3 See complete solution.

8E.4 $\tau = (\frac{1}{2}\ell - x)mg\sin\theta$; $\alpha = \tau/I$ where I is as given in 8D.2 above.

Yes (for small angles to vertical).

8E.5 See complete solution.

8E.6 $a = (m_1 g\sin\theta_1 - m_2 g\sin\theta_2)/(m_1 + m_2 + \frac{1}{2}M)$, and the angular acceleration $\alpha = a/R$.

8E.7 See complete solution.

8E.8 See complete solution.

8E.9 (a) The time taken to roll down the slope is $t = \sqrt{\dfrac{2(1+k^2)\ell}{g\sin\theta}}$, where $k = \dfrac{I}{MR^2}$. The values of k are: uniform rubber ball, $\frac{2}{5}$; hoop, 1; uniform cylindrical log, $\frac{1}{2}$; soccer ball (hollow sphere), $\frac{2}{3}$; wheel (thick cylinder), $\frac{5}{8}$.

(b) You need to minimize the rotational kinetic energy for a given linear speed. An object that can slide without friction would clearly win the race, but since all the objects currently in play roll without slipping, the hope for a frictionless object seems unrealistic. But a wheeled vehicle, say a heavy cart with light wheels, should do fine: the rotational kinetic energy of the wheels is small compared to the kinetic energy of the whole cart. If, however, the rules of the game state that the whole object has to roll, you need something which has most of its mass close to its axis of rotation. An example would be a wheel with a very light rim and spokes, and a heavily ballasted central hub. To see how this works, suppose your wheel consists of a rim with radius R and mass $0.1M$, and a uniform cylindrical hub with radius $0.1R$ and mass $0.9M$ (assume we can neglect the mass of the spokes). Its moment of inertia is $I = (0.1M)R^2 + \frac{1}{2}(0.9M)(0.1R)^2 = 0.1045MR^2$, so $k = 0.1045$, much less than for any of the objects in part (a): your wheel should win comfortably.

8E.10 (a) 63 J; (b) $J = 45$ N $\cdot$ s in the direction of motion; $\tau = 4.4$ N $\cdot$ m.