ESSENTIALS OF INTRODUCTORY CLASSICAL MECHANICS

Sixth Edition

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CHAPTER 9

ROTATION IN THREE DIMENSIONS

OVERVIEW

In the previous chapter we discussed rotations of a rigid body about a fixed axis—that is, the case in which a rigid body rotates about an axis that is fixed in advance by the construction of the physical system. Fixed-axis rotation includes many interesting applications, from pulleys to automobile wheels and airplane propellers. In this chapter, however, we will broaden the discussion to situations for which the rotation axis is not fixed.

To treat such cases, we will introduce the full vector formalism for describing rotational motion. Quantities that were defined as scalars in the previous chapter—such as angular velocity, angular momentum, and torque—will be generalized in this chapter to become vectors. Since the vector formalism is more general than the scalar formalism, most physicists would in fact use the vector formalism even when describing rotation about a fixed axis. If the rotation axis is fixed, each scalar quantity defined in the previous chapter is equal to the component of the corresponding vector quantity along the axis of rotation.

The reader should be prepared for the fact that rotation in three dimensions is a complicated subject. The cause of the complexity lies in the basic geometry. If you translate an object by 1 m in the x-direction and then 2 m in the y-direction, the object ends up at the same place as it would if you performed the operations in the opposite order. Translations commute. However, if you rotate an object by 90° about the x-axis and then 90° about the y-axis, it will ***not*** end up in the same orientation as it would if you performed the operations in the opposite order. (If you are not convinced, try it.) Rotations do not commute, and therefore one cannot hope to separately describe the rotational motion about the x-, y-, and z-axes.

The ability to describe rotations with no fixed axis has several important practical applications. First, we explore the conditions under which the forces and torques on a given rigid body cancel, so that the body remains in equilibrium. This branch of mechanics, known as ***statics***, has obvious relevance to fields such as construction and architecture. We also apply the vector description of rotational motion to the case of planetary orbits, and to the classic example of three-dimensional rotation—the gyroscope.

When you have completed this chapter you should:

- ✔ distinguish between rotation about a fixed axis and rotation for which the axis is free to move;
- ✔ understand the vector definitions of angular velocity, angular momentum, and torque;
- ✔ be able to do calculations involving equilibrium conditions for simple rigid bodies;
- ✔ recognize how conservation of angular momentum can be applied to the motion of point particles;
- ✔ be able to do simple calculations involving rotation in three dimensions.

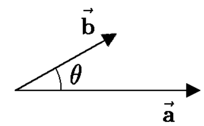
ESSENTIALS

In the previous chapter we studied the motion of rigid bodies moving in a plane, and rotating about an axis perpendicular to that plane. This is clearly not the most general type of rotation — for example, a boat in rough seas can rotate about an axis running from bow to stern, a horizontal axis perpendicular to this (i.e. running across the middle of the boat), or a vertical axis. In aircraft and boats these motions are referred to as roll, pitch and yaw, respectively. In general the motion of the boat will be a combination of these. We can regard rotations about these three perpendicular axes as components of the general rotation of the boat, just as we regard x, y and z as components of its position in space relative to a point.

To discuss rotations in three dimensions with no fixed axis, we will need one new mathematical definition: the vector cross product. Given two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, the vector cross product $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is another vector, which can be defined geometrically by specifying its magnitude and direction:

1) $|\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \sin\theta$, where θ is the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. More precisely θ is defined as the smallest angle by which one vector can be rotated so that it is parallel to the other, so $0 \le \theta \le 180°$, and therefore $\sin\theta \ge 0$.

2) The direction of $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. The choice between the two opposite directions meeting this description is made by a convention called the right-hand rule: if the knuckles of the right hand are rotated from $\vec{\mathbf{a}}$ to $\vec{\mathbf{b}}$, the direction in which the thumb points is the direction of the cross product.



$\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is perpendicular to the page, and out of the page.

The cross product can equivalently be defined in terms of the coordinates of the vectors. Writing $\vec{\mathbf{a}} = [a_x, a_y, a_z]$ and $\vec{\mathbf{b}} = [b_x, b_y, b_z]$, then $\vec{\mathbf{c}} \equiv \vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is given by

$$\begin{aligned} c_x &= a_y b_z - a_z b_y \\ c_y &= a_z b_x - a_x b_z \\ c_z &= a_x b_y - a_y b_x \ . \end{aligned}$$

Note that the second and third lines can each be obtained from the previous one by a cyclic permutation: $x \to y$, $y \to z$, and $z \to x$. Alternatively, if you are familiar with determinants, the equations above can be written as

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} ,$$

where $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ are unit vectors in x, y, and z directions respectively. (If you are not familiar with determinants, just use the previous set of equations.)

There are a number of properties of the cross product that are useful to know:

1) Coordinate independence. This is obvious from the geometric definition, which makes no reference to coordinates, but it is not obvious from the coordinate definition. Nonetheless, if I choose a coordinate system that is rotated relative to yours by any angle about any axis, and we both calculate the cross product $\vec{\mathbf{c}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}}$ using the coordinate definition, our resulting vectors will agree. Since we are using different coordinate systems, my vector components $a_x, a_y, a_z, b_x, b_y, b_z, c_x, c_y$, and c_z will all be different from yours, but if we were to construct our vectors in physical space, as wooden arrows for example, they would line up exactly.

2) Distributive law: $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \vec{\mathbf{c}}$.

3) Anticommutivity: $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = -\vec{\mathbf{b}} \times \vec{\mathbf{a}}$.

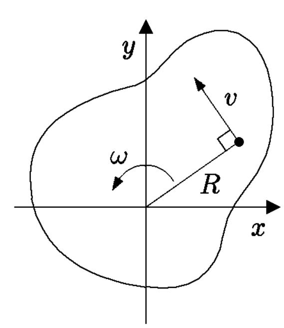
4) If α is any scalar, then $(\alpha\vec{\mathbf{a}}) \times \vec{\mathbf{b}} = \alpha(\vec{\mathbf{a}} \times \vec{\mathbf{b}})$.

Although the definition of the cross product seems at first to be arbitrary (Why $\sin\theta$, and not $\tan\theta$? Why is the cross product perpendicular to the two vectors?), it is in fact highly constrained. If one wants to define a cross product obeying properties (1) and (2) above, it can be shown that the definition is unique, up to a possible redefinition by multiplying the old definition by a constant.

Armed with the new mathematical tool of the cross product, we can now generalize the definitions of each of the three key quantities introduced in the previous chapter: angular velocity, angular momentum, and torque.

If we imagine holding one point of a rigid body fixed, then the most general possible motion it can undergo is by definition a rotation. The purpose of introducing the angular velocity vector is to provide an economical description of this motion. Since the axis of rotation is not fixed, the angular velocity vector should describe not only the speed of the rotation, but also the direction of its axis.

In the previous chapter we discussed rotation about a fixed axis—the z-axis for example—as illustrated at the right. In this case the velocity of any atom in the rigid body is given by



$$|\vec{\mathbf{v}}| = R\omega_{\text{axis}} ,$$

where R is the distance of the atom from the axis of rotation, and ω_{axis} is the angular velocity about the axis. (We will consistently attach the subscript "axis" to the scalar quantities defined in the previous chapter, to distinguish them from the analogous vector quantities that we are defining in this chapter.) The direction of the velocity is tangential, meaning that it is perpendicular to the radial vector and perpendicular to the axis of rotation. Using the cross product, the magnitude and direction of $\vec{\mathbf{v}}$ can be expressed in one simple formula. To do this, we define the vector $\vec{\omega}$ by

1) $|\vec{\omega}| = \omega_{\text{axis}} =$ rotation rate, measured for example in radians per second.

2) The direction of $\vec{\omega}$ is ***along the axis*** of the rotation. The choice between the two directions along the axis is made by another right-hand rule convention: if the knuckles of the right hand are curled in the direction of the motion, $\vec{\omega}$ points the same direction as the thumb. Equivalently, if you look at the object along the axis of rotation, $\vec{\omega}$ points toward you if the rotation appears counterclockwise, and away from you if the rotation appears clockwise.

For motion of the rigid body with one point held fixed, the velocity of any atom is then given by

$$\boxed{\vec{\mathbf{v}} = \vec{\omega} \times \vec{\mathbf{r}}\ ,}$$

where $\vec{\mathbf{r}}$ is the displacement vector of the atom, measured from the fixed point. This equation reduces to the formula from the previous chapter when the object rotates about the z-axis, so it is certainly valid in that case. Wherever the axis of rotation points, however, someone can choose it as her z-axis. The formula will then necessarily hold in her coordinate system. Since the cross-product is coordinate independent, the formula must hold in all coordinate systems. This formula compactly describes both the magnitude and direction of $\vec{\mathbf{v}}$, and allows one to describe rotation even when the axis of rotation, and hence $\vec{\omega}$, changes with time. While no atom moves in the direction of $\vec{\omega}$, the velocity of any atom is easily expressed in terms of $\vec{\omega}$ by the formula above.

If a rigid body moves without even one point held fixed, such as a football flying through the air and spinning as it moves, the description is still simple. Choose a reference point P on the body, and define $\vec{\omega}$ as above, in the frame in which P is fixed. (Usually it is most convenient to choose P to be the center of mass.) If $\vec{\mathbf{r}}_P$

and $\vec{\mathbf{v}}_P$ are the position and velocity of this reference point, then the velocity of any atom is given by

$$\boxed{\vec{\mathbf{v}} = \vec{\mathbf{v}}_P + \vec{\omega} \times \left(\vec{\mathbf{r}} - \vec{\mathbf{r}}_P\right) \ .}$$

Problem 9D.8

As long as one uses only inertial frames of reference, it can be shown that the value of the angular velocity vector $\vec{\omega}$ depends only on which rigid body is being described. It does not depend on the choice of reference point P, the origin of the coordinate system, or even the velocity of the coordinate system.

Another simplifying feature of angular velocity vectors is their additivity. If a gear wheel turns in the engine of a boat, which is itself being tossed about by a turbulent ocean, which in turn is rotating with the Earth about its axis, then it can be shown that the total angular velocity $\vec{\omega}$ of the gear wheel is the vector sum of the angular velocity of the Earth, the angular velocity of the boat relative to the Earth, and the angular velocity of the gear wheel relative to the boat.

To generalize the concept of angular momentum, recall that for rotation about a fixed axis, we defined the angular momentum about the axis as

$$\begin{aligned} L_{\text{axis}} &= I\omega \\ &= \sum_i m_i R_i^2 \omega \\ &= \sum_i m_i R_i v_i \ , \end{aligned}$$

where m_i, R_i, and v_i are respectively the mass, the distance from the axis, and the speed of the i^{th} atom. This formula can be generalized to define the vector angular momentum $\vec{\mathbf{L}}$ about the origin by writing

$$\vec{\mathbf{L}} = \sum_i m_i \vec{\mathbf{r}}_i \times \vec{\mathbf{v}}_i \ ,$$

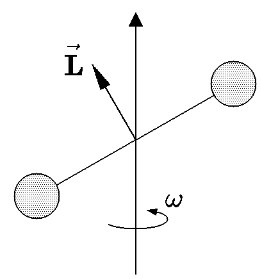
where $\vec{\mathbf{r}}_i$ is the displacement vector of the i^{th} atom, and $\vec{\mathbf{v}}_i$ is its velocity vector. Since $m_i \vec{\mathbf{v}}_i = \vec{\mathbf{p}}_i$, where $\vec{\mathbf{p}}_i$ is the momentum of the i^{th} atom, one usually writes

$$\boxed{\vec{\mathbf{L}} = \sum_i \vec{\mathbf{r}}_i \times \vec{\mathbf{p}}_i \ .}$$

Unlike the angular velocity, the angular momentum usually depends on the choice of the origin, since the displacement vectors $\vec{\mathbf{r}}_i$ are measured from this origin. If the origin is called Q, $\vec{\mathbf{L}}$ is called the angular momentum ***about the point*** Q. However, for the special case in which the total linear momentum of the rigid body $\vec{\mathbf{P}}_{\rm tot} = 0$, $\vec{\mathbf{L}}$ has the same value about all points. One can verify that $L_{\rm axis}$ (the scalar angular momentum for rotation about a fixed axis as defined in the previous chapter) is equal to the component of $\vec{\mathbf{L}}$ in the direction of the axis, provided that the point about which $\vec{\mathbf{L}}$ is calculated lies on the axis about which $L_{\rm axis}$ is calculated.

While $\vec{\omega}$ points along the instantaneous axis of rotation, $\vec{\mathbf{L}}$ does not always point in this direction. If the object is symmetric about the axis of rotation, as is often the case in simple problems, then $\vec{\mathbf{L}}$ does point along the axis. If the object is asymmetric, however, then $\vec{\mathbf{L}}$ need not point along the axis. An example is the obliquely angled dumbbell shown rotating about the vertical axis at the right, for which the direction of $\vec{\mathbf{L}}$ is shown. As the object rotates about the vertical axis, the vector $\vec{\mathbf{L}}$ turns with it. For such an asymmetric object $L_{\rm axis}$ is equal to the component of $\vec{\mathbf{L}}$ along the axis of rotation as always, but $L_{\rm axis}$ is not in this case equal to the magnitude of $\vec{\mathbf{L}}$. Because $\vec{\mathbf{L}}$ and $\vec{\omega}$ are not always parallel, the concept of a moment of inertia for general three dimensional rotations is quite complicated, and will not be discussed here.

Problems 9B.1, 9D.1, 9D.2



Finally, we wish to find a vector generalization of the definition of torque. Since in the previous chapter $\tau_{\rm axis}$ was found to be equal to $dL_{\rm axis}/dt$, a natural way to proceed is to differentiate our expression for the vector angular momentum $\vec{\mathbf{L}}$. The differentiation of the cross product is completely analogous to the differentiation of an ordinary product, as can be verified by using the component definition to express the components of the cross product in terms of ordinary products of functions. Then

Problem 9C.2

Feynman Lectures on Physics, Vol 1, pages 20-1 to 20-4.

$$\frac{d\vec{\mathbf{L}}}{dt} = \sum_i \left[\frac{d\vec{\mathbf{r}}_i}{dt} \times \vec{\mathbf{p}}_i + \vec{\mathbf{r}}_i \times \frac{d\vec{\mathbf{p}}_i}{dt} \right] .$$

But $d\vec{\mathbf{r}}_i/dt$ is the velocity $\vec{\mathbf{v}}_i$, which is parallel to the momentum $\vec{\mathbf{p}}_i$. Since the cross product of two parallel vectors is zero (because $\sin\theta = 0$), the first term vanishes. Using $d\vec{\mathbf{p}}_i/dt = \vec{\mathbf{F}}_i$, where $\vec{\mathbf{F}}_i$ is the total force acting on the $i^{\rm th}$ atom, the second term can be written more simply. Defining the resultant derivative to be the vector torque $\vec{\tau}$, we have

$$\boxed{\vec{\tau} = \frac{d\vec{\mathbf{L}}}{dt} = \sum_i \vec{\mathbf{r}}_i \times \vec{\mathbf{F}}_i \ .}$$

Problem 9D.3

Like the angular momentum $\vec{\mathbf{L}}$, the torque $\vec{\tau}$ usually depends on the choice of the origin, since the displacement vectors $\vec{\mathbf{r}}_i$ are measured from this origin. If the origin is called Q, $\vec{\tau}$ is called the torque ***about the point*** Q. However, for the special case in which the total force $\vec{\mathbf{F}}_{\text{tot}}$ acting on the rigid body is zero, $\vec{\tau}$ has the same value about all points. As with angular momentum, there is a relation between the vector quantity defined here and the scalar quantity defined in the previous chapter. The scalar quantity τ_{axis} defined in the previous chapter is equal to the component of $\vec{\tau}$ in the direction of the axis, provided that the point about which $\vec{\tau}$ is calculated lies on the axis about which τ_{axis} is calculated. Problem 9C.1

In the previous chapter we found that if the internal forces between any two masses in the system are equal, opposite, and directed along the line joining the two masses, then the internal torques about the fixed axis all cancel. The same result holds for the full vector torque. To see this, let $\vec{\tau}_{ij}$ denote the torque on particle i due to particle j, and let $\vec{\mathbf{F}}_{ij}$ denote the force on particle i due to particle j. Then

$$\vec{\tau}_{12} + \vec{\tau}_{21} = \vec{\mathbf{r}}_1 \times \vec{\mathbf{F}}_{12} + \vec{\mathbf{r}}_2 \times \vec{\mathbf{F}}_{21} \ .$$

Using $\vec{\mathbf{F}}_{21} = -\vec{\mathbf{F}}_{12}$,

$$\begin{aligned}\vec{\tau}_{12} + \vec{\tau}_{21} &= \vec{\mathbf{r}}_1 \times \vec{\mathbf{F}}_{12} - \vec{\mathbf{r}}_2 \times \vec{\mathbf{F}}_{12} \\ &= \left(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\right) \times \vec{\mathbf{F}}_{12} \ .\end{aligned}$$

Finally, using the assumption that $\vec{\mathbf{F}}_{12}$ is directed along the line joining the masses, and is therefore parallel to $\left(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\right)$, we find

$$\vec{\tau}_{12} + \vec{\tau}_{21} = 0 \ .$$

Thus, the total torque is the same as the total external torque. ***If there are no external torques, then the total torque vanishes, and the total angular momentum vector $\vec{\mathbf{L}}$ is conserved.*** Conservation of angular momentum ranks with the conservation of energy and of linear momentum as a basic principle of mechanics; like the other two conservation laws, it remains true even when we include relativity and quantum mechanics.

One application of this vector formalism is the determination of the conditions under which a particular object or structure will ***not*** move, rather than what will happen if it does move. This branch of mechanics is called ***statics***. (The study of motion is ***kinematics***, and ***dynamics*** involves the causes of motion — forces and interactions.)

For point particles we already know that an initially stationary particle will remain so if no net force acts on it. If we extend our consideration to rigid bodies we must also ensure that the object does not rotate, so we must also require that there be no net torque. A possible problem here is that we do not know what reference point to calculate the torque about. Fortunately, in a statics problem the total force acting on the rigid body is always zero, and in that case the vector torque $\vec{\tau}$ is the same for all reference points. Problems 9A

Since many of our problems will involve the uniform acceleration of gravity near the surface of the Earth, it is useful to examine the torque in this case. Since the gravitational force on the i^{th} atom is given by $\vec{\mathbf{F}}_i = m_i\vec{\mathbf{g}}$, the general formula for the vector torque simplifies as

$$\begin{aligned}
\vec{\tau}_{\text{grav}} &= \sum_i \vec{\mathbf{r}}_i \times \vec{\mathbf{F}}_i \\
&= \sum_i \vec{\mathbf{r}}_i \times m_i\vec{\mathbf{g}} \\
&= \left(\frac{1}{M_{\text{tot}}}\sum_i m_i\vec{\mathbf{r}}_i\right) \times M_{\text{tot}}\vec{\mathbf{g}} \\
&= \vec{\mathbf{r}}_{\text{cm}} \times \vec{\mathbf{F}}_{\text{tot}} \ .
\end{aligned}$$

Thus, for purposes of calculating the torque, the force associated with a uniform gravitational acceleration can be treated as if the total force were applied at the center of mass.

Notice that our new definitions of angular momentum and torque allow us to apply these concepts to circumstances where we do not seem to have 'rotational motion' in the usual sense. For example, a point particle moving in a straight line has a well-defined position vector $\vec{\mathbf{r}}$ (relative to a reference point off to one side of its trajectory), momentum $\vec{\mathbf{p}}$ and perhaps some applied force $\vec{\mathbf{F}}$. We can therefore assign it an angular momentum $\vec{\mathbf{L}}$ and an applied torque $\vec{\tau}$ about this reference point. Further, if the applied torque happens to be zero the angular momentum we calculate will be conserved. This looks very strange at first sight, but it is not really unreasonable—if we consider a planet orbiting a star, we would certainly concede that it is performing rotational motion and should have angular momentum, but at any given instant it simply has a certain velocity and a certain position relative to the star, just as it would have if it were traveling in a straight line.

Since we can have a net force without a net torque, angular momentum may be conserved in situations where linear momentum is changing. An important example of this is a ***central force***, in which the force on a particle is always directed toward (or perhaps away from) a specific point P. In such a case $\vec{\mathbf{r}}$ (the position vector of the particle relative to P) is parallel to $\vec{\mathbf{F}}$, and therefore the torque acting on the particle about P vanishes. The orbits of the planets or comets about the Sun are examples of central force motion, in the approximation that the Sun is stationary in an inertial frame and that the interactions between the planets can be ignored. In such cases the angular momentum about the source P of the force, although not about any other point, is conserved.

Problems 9B.2, 9B.3, 9B.4, and 9B.5

The vector nature of angular momentum is a physical reality, not just a mathematical convenience. This can be most clearly seen in cases where a rotating object behaves in an unexpected way because of the magnitude and direction of its angular momentum vector. One of the most elegant illustrations of this is the precession of gyroscopes, where the action of gravity seems to cause the spinning gyroscope to move sideways instead of downward. This behavior is a direct consequence of the vector nature of torque and angular momentum. The properties of gyroscopes are explored in Problems 9D.4 and 9D.5.

Problems 9D.4 and 9D.5

Finally, we can describe in principle the treatment of combined translational and rotational motion of rigid bodies in three dimensions. As in the two-dimensional case discussed in the previous chapter, combined translational and rotational motion can be treated by separately describing the motion of the center of mass, and the rotation about the center of mass. As we showed in Chapter 5, the translational motion of the center of mass of any system is controlled by the total force acting on the system:

$$\boxed{\sum \vec{\mathbf{F}}^{\text{ext}} = M\vec{\mathbf{a}}_{\text{cm}} = \frac{\mathrm{d}\vec{\mathbf{p}}}{\mathrm{d}t} \qquad \text{(translational)}\,,}$$

where M is the mass of the rigid body, $\vec{\mathbf{a}}_{\text{cm}}$ is the acceleration of its center of mass, and $\vec{\mathbf{p}}$ is its momentum. The rotation of the rigid body about its center of mass is governed by:

$$\boxed{\sum \vec{\boldsymbol{\tau}}^{\text{ext}} = \frac{\mathrm{d}\vec{\mathbf{L}}_{\text{cm}}}{\mathrm{d}t} \qquad \text{(rotational)}\,,}$$

where $\sum \tau^{\text{ext}}$ is the total external torque calculated about the center of mass, and $\vec{\mathbf{L}}_{\text{cm}}$ describes the angular momentum about the center of mass. As in the two-dimensional case, the rotational equation is most easily justified by working in the possibly non-inertial frame in which the center of mass is at rest and at the origin. Since the fictitious forces in such a frame are equivalent to those produced by a uniform gravitational field, the torque caused by the fictitious forces can be calculated as if the entire force acted directly on the center of mass. Thus the torque ***about*** the center of mass, caused by the fictitious forces, is zero. Only physical torques contribute to the left hand side of the above equation.

In considering combined translation and rotational motion, it is useful to know that the total angular momentum of any system, whether it is rigid or not, can be decomposed in a simple way. The total angular momentum $\vec{\mathbf{L}}$ of the system, about any chosen origin, can be written as

$$\boxed{\vec{\mathbf{L}} = \vec{\mathbf{r}}_{\rm cm} \times \vec{\mathbf{p}}_{\rm tot} + \sum_i \vec{\mathbf{r}}_{{\rm rel},i} \times m_i \vec{\mathbf{v}}_{{\rm rel},i} \ ,}$$

where $\vec{\mathbf{r}}_{{\rm rel},i} = \vec{\mathbf{r}}_i - \vec{\mathbf{r}}_{\rm cm}$ is the position of the $i^{\rm th}$ particle relative to center of mass, and similarly $\vec{\mathbf{v}}_{{\rm rel},i} = \vec{\mathbf{v}}_i - \vec{\mathbf{v}}_{\rm cm}$ is the velocity of the $i^{\rm th}$ particle relative to the velocity of the center of mass. The equation can be summarized by saying that the total angular momentum is the angular momentum of the center of mass, plus the angular momentum about the center of mass. Similarly the total torque acting on the system, about any chosen origin, can be decomposed as

$$\boxed{\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}}_{\rm cm} \times \vec{\mathbf{F}}_{\rm tot} + \sum_i \vec{\mathbf{r}}_{{\rm rel},i} \times \vec{\mathbf{F}}_i \ ,}$$

where $\vec{\mathbf{F}}_i$ is the external force acting on the $i^{\rm th}$ particle, and $\vec{\mathbf{F}}_{\rm tot}$ is the total external force on the system. Since $\vec{\mathbf{r}}_{\rm cm}$ is the only quantity in this formula that depends on the choice of origin, all dependence on the choice of origin disappears if $\vec{\mathbf{F}}_{\rm tot} = 0$.

SUMMARY

* To fully describe rotations in three dimensions, the three quantities angular velocity, angular momentum, and torque must be defined as vectors.
* If a rigid body rotates with one point held fixed, then the angular velocity $\vec{\omega}$ is defined to point along the instantaneous axis of rotation, with a magnitude equal to the scalar angular velocity discussed in the previous chapter. The choice of the two directions along the axis is determined by the right-hand rule convention. The velocity of any atom on the rigid body is given by $\vec{\mathbf{v}} = \vec{\omega} \times \vec{\mathbf{r}}$, where $\vec{\mathbf{r}}$ is the displacement of the atom measured from the fixed point.
* The vector angular momentum of a particle about a point P is given by $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$, where $\vec{\mathbf{r}}$ is the displacement of the particle measured from the point P, and $\vec{\mathbf{p}}$ is the momentum.
* The vector torque exerted on a particle about a point P is given by $\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$, where $\vec{\mathbf{r}}$ is the displacement of the particle measured from the point P, and $\vec{\mathbf{F}}$ is the applied force. The torque is equal to the rate of change of the angular momentum.
* Angular momentum is a conserved quantity: that is, the angular momentum of a system (about any point) does not change in the absence of an external torque.
* An initially stationary rigid body remains in equilibrium if there is no net force and no net torque acting on it. The torque may be calculated about any point.
* Combined translational and rotational motion of a rigid body can be described by separately considering the translational motion of the center of mass, and the rotation of the object about the center of mass.
* Physical concepts introduced in this chapter: angular velocity, angular momentum, and torque as vectors; central forces.
* Mathematical concepts introduced in this chapter: vector (or cross) product.

* Equations introduced in this chapter:

$$c_x = a_y b_z - a_z b_y \; ; \quad c_y = a_z b_x - a_x b_z \; ; \quad c_z = a_x b_y - a_y b_x \; .$$
(vector cross product, component form);

$$|\vec{\mathbf{c}}| = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\sin\theta$$
(magnitude of vector cross product);

$$\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$$
(velocity of atom in rotating body with a fixed point);

$$\vec{\mathbf{v}} = \vec{\mathbf{v}}_P + \vec{\boldsymbol{\omega}} \times \left(\vec{\mathbf{r}} - \vec{\mathbf{r}}_P\right)$$
(velocity of atom in rotating body, general case);

$$\vec{\mathbf{L}} = \sum_i \vec{\mathbf{r}}_i \times \vec{\mathbf{p}}_i$$
(angular momentum, as vector product);

$$\vec{\boldsymbol{\tau}} = \sum_i \vec{\mathbf{r}}_i \times \vec{\mathbf{F}}_i$$
(vector torque, as vector product);

$$\vec{\boldsymbol{\tau}} = \frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t}$$
(torque equation);

$$\left.\begin{aligned} \sum \vec{\mathbf{F}}^{\text{ext}} &= M\vec{\mathbf{a}}_{\text{cm}} = \frac{\mathrm{d}\vec{\mathbf{p}}}{\mathrm{d}t} \\ \sum \vec{\boldsymbol{\tau}}^{\text{ext}} &= \frac{\mathrm{d}\vec{\mathbf{L}}_{\text{cm}}}{\mathrm{d}t} \end{aligned}\right\}$$
(combined translational and rotational motion);

$$\vec{\mathbf{L}} = \vec{\mathbf{r}}_{\text{cm}} \times \vec{\mathbf{p}}_{\text{tot}} + \sum_i \vec{\mathbf{r}}_{\text{rel},i} \times m_i \vec{\mathbf{v}}_{\text{rel},i}$$
(angular momentum decomposition);

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}}_{\text{cm}} \times \vec{\mathbf{F}}_{\text{tot}} + \sum_i \vec{\mathbf{r}}_{\text{rel},i} \times \vec{\mathbf{F}}_i$$
(torque decomposition).

PROBLEMS AND QUESTIONS

By the end of this chapter you should be able to answer or solve the types of questions or problems stated below.

Note: throughout the book, in multiple-choice problems, the answers have been rounded off to 2 significant figures, unless otherwise stated.

At the end of the chapter there are answers to all the problems. In addition, for problems with an (H) or (S) after the number, there are respectively hints on how to solve the problems or completely worked-out solutions.

9A STATIC EQUILIBRIUM

9A.1 You're helping a friend move, and the two of you carry a desk out to your truck. The desk is 1.5 m long, and you find that to lift it you have to exert an upward force of 300 N, while your friend only has to apply 200 N. Taking $g = 10$ m/s^2, what is the mass of the desk?

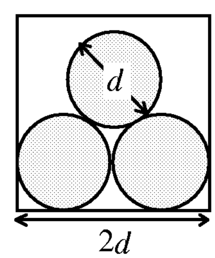
(a) 25 kg; (b) 30 kg; (c) 50 kg; (d) none of these.

If you're each holding on to a short side of the desk, how far from your end is its center of mass?

(a) 0.5 m; (b) 0.6 m; (c) 0.9 m; (d) 1 m.

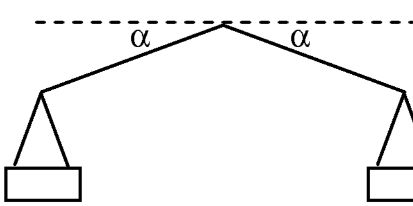
9A.2 A uniform steel girder 10 m long and having a mass of 1500 kg is placed on the flat roof of a building such that part of it sticks out beyond the edge of the roof. As a result of a bet, a 100 kg construction worker walks out and stands at the very end of the beam. What is his maximum possible distance from the edge of the building?

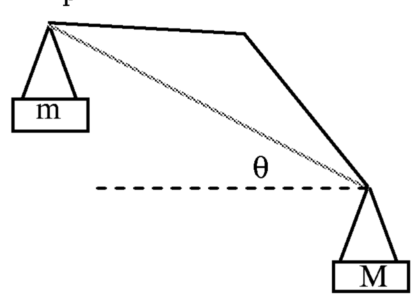
(a) 5.0 m; (b) 4.7 m; (c) 5.8 m; (d) 4.2 m.



9A.3 A rectangular bin contains three identical cylindrical bottles, each of mass 0.7 kg, stacked horizontally as shown. The two lower bottles do not quite touch. Find all the forces acting on each bottle, assuming frictional effects can be neglected.

9A.4 (S) A ball of radius r and mass m is placed against a step of height h. What is the minimum horizontal force F (applied to the center of mass) required to start the ball rolling over the step?



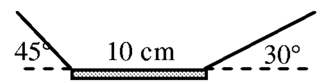


9A.5 (S) A beam balance consists of two pans suspended from a rod which is bent in the middle as shown. The rod is free to pivot about the bend. If two different masses M and m

9A.5, continued:

($M > m$) are placed in the pans, at what angle θ does the balance come to rest? How does its behavior differ from that of a straight rod like a seesaw?

9A.6 (H) A non-uniform horizontal rod is supported by two massless strings, one making an angle of 30° with the horizontal and one an angle of 45°. The rod is 10 cm long. Where is its center of mass?

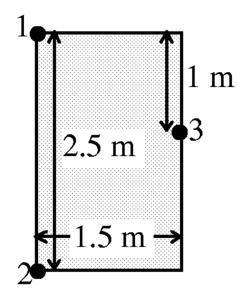


9A.7 (H) A uniform ladder of mass m and length ℓ rests against a smooth wall. A do-it-yourself enthusiast of mass M stands on the ladder a distance d from the bottom. If the ladder makes an angle θ with the ground, what is the minimum coefficient of friction required between the ladder and the ground in order that the ladder will not slip? Assume that there is negligible friction between the ladder and the wall.

Suppose that the homeowner in question is attempting to do some emergency repairs after her house has been damaged by a storm. Her 3 m ladder is not really long enough for the task, and she has to stand on the top rung, 2.8 m from the base, in order to do the job. Her mass is 70 kg, the mass of the aluminum ladder is negligible, and the angle the ladder makes with the horizontal is 70°. Unfortunately, due to the rain, the coefficient of friction between the ladder and her paved yard is only 0.20. Will she slip? Will the situation be improved if her friend, who has a mass of 90 kg but is afraid of heights, stands on the bottom rung of the ladder, 0.2 meters from the base, while she climbs to the top?

9A.8 (S) A table consists of a uniform circular top of mass M and radius r, supported by three equally spaced legs of negligible mass situated at the rim of the table. A coffee pot of mass m is located midway between two adjacent legs. Calculate the normal force exerted by the floor on each leg of the table. What if the table had four equally spaced legs?

9A.9 (H) Three people are carrying a rectangular slab of wood weighing 800 N. The slab is 2.5 m long and 1.5 m wide and of uniform thickness. Two of the people hold the slab at each end of a long side, while the third holds the other long side, 1 m from a corner. Assuming the slab is level and they all lift vertically, find the force exerted by each person. How do these forces depend on the distance between the third carrier and the corner of the slab?



9B ANGULAR MOMENTUM AND CENTRAL FORCES

9B.1 (S) A symmetrical rigid body spins about its axis of symmetry. In which direction does its angular momentum vector point? (The body's center of mass is stationary.)

9B.2 (S) One of the first achievements of the modern scientific method was Johannes Kepler's analysis of Tycho Brahe's observations of planetary positions. Kepler deduced three empirical laws governing planetary motion, from which Newton subsequently developed his law of universal gravitation.

9B.2, continued:

Kepler's second law states that "a line drawn from the Sun to any planet sweeps out equal areas in equal times". Show that this law implies that the angular momentum of the planet due to its orbital motion is constant.

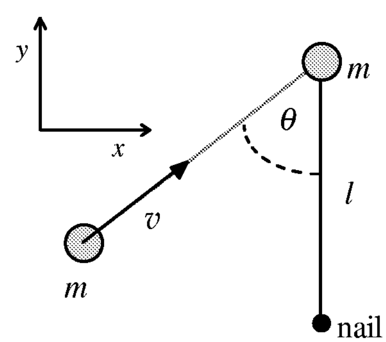
9B.3 (S) Comets, in contrast to planets, follow very elongated elliptical orbits, so that their distance from the Sun varies greatly over the course of the orbit. Suppose that a short-period comet has an orbit whose closest approach to the Sun (or perihelion) is 5×10^{10} m, the furthest point (or aphelion) being 2×10^{12} m. What is its speed at perihelion?

9B.4 (H) Suppose that we wished to dispose of nuclear waste by launching it into space and 'firing it into the Sun'. One simple plan would be to start by putting the rocket into a circular orbit about the Sun at the same radius (and hence the same orbital speed) as the Earth. The rocket would then fire its engines in reverse for a short burst, slowing the rocket just enough so that the resulting orbit would graze the surface of the Sun, causing the rocket to be vaporized. By how much would the speed have to be changed? For comparison, if the rocket fired its engines forward for a short burst, by how much would the speed have to be changed for the rocket to leave the solar system altogether? [The Earth's orbit has a radius of 150 million kilometers and the Sun's radius is about 700 thousand kilometers. The mass of the Sun is 2.0×10^{30} kg, and the value of the gravitational constant G is $6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$.]

Challenge Problem:

The method of reaching the Sun described above is far from being the most fuel-efficient. Find a flight plan by which the rocket could reach the Sun with a significantly smaller change in speed required from the engines, and thus with significantly less fuel consumption.

9B.5 On a frictionless table with its surface in the horizontal xy plane, a small object of mass m is attached by a nail to a taut horizontal massless string of length ℓ. Initially the object is at rest and the string lies along the y axis. A second identical object slides on the table with speed v in a direction making an angle θ to the string, as shown. On impact the objects stick together and move off with the string remaining taut.



(a) After impact, what is the magnitude of the angular momentum vector of the two-object system about the nail?

(b) What is the direction of this angular momentum vector?

9C VECTOR TORQUE

9C.1 (H) A particle of mass m at position $[x, y, 0]$ with respect to some point O is acted on by a force $\vec{\mathbf{F}}$ given by $[F_x, F_y, 0]$. Using the definition of the torque about an axis given in Chapter 8, namely $\tau_{\text{axis}} = F_\perp R$, where $F_\perp$ is the counterclockwise tangential component of the applied force and R is the radial distance to the axis of rotation, show (without using vector products) that the torque on the particle about the z-axis is

$$\tau_{\text{axis}} = xF_y - yF_x \ .$$

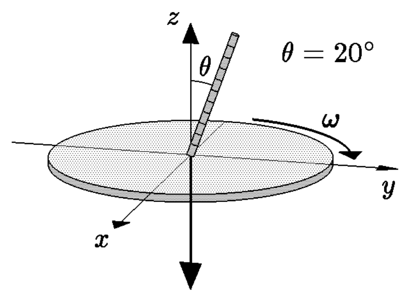
That is, prove that the scalar torque defined in Chapter 8 is equal to the z-component of the vector torque defined in Chapter 9.

9C.2 (S) Use the component form of the vector product to verify that $\vec{\boldsymbol{\tau}} = \dfrac{d\vec{\mathbf{L}}}{dt}$.

9C.3 Explain, in 100 words or less, the distinction between torque about an ***axis*** and torque about a ***point***.

9D MOTION OF A RIGID BODY IN THREE DIMENSIONS

9D.1 (S) A uniform rod is fixed to a rotating horizontal turntable so that its lower end is on the axis of the turntable and it makes an angle of 20° to the vertical. (It is thus rotating with uniform angular velocity about an axis passing through one end and inclined at 20° to the direction of the rod.) If the turntable is rotating clockwise as seen from above, what is the direction of the rod's angular velocity vector?



(a) vertically downwards; (b) down at 20° to the vertical;
(c) up at 20° to the vertical; (d) vertically upwards.

What is the direction of its angular momentum vector (calculated about its lower end)?

(a) vertically downwards; (b) down at 20° to the horizontal;
(c) up at 20° to the horizontal; (d) vertically upwards.

Is there a torque acting on it, and if so in what direction?

(a) yes, vertically; (b) yes, horizontally;
(c) yes, at 20° to the horizontal; (d) no.

9D.2 (S) A conical pendulum consists of a small heavy sphere of mass m attached to a string of length ℓ and negligible mass. The string makes an angle θ with the vertical and the bob describes a circular path with constant speed v.

(a) What is the angular momentum of the bob about a vertical axis through the point of suspension? What torque is exerted about this axis?

(b) What is the angular momentum vector of the bob with respect to the point of suspension itself? What torque is exerted about this point?

9D.3 (H) A ruler 31 cm long and 2 cm wide is resting on an air table, so there is no frictional force acting. You are asked to apply two 1 N forces to the ruler (in the plane of the air table—no vertical forces allowed!) such that (a) there is a net force on the ruler, but no net torque; (b) there is a net torque, but no net force; (c) there is neither a net force nor a net torque; (d) there is a net torque ***and*** a net force. In each case, what will the ruler do? (Recall that any two-dimensional motion can be represented as translation

9D.3, continued:

plus rotation about the center of mass, so you can regard the ruler's center of mass as the axis for your torque.)

Suppose you choose to apply your two 1 N forces parallel to one another and perpendicular to the long side of the ruler, one at its midpoint and one at one end. Which of the cases (a)–(d) is this? If you apply these forces for 0.001 s and then let the ruler move freely, what will its final linear and angular velocity be if its mass is 50 g? Make a scale drawing of the ruler's motion by plotting its position and orientation at $t = 0$, 10, 12, 14, 16, 18 and 20 s, defining $t = 0$ as the time the forces were first applied.

9D.4 (S) A gyroscope is a massive rapidly spinning wheel mounted on an axle of negligible mass. If such a device is held with the axle horizontal and supported at one end by a vertical post, what will happen when it is released? Assume the wheel has mass m and moment of inertia I, that it is spinning with angular velocity ω, and that the distance between the wheel and the supported end of the axle is ℓ.

9D.5 (H) A frequent exhibit in science museums is a gyroscope made of a bicycle wheel with a weighted rim and handles along the axle. Visitors to the exhibit take the spinning wheel with its axis horizontal and sit in a chair which is suspended from a frictionless pivot so that it is free to swing or rotate in any direction. They are then instructed to turn the wheel so that its axis is vertical. What happens? Explain what is going on in terms of angular momentum and applied torques.

9D.6 (H) An aircraft lands at a speed v_0. Before it touches down, its wheels are not rotating. Describe in words what happens when the wheels touch the ground. Assuming that each wheel has radius R and moment of inertia I and supports a weight Mg, and that the pilot does not apply reverse thrust until the aircraft is no longer skidding, how fast is the plane moving when it stops skidding?

9D.7 Discuss the relevance of angular momentum in the following examples:

(a) rifles, which have grooved barrels so that the bullet emerges spinning about its long axis, as opposed to smooth-bore guns, whose bullets don't spin;

(b) frisbees;

(c) tail rotors on helicopters;

(d) navigational gyroscopes (mounted on frictionless gimbals, which allow their axes to rotate freely relative to the mounting) in aircraft and satellites;

(e) stabilizing gyroscopes (large flywheels mounted on a fixed axis) on ships.

9D.8 (S) Show that the equation relating the velocity of a point on a rigid body to the body's angular velocity

$$\vec{\mathbf{v}} = \vec{\mathbf{v}}_P + \vec{\omega} \times (\vec{\mathbf{r}} - \vec{\mathbf{r}}_P),$$

where P is a reference point on the body, is indeed independent of the choice of reference point P, the origin of the coordinate system, and the velocity of the coordinate system, as stated in the text.

9D.9 A meteoroid in empty space rotates about its center of mass. We choose a coordinate system in which the center of mass is at rest at the origin, and the angular velocity at the time of interest points along the z-axis, so that $\vec{\omega} = [0, 0, \omega_z]$. A small particle, or ***chondrule***, of the mineral olivine is embedded in the meteroid at position $\vec{\mathbf{r}}_0 = [x_0, 0, z_0]$. The chondrule has mass M. (Note that the chondrule is part of the meteoroid, so its mass and position have been included in the calculation of the meteoroid's center of mass.)

(a) What are the components of the velocity vector $\vec{\mathbf{v}}$ of the chondrule?

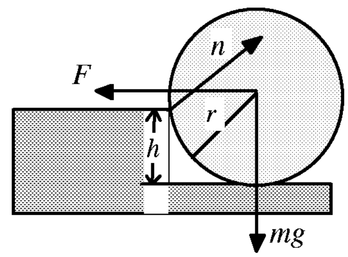
(b) What is the angular momentum of the chondrule (i) about the z-axis, i.e. the instantaneous axis of rotation; (ii) about the center of mass of the meteoroid?

(c) The overall angular momentum vector of the meteoroid about its center of mass makes an angle θ with the angular velocity vector. Describe in words the motion of the meteoroid as seen by a passing space probe.

COMPLETE SOLUTIONS TO PROBLEMS WITH AN (S)

9A.4 *A ball of radius r and mass m is placed against a step of height h. What is the minimum horizontal force F (applied to the center of mass) required to start the ball rolling over the step?*

Conceptualize



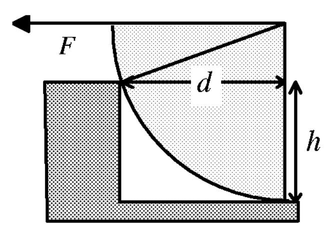
If we consider the case where the ball simply rests on the lower level with no applied force $\vec{\mathbf{F}}$, then clearly there will be a force $m\vec{\mathbf{g}}$ downwards and a normal force $\vec{\mathbf{N}}$ upwards; with no other forces acting, these will be of equal magnitude. As we apply a gradually increasing force $\vec{\mathbf{F}}$, the ball will be pulled against the step, which will therefore begin to exert a contact force $\vec{\mathbf{n}}$. This has some upward component, so the magnitude of $\vec{\mathbf{N}}$ decreases as $\vec{\mathbf{n}}$ increases. When the ball eventually starts to roll over the step, it loses contact with the lower level, and $\vec{\mathbf{N}}$ is then obviously zero. It follows that the limiting case occurs when the ball is still stationary and in contact with the lower level, but the value of $\vec{\mathbf{N}}$ is zero, and the upward component of $\vec{\mathbf{n}}$ balances $m\vec{\mathbf{g}}$. Increasing $|\vec{\mathbf{F}}|$ beyond this will require that $|\vec{\mathbf{n}}|$ increase (the ball cannot move horizontally, so the horizontal component of $\vec{\mathbf{n}}$ must balance $\vec{\mathbf{F}}$) and there will then be a net upward force on the ball to lift it over the step.

Formulate and Solve

In the limiting case the ball is stationary, so there is no net force and no net torque. The force equations involve the unknown magnitude and direction of $\vec{\mathbf{n}}$, but if we consider torque ***about the point of contact with the step*** these are eliminated from the torque equation. This gives

$$\tau = F(r-h) - mgd\ ,$$

where d is the horizontal distance between the center of the ball and the vertical part of the step. Constructing a right-angled triangle between the center of the ball, the point of the step, and the line of $\vec{\mathbf{F}}$ gives us



$$d^2 = r^2 - (r-h)^2 = h(2r-h)$$

and so

$$F = mg\frac{\sqrt{h(2r-h)}}{r-h}\ .$$

Scrutinize

The dimensions of our solution are clearly correct, since the term multiplying mg is dimensionless. We can do a further check by considering special situations: if $h \to 0$, $F \to 0$ (no force is required to roll the ball over a nonexistent step!), and if $h \to r$, $F \to \infty$ (if $h = r$ the applied force acts through the point of contact, and hence exerts

9A.4, continued:

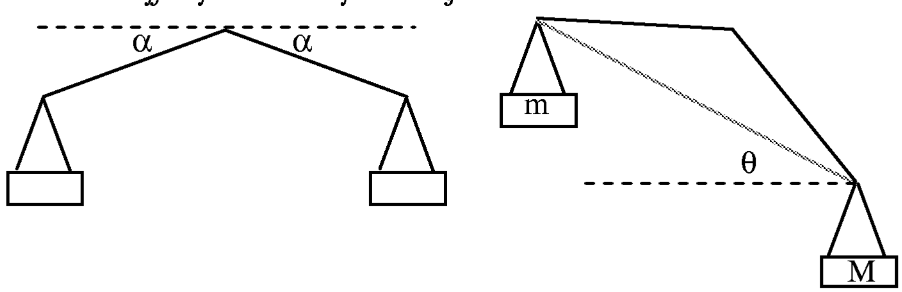
no torque about that point, so regardless of its magnitude it cannot cause the ball to roll over the step).

If $h > r$ we get a negative force, which is clearly not a solution (pulling the ball away from the step is not going to work!). Common sense agrees that if our force is actually applied below the level of the step it is never going to succeed in pulling the ball over the top, and if we consider our diagram we see that for such a situation the torque exerted by $\vec{\mathbf{F}}$ is not $F(r-h)$ as we have assumed, because the point of contact between ball and step is no longer the top of the step, but rather the leftmost point of the ball. This is exactly the same as the case $h = r$, and we saw above that $F \to \infty$ in this case.

Learn
Many problems involving limiting cases can be regarded as statics problems, because the limiting case often corresponds to a stationary object with zero net force acting on it. This is frequently the case for friction problems, for example.

Note that in this example a good choice of reference axis for the torques reduced the number of equations we needed from three to one, by eliminating any need to calculate $\vec{\mathbf{n}}$.

9A.5 *A beam balance consists of two pans suspended from a rod which is bent in the middle as shown. The rod is free to pivot about the bend. If two different masses M and m ($M > m$) are placed in the pans, at what angle θ does the balance come to rest? How does its behavior differ from that of a straight rod like a seesaw?*



Conceptualize
When the balance is at rest there must be no net force on it (or it would move) and also no net torque (or it would turn). In this case the net force is always zero (a contact force from the pivot is balancing the gravitational force on the pans), so we are only interested in the net torque. The gravitational force always acts through the center of mass, as shown in the *Essentials,* so we can disregard the shape of the pans and simply treat them as point masses.

Formulate
We have three forces acting ($M\vec{\mathbf{g}}$, $m\vec{\mathbf{g}}$, and a contact force from the pivot), but by taking the pivot as the reference axis we can eliminate its contact force from the torque equation. Straightforward trigonometry tells us that the right-hand arm of the balance makes an angle $(\theta + \alpha)$ with the horizontal and the left-hand arm an angle $(\alpha - \theta)$. The torque around the pivot is therefore

$$\tau = Mg\ell\cos(\theta + \alpha) - mg\ell\cos(\alpha - \theta) \ ,$$

9A.5, continued:

writing ℓ for the length of each arm of the balance. Hence for equilibrium

$$M\cos(\theta+\alpha) = m\cos(\alpha-\theta)\ .$$

Solve

To solve this we make use of the trigonometric identity

$$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

and write

$$(M-m)\cos\theta\cos\alpha = (M+m)\sin\theta\sin\alpha\ ,$$

from which

$$\tan\theta = \frac{1}{\tan\alpha}\left(\frac{M-m}{M+m}\right)\ .$$

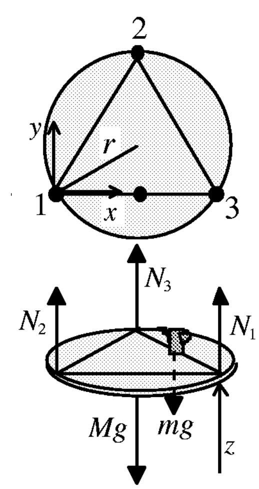
A straight rod corresponds to the case $\alpha = 0$, and our condition for equilibrium is then $M\cos\theta = m\cos\theta$. This plainly cannot be met unless $M = m$ or $\theta = 90°$. For unequal masses the balance will therefore tilt until it hangs vertically with M below m (if the design of the pivot allows this). For equal masses ***any*** value of θ will do—if the balance is held stationary at any angle it will remain so when released. The bent balance, on the other hand, will swing back to the horizontal position if released with equal masses in the pans.

Scrutinize

The equation is clearly dimensionally correct, and in considering the straight rod we have already checked one extreme value. We can also consider $M = m$, in which case we should expect the balance to hang level, as indeed it does ($\tan\theta = 0$). For very large M we would expect the balance to tilt so that the pan containing M hung straight downwards, which requires $\theta = 90° - \alpha$, and this is what our formula gives ($\tan(90° - \phi) = 1/\tan\phi$ for any angle ϕ).

9A.8 *A table consists of a uniform circular top of mass M and radius r, supported by three equally spaced legs of negligible mass situated at the rim of the table. A coffee pot of mass m is located midway between two adjacent legs. Calculate the normal force exerted by the floor on each leg of the table. What if the table had four equally spaced legs?*

Conceptualize



This is again a problem in which the net force and the net torque will be zero. Since the table legs are massless, there must be no net force on them—a net force on a massless object gives infinite acceleration—and so the force that each of them applies to the tabletop is the same as the force that the floor applies to them, and likewise for the force applied to the leg by the tabletop. The coffeepot is small in dimensions compared to the tabletop, so we can treat it as a point mass. All our forces (the weights of the table and coffeepot and the three normal forces) are in the z direction, so we have only one force equation, but we have two components of torque corresponding to rotation about the x-axis and the y-axis respectively.

9A.8, continued:

Formulate ***Three legs:***
There is no obvious 'best' choice of reference point: we can choose the top of leg 1, which at least removes its normal force from the torque equations. With this choice of coordinate system, our three equations are

$$F_z = N_1 + N_2 + N_3 - (M+m)g$$
$$\tau_y = \frac{\sqrt{3}}{2}rMg + \frac{\sqrt{3}}{2}rmg - \frac{\sqrt{3}}{2}rN_2 - \sqrt{3}\,rN_3$$
$$\tau_x = \frac{1}{2}rMg - \frac{3}{2}rN_2 \ .$$

Solve
The equilibrium conditions for the table are thus

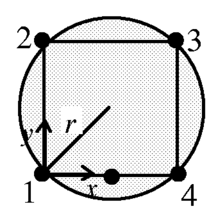
$$(M+m)g = N_1 + N_2 + N_3$$
$$(M+m)g = N_2 + 2N_3$$
$$Mg = 3N_2 \ ,$$

which are easily solved to give

$$N_1 = N_3 = \left(\frac{1}{3}M + \frac{1}{2}m\right) g \ ;$$
$$N_2 = \frac{1}{3}Mg \ .$$

Scrutinize
As we might expect from the symmetry of the situation, the weight of the table is shared equally by all three legs, while the weight of the coffeepot is taken by the two legs between which it sits.

Formulate ***Four legs:***
The equations for the four-legged table are very similar:



$$F_z = N_1 + N_2 + N_3 + N_4 - (M+m)g$$
$$\tau_y = \frac{1}{\sqrt{2}}rMg + \frac{1}{\sqrt{2}}rmg - \frac{2}{\sqrt{2}}rN_3 - \frac{2}{\sqrt{2}}rN_4$$
$$\tau_x = \frac{1}{\sqrt{2}}rMg - \frac{2}{\sqrt{2}}rN_2 - \frac{2}{\sqrt{2}}rN_3 \ .$$

However, here we have a potential problem—there are still only three equations, but we have added another unknown force. We will not be able to solve this problem completely using only these equations.

9A.8, continued:

Solve

If we continue nonetheless, we obtain the equilibrium conditions

$$\begin{aligned}(M+m)g &= N_1 + N_2 + N_3 + N_4 \\ (M+m)g &= 2N_3 + 2N_4 \\ Mg &= 2N_2 + 2N_3 \ ,\end{aligned}$$

from which we find

$$\begin{aligned}N_1 &= \left(\frac{1}{2}M + m\right) g - N_4 \\ N_2 &= N_4 - \frac{1}{2}mg \\ N_3 &= \frac{1}{2}(M+m)g - N_4 \ .\end{aligned}$$

This is as far as we can go without making additional assumptions. In this simple case, we can argue that if the floor is uniform the symmetry of the situation implies that $N_1 = N_4$, which gives us the fourth equation we need to deduce that

$$\begin{aligned}N_1 &= \left(\frac{1}{4}M + \frac{1}{2}m\right) g = N_4 \\ N_2 &= \frac{1}{4}Mg = N_3 \ .\end{aligned}$$

However, this clearly would not work if the coffeepot had been less symmetrically located, or if we had an uneven floor or legs of slightly different lengths. (For example, a solution with $N_2 = 0$, corresponding to a short leg 2 which does not touch the ground, is perfectly possible.) In such a case we would have to admit that the problem is not completely soluble in terms of rigid bodies. Such problems are called ***underdetermined*** (or sometimes ***indeterminate***).

Scrutinize

The underdetermined nature of the four-legged table problem is clear to anyone who has ever sat at a table with legs of slightly unequal length (or situated on a slightly uneven floor). Such tables commonly have two more-or-less stable states, each with one leg not touching the ground, and will go from one to the other whenever a small off-center force is applied. This is because such a table is a very good approximation to a rigid body. Less rigid objects do not behave in this way, as discussed below.

Learn

It is apparent from everyday experience that many apparently underdetermined systems do in fact have perfectly well-defined solutions. For example, a car with an asymmetric weight distribution is completely equivalent in rigid-body terms to our four-legged table, but it is obvious that in fact such a car does have a unique stable equilibrium position—if we drove it on to a weighbridge segmented so that the weight on each

9A.8, continued:

wheel was determined separately, we would get a definite value for the downward force on each wheel, and we would expect that answer to remain the same if we repeated the measurement. What happens in this case is that the springs and tires of the car compress until the unequal loadings on the four wheels are balanced by unequal spring forces. We don't know how to deal with this because the car is not now behaving as a completely rigid body—parts of it are deforming and changing their relative positions. Fortunately, most solid objects can be regarded as very-nearly-rigid bodies, and small deformations of the object behave like compression or extension of a spring: a restoring force acts which is proportional to the amount of deformation and directed so as to return the object to its original state. Hence we can solve such problems by combining rigid-body techniques with spring forces, involving some new terminology (introducing the equivalents of the spring constant k for various kinds of deformation) but no new physical principles. Since this book is intended to be introductory, we will not pursue the details of this any further.

9B.1 *A symmetrical rigid body spins about its axis of symmetry. In which direction does its angular momentum vector point? (The body's center of mass is stationary.)*

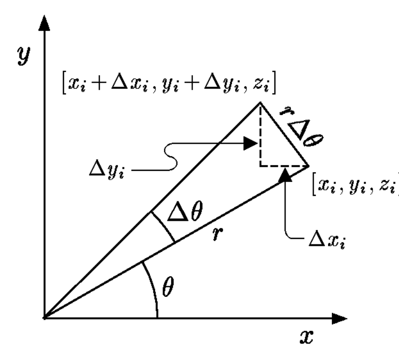
Conceptualize

We calculate the angular momentum of an extended object by summing (integrating, actually) over its volume. To do this we need to know the momentum of a small mass element as a function of its distance from the axis of rotation (we shall then be able to calculate $\vec{\mathbf{r}} \times \vec{\mathbf{p}}$ in terms of $\vec{\mathbf{r}}$). Therefore the first stage of the solution is to obtain an expression for the linear velocity of such a mass element in terms of the angular velocity of the body. We can then construct its angular momentum, and the wording of the question suggests that the symmetry of the body will lead to a solution.

Formulate

Take the axis about which the body spins to be the z-axis and consider a small element of mass m_i at position vector $[x_i, y_i, z_i]$. After a time Δt it will have position vector $[x_i + \Delta x_i, y_i + \Delta y_i, z_i]$, where we can see from the diagram that

$$\Delta x_i = -r\Delta\theta \sin\theta = -y_i \Delta\theta$$
$$\Delta y_i = +r\Delta\theta \cos\theta = +x_i \Delta\theta \ .$$



Since $\frac{\Delta\theta}{\Delta t} \to \frac{\mathrm{d}\theta}{\mathrm{d}t}$ as Δt tends to zero,

$$v_{x,i} = \frac{\mathrm{d}x_i}{\mathrm{d}t} = -y_i \frac{\mathrm{d}\theta}{\mathrm{d}t} = -\omega y_i$$
$$v_{y,i} = \frac{\mathrm{d}y_i}{\mathrm{d}t} = +x_i \frac{\mathrm{d}\theta}{\mathrm{d}t} = +\omega x_i \ .$$

The angular momentum of this small element of mass about the body's center of mass is therefore

$$\vec{\mathbf{L}}_i = \vec{\mathbf{r}}_i \times m_i \vec{\mathbf{v}}_i = m_i [y_i v_{z,i} - z_i v_{y,i} \ , \ z_i v_{x,i} - x_i v_{z,i} \ , \ x_i v_{y,i} - y_i v_{x,i}]$$
$$= m_i \omega [-z_i x_i, -z_i y_i, (x_i^2 + y_i^2)]$$

9B.1, continued:

With this information we can solve the problem.

Solve
The body is symmetrical about its axis of rotation. Therefore there must be an identical element of mass m_i at position vector $[-x_i, -y_i, z_i]$. The angular momentum of this mass element is, by the same logic,

$$\vec{\mathbf{L}}_{i'} = m_i\omega[+z_i x_i, +z_i y_i, (x_i^2 + y_i^2)] \ .$$

Adding these two contributions, the x- and y-components cancel out and the net angular momentum points along the z-axis. We can repeat this process for every pair of mirror-image mass elements in the entire body, so our conclusion is that the angular momentum vector of the whole object points along the z-axis, i.e. along the axis of rotation.

Scrutinize
We could actually have deduced the answer to this question without doing any calculation at all. We know our body has an axis of symmetry: that is, there is an axis for which, if the body is rotated about this axis, it will rotate back into itself (i.e. its appearance after this rotation is indistinguishable from its appearance before the rotation). Suppose $\vec{\mathbf{L}}$ does ***not*** point along the axis of symmetry: then after the rotation we have a body which looks the same as before, is rotating in the same way as before (as the body is spinning about its axis of symmetry, our rotation does not change $\vec{\omega}$), and yet has a different angular momentum vector. This is clearly unphysical, and the only way to avoid it is to conclude that $\vec{\mathbf{L}}$ ***does*** point along the axis of symmetry.

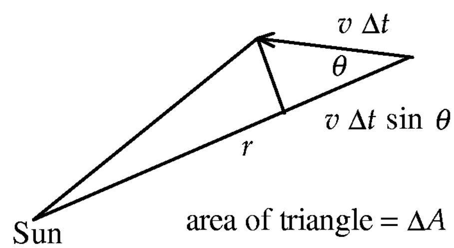
Learn
Axes of rotation with the properties of the one in this problem (i.e. they pass through the body's center of mass and the angular momentum vector of the body is along the axis of rotation) are called ***principal axes***. It turns out that ***any*** rigid body, even if not symmetrical, has three mutually perpendicular principal axes (for highly symmetrical objects like spheres there may be more than one possible set of three). This result has interesting consequences. If the angular momentum vector points along the angular velocity vector, no torque is required to keep the object spinning, and consequently once set spinning it will (in the absence of external forces) continue to do so. If, however, it is set spinning about an axis which is not a principal axis, so that the angular momentum does not point along the axis of rotation, a torque is required to keep the body rotating around this axis, because the direction of the angular momentum keeps changing. The result is that a body subject to no net torque (e.g. a projectile) can have stable rotational motion about a time-independent axis only if the axis is a principal axis. If one starts an isolated body spinning with an angular velocity that is not aligned with a principal axis, the direction of the ***angular momentum vector*** must remain fixed (because no torque is acting, so $\vec{\mathbf{L}}$ is constant), and since this is not parallel to the angular ***velocity*** vector, it follows that the axis of rotation must change with time. The resulting motion is quite similar to the precession of a gyroscope (see problem 9D.4) and is called ***torque-free precession***. You can see this effect in the "tumbling" of a thrown football or the wobbling of a badly thrown frisbee.

9B.2 *One of the first achievements of the modern scientific method was Johannes Kepler's analysis of Tycho Brahe's observations of planetary positions. Kepler deduced three empirical laws governing planetary motion, from which Newton subsequently developed his law of universal gravitation.*

Kepler's second law states that "a line drawn from the Sun to any planet sweeps out equal areas in equal times". Show that this law implies that the angular momentum of the planet due to its orbital motion is constant.

Conceptualize

The geometry of the situation is shown in the diagram. As the gravitational force acts along the line of $\vec{\mathbf{r}}$, there are only two independent unit vectors in this problem, $\hat{\mathbf{r}}$ and $\hat{\mathbf{v}}$. These define a plane, and all the motion of the planet will take place in this plane (there is no component of force acting to give it an acceleration out of the plane). This is therefore a two-dimensional problem, and the angular momentum will have only one component, in the direction perpendicular to the orbital plane. To solve the problem we must work out the rate at which the radius vector sweeps out area (the ***areal velocity***), and relate this to the angular momentum.



Formulate and Solve

Assume that at a given time our planet is a distance r from the Sun and is moving with velocity $\vec{\mathbf{v}}$. In a short time interval Δt it moves a distance $v\Delta t$ and the line joining it and the Sun sweeps out a small triangular area $\Delta A = \frac{1}{2}rv\Delta t\sin\theta$, where θ is the angle between $\vec{\mathbf{v}}$ and $\vec{\mathbf{r}}$. Now the angular momentum of the planet is just $mrv\sin\theta$, where m is the mass of the planet, so

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \lim_{\Delta t\to 0}\frac{\Delta A}{\Delta t} = \frac{L}{2m}$$

and we conclude that Kepler's second law, $\mathrm{d}A/\mathrm{d}t =$ constant, implies constant angular momentum (and vice versa).

Scrutinize

We have not used any information about the gravitational force in this derivation ***except*** the fact that it acts along the line joining the two bodies: it is a ***central force***. The equal-area law will in fact hold for ***any*** central force, since if the force between two objects is directed along the line joining them there is no torque (taking either body as reference point), and thus no change in angular momentum.

Learn

We conclude that Kepler's second law tells us very little about gravity, just that it is a central force. However, Kepler's first and third laws, which describe the shape of planetary orbits and the relation between orbital size and period respectively, are much more restrictive. These laws allowed Newton to deduce the detailed form of the law of gravity.

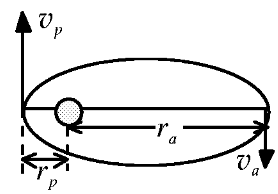
9B.3 *Comets, in contrast to planets, follow very elongated elliptical orbits, so that their distance from the Sun varies greatly over the course of the orbit. Suppose that a short-period comet has an orbit whose closest approach to the Sun (or perihelion) is* 5×10^{10} *m, the furthest point (or aphelion) being* 2×10^{12} *m. What is its speed at perihelion?*

Conceptualize

This problem seems well suited to a conservation-law approach. We have seen in Problem 9B.2 that angular momentum is conserved in planetary orbits, and—since gravity, a conservative force, is the only force acting—we expect mechanical energy to be conserved as well. The comet's linear momentum is not conserved, because there is a net force acting on it. As we saw in 9B.2, orbital angular momentum has only one non-zero component, so we will have two equations, one for energy and one for the z-component of angular momentum.

Formulate

Recalling from Chapter 4 that the standard expression for gravitational potential energy is $-GMm/r$, where r is the distance between the two masses M and m and the zero of potential energy is defined to be infinite separation of the masses, conservation of energy gives



$$v_p^2 - \frac{2GM}{r_p} = v_a^2 - \frac{2GM}{r_a}$$

(where we have divided out the common factor of $m/2$). Since at perihelion and aphelion the velocity of the comet is at right angles to the line joining it to the Sun, its angular momentum $L = mvr$ in both cases, so conservation of angular momentum means that

$$v_p r_p = v_a r_a \ .$$

Solve

We use the second equation to eliminate v_a in the first, giving

$$v_p^2 - \frac{2GM}{r_p} = \frac{v_p^2 r_p^2}{r_a^2} - \frac{2GM}{r_a} \ ,$$

i.e.

$$v_p^2 \left(\frac{r_a^2 - r_p^2}{r_a^2} \right) = 2GM \left(\frac{r_a - r_p}{r_a r_p} \right)$$

hence

$$v_p = \sqrt{\frac{2GMr_a}{r_p(r_a + r_p)}} = 72 \text{ km/s} \ .$$

Scrutinize

We can check this equation by setting $r_a = r_p$, i.e. a circular orbit. The equation then reduces to $v = \sqrt{GM/r}$, which is what we get by considering

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \ .$$

9B.3, continued:

The circular orbit speed for $r = r_p$ is 52 km/s, less than the comet's speed at this point. This is as we expect, because the comet is going to move outwards (like a car taking a bend too fast). Conversely, the comet's speed at aphelion is $v_a = v_p r_p / r_a = 1.8$ km/s, less than the circular speed of 8.2 km/s.

Learn
As with most problems where conservation laws can be applied, this is much easier than trying to solve the relevant differential equations. Note that the total energy of the comet is $\frac{1}{2}mv^2 - GMm/r = -GMm/(r_a + r_p)$. The equivalent expression for a circular orbit is $-GMm/2r$ (see problem 6.2), so the radius of the circular orbit with the same total energy as the elliptical orbit is $\frac{1}{2}(r_a + r_p)$, or half the long axis of the ellipse. This quantity appears repeatedly in dynamical calculations of orbital motion and is referred to as the ***semi-major axis*** of the ellipse.

9C.2 *Use the component form of the vector product to verify that* $\vec{\boldsymbol{\tau}} = \dfrac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t}$.

Conceptualize
This problem needs little conceptualization, as it is very specific.

Formulate
Torque $\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \vec{\mathbf{r}} \times m\vec{\mathbf{a}}$, and angular momentum $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = \vec{\mathbf{r}} \times m\vec{\mathbf{v}}$. We can approach the 'target' equation from either direction, but it is probably simplest to take the component form of $\vec{\mathbf{L}}$ and differentiate.

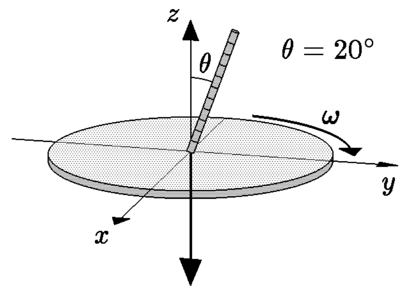
Solve
The component form of the vector product tells us that $L_x = yp_z - zp_y$. Writing this in terms of mass and velocity gives us

$$\begin{aligned}
\frac{\mathrm{d}L_x}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(myv_z - mzv_y) \\
&= m\frac{\mathrm{d}y}{\mathrm{d}t}v_z + my\frac{\mathrm{d}v_z}{\mathrm{d}t} - m\frac{\mathrm{d}z}{\mathrm{d}t}v_y - mz\frac{\mathrm{d}v_y}{\mathrm{d}t} \\
&= mv_yv_z + y(ma_z) - mv_zv_y - z(ma_y) \\
&= yF_z - zF_y = \tau_x\ .
\end{aligned}$$

The same holds for the y- and z-components, thus verifying the relationship we deduced earlier using vector notation.

Learn
This proof demonstrates that we can always manipulate the vector product by using its coordinate form. Pure vector algebra may be more elegant, but it is never ***necessary*** to use it.

9D.1 *A uniform rod is fixed to a rotating horizontal turntable so that its lower end is on the axis of the turntable and it makes an angle of 20° to the vertical. (It is thus rotating with uniform angular velocity about an axis passing through one end and inclined at 20° to the direction of the rod.) If the turntable is rotating clockwise as seen from above, what is the direction of the rod's angular velocity vector?*



(a) *vertically downwards;* (b) *down at 20° to the vertical;*
(c) *up at 20° to the vertical;* (d) *vertically upwards.*

What is the direction of its angular momentum vector (calculated about its lower end)?

(a) *vertically downwards;* (b) *down at 20° to the horizontal;*
(c) *up at 20° to the horizontal;* (d) *vertically upwards.*

Is there a torque acting on it, and if so in what direction?

(a) *yes, vertically;* (b) *yes, horizontally;*
(c) *yes, at 20° to the horizontal;* (d) *no.*

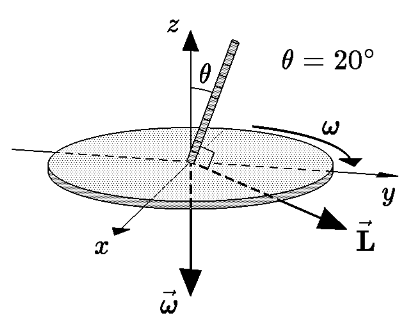
Conceptualize

The problem asks for the direction of the rod's angular velocity vector, but the motion of the rod is complicated, and a little hard to visualize. However, we are told that the rod is fixed to the turntable, which means that the rod and turntable together form a single rigid body, and therefore must have a common angular velocity. The turntable rotates in the horizontal plane, so the axis is vertical. By definition the vector angular velocity $\vec{\omega}$ points along the axis of rotation, so the only issue will be to use the right-hand rule to determine if it points up or down.

For angular momentum and torque ***about a point***, we must use the vector definitions. As we know more about the motion of the rod than we do about the forces acting on it, it makes sense to calculate the angular momentum vector $\vec{\mathbf{L}}$ first, and then derive the torque from the rate of change of $\vec{\mathbf{L}}$.

Formulate

Assume that at time $t = 0$ the rod is in the yz-plane, and consider a small segment of the rod at a distance r from its lower end. The y-coordinate of this segment is $r \sin\theta$, where $\theta = 20°$, and the x-coordinate is zero. At time t the rod will have rotated through an angle ωt about the vertical (z) axis. Looking from above, we will use our standard convention that angles and angular velocities are taken as positive when counterclockwise, so the clockwise motion in this case implies that $\omega < 0$. The position vector of the segment is then



$$\vec{\mathbf{r}} = [-\rho \sin\omega t,\ \rho\cos\omega t,\ r\cos\theta]\ ,$$

9D.1, continued:

where $\rho \equiv r \sin\theta$. The velocity of this segment is

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = -\omega\rho[\cos\omega t,\, \sin\omega t, 0] \ ,$$

so its angular momentum is

$$\begin{aligned} \overrightarrow{\Delta\mathbf{L}} &= \vec{\mathbf{r}} \times (\Delta m\,\vec{\mathbf{v}}) = -\omega\rho\,\Delta m\,[-r\cos\theta\sin\omega t,\ r\cos\theta\cos\omega t,\ -\rho] \\ &= -\omega r^2 \sin\theta\,\Delta m\,[-\cos\theta\sin\omega t,\ \cos\theta\cos\omega t,\ -\sin\theta] \ , \end{aligned}$$

where Δm is the mass of the segment, and for the z-component we have used the trigonometric identity $\sin^2\alpha + \cos^2\alpha = 1$. To find the angular momentum for the whole rod, we would integrate this expression along the length of the rod. However, since this problem asks only about directions, we do not need to carry out this integration. From the above expression we see that the direction of $\overrightarrow{\Delta\mathbf{L}}$ does not depend on r, so the integration will not affect the direction:

$$\vec{\mathbf{L}} \propto -\omega[-\cos\theta\ \sin\omega t,\ \cos\theta\ \cos\omega t,\ -\sin\theta] \ ,$$

where the proportionality constant is independent of time.

The net torque is the time derivative of $\vec{\mathbf{L}}$, so

$$\vec{\boldsymbol{\tau}} = \frac{d\vec{\mathbf{L}}}{dt} \propto \omega^2[\cos\theta\ \cos\omega t,\ \cos\theta\ \sin\omega t,\ 0] \ .$$

Solve

The turntable rotates clockwise as viewed from above, so from the definition of $\vec{\omega}$, the angular velocity vector must point downwards, along the negative z-axis. The answer to the first question is ***(a)***.

The torque is clearly nonzero, so a torque acts on the rod, but its z-component *is* zero, so it acts horizontally. The answer to the third question is ***(b)***.

If we consider time $t = 0$, the angular momentum vector is in the direction $[0, \cos\theta, -\sin\theta]$, where $\theta = 20°$. It clearly points downwards (the z component is negative) at $20°$ to the horizontal. The answer to the second question is ***(b)***.

Scrutinize

If you are good at visualizing things in three dimensions, you can work out the direction of the angular momentum vector without resorting to the coordinate form of the cross product. At the time when the rod is in the yz-plane, the instantaneous velocity of any point on the rod is in the positive x-direction, out of the page (since we are dealing with clockwise rotation). The vector $\vec{\mathbf{r}}$ is in the plane of the page, pointing $20°$ to the right of the vertical axis. Using the right-hand rule, the vector $\vec{\mathbf{L}} \propto \vec{\mathbf{r}} \times \vec{\mathbf{v}}$ points downward and towards the right, at $20°$ below the horizontal. As the rod rotates, the vertical

9D.1, continued:

component of $\vec{\mathbf{L}}$ remains the same, but the horizontal component sweeps out a circle, remaining constant in magnitude but changing direction: therefore there is no vertical component of torque (L_z remains fixed) but there are x- and y-components (L_x and L_y change with time). Hence the torque acts horizontally.

Learn

Notice that this system has zero angular acceleration ($\vec{\omega}$ is constant), but a nonzero net torque. The simple relation between torque and angular acceleration ***does not hold*** for vector torques, although you can always rely on the scalar relation $\tau_{\text{axis}} = I\alpha_{\text{axis}}$ for cases of rotation about a fixed axis. Recall that the scalar quantity τ_{axis} is not the magnitude of the $\vec{\tau}$, but rather its z-component τ_z, so in this case both sides of the equation $\tau_{\text{axis}} = I\alpha_{\text{axis}}$ are zero.

Why did we not use $\vec{\mathbf{r}} \times \vec{\mathbf{F}}$ to calculate the torque? After all, the unknown contact force exerted by the turntable through the weld at the base of the rod acts through the origin, and therefore contributes no torque—or does it? If we consider the rod when it is stationary, we can see that this would present a problem: there would be nothing to oppose the torque exerted by gravity about the contact point, which has magnitude $\frac{1}{2}MgR\sin\theta$, where M is the total mass of the rod and R is its total length. But this cannot be: the net torque on a stationary object, calculated about any point, must be zero.

The only way out of this impasse is to recognize that the weld is not a geometrical point: it has some small spatial extent. The torque exerted by gravity is cancelled by an upward contact force acting through the near side of the weld (i.e. on the same side of the weld as the body of the rod) together with a downward contact force acting on the far side of the weld. Because they act so close to the origin, each of these forces must be much larger in magnitude than Mg (to satisfy the condition of zero net ***force***, the difference between them must be Mg upwards). The weld therefore has to withstand forces much larger than the weight of the rod, so the engineering requirements for joints of this kind are much more stringent than one might naïvely expect.

One thorny aspect of this problem is the choice of signs that it forces us to consider: the angular velocity is clockwise, while we conventionally define counterclockwise as positive. Here we took ω to be negative, but we could alternatively have changed our conventions. Changing conventions makes it easier to keep track of which way the turntable is really rotating, but it requires you to rethink the signs that appear in your equations. Either approach requires care. One advantage of becoming accustomed to working with negative values is that one often needs this technique for more complicated problems. If there were two objects rotating about the same axis but in opposite directions, we would certainly not want to use different conventions for each so that both angular velocities would be positive.

9D.2 *A conical pendulum consists of a small heavy sphere of mass m attached to a string of length ℓ and negligible mass. The string makes an angle θ with the vertical and the bob describes a circular path with constant speed v.*

9D.2, continued:

(a) *What is the angular momentum of the bob about a vertical axis through the point of suspension? What torque is exerted about this axis?*

(b) *What is the angular momentum vector of the bob with respect to the point of suspension itself? What torque is exerted about this point?*

Conceptualize

The first part of this problem uses the 'scalar' quantities of torque and angular momentum about an axis. These are evaluated using quantities defined in the plane in which the bob moves. Even without calculation one can see that in this plane the bob behaves as if it were subject to a central force (its acceleration is directed towards the center of the circle in which it moves), and so we predict that the angular momentum will turn out to be constant, and the torque to be zero.

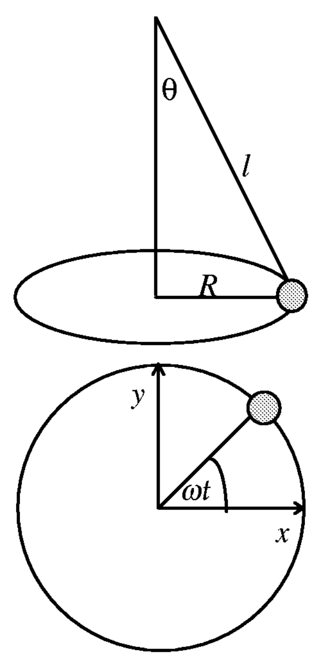
The second part of the problem is three-dimensional, as the suspension point is not in the plane of the bob's motion, and will therefore have to be solved using vector products.

Formulate and Solve (a)

The bob is describing a circle of radius $R = \ell \sin\theta$. Its angular velocity is therefore $\omega = v/R$ and its angular momentum about a vertical axis through the center of the circle is $I\omega = mRv$. This is constant, so there can be no torque about this axis, as we expected from the arguments above.

Formulate (b)

Now consider the situation relative to the point of suspension. Define a coordinate system with the origin at the suspension point and the z-axis vertically upwards. Assume that at time $t = 0$ the bob is in the xz-plane, i.e. its y-coordinate is zero. At time t it has moved through an angle ωt, where ω is its angular velocity about the z-axis, and its position relative to the suspension point is



$$x = R\cos\omega t \ , \quad y = R\sin\omega t \ , \quad z = -Z \ ,$$

where $Z = \ell\cos\theta$. Its velocity is

$$\vec{\mathbf{v}} = \left[\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right]$$

and its momentum $m\mathbf{v}$ is therefore

$$p_x = -\omega m R\sin\omega t \ , \quad p_y = \omega m R\cos\omega t \ , \quad p_z = 0 \ .$$

9D.2, continued:

Solve (b)
Forming the vector product of $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$ gives us the angular momentum

$$\begin{aligned} L_x &= yp_z - zp_y = \omega mRZ\cos\omega t \ ; \\ L_y &= zp_x - xp_z = \omega mRZ\sin\omega t \ ; \\ L_z &= xp_y - yp_x = \omega mR^2(\cos^2\omega t + \sin^2\omega t) \\ &= \omega mR^2 = I\omega \ . \end{aligned}$$

Note that this is ***not*** along the z-axis. It is also not constant, implying the existence of a torque about this point. Differentiating $\vec{\mathbf{L}}$ with respect to time gives us $\tau_z = 0$, since L_z is constant, and

$$\begin{aligned} \tau_x &= -\omega^2 mRZ\sin\omega t \ ; \\ \tau_y &= \omega^2 mRZ\cos\omega t \ ; \end{aligned}$$

The magnitude of the total torque is $\omega^2 mRZ$.

Scrutinize
The result of part (b) is, in fact, exactly what we would expect: the centripetal force calculated from the acceleration required for the bob's circular motion is $mv^2/R = m\omega^2 R$, and its line of action is a perpendicular distance Z from the suspension point, so the torque should indeed be $\omega^2 mRZ$.

The results of part (a) and part (b) should be consistent, as they refer to the same situation. We can check this by looking at the z-component of our angular momentum vector—the 'scalar' angular momentum around an axis is simply the component of vector angular momentum parallel to that axis. So the z-component of $\vec{\mathbf{L}}$ should be mvR, and indeed it is.

Learn
This example clearly demonstrates the important differences between torque and angular momentum about an ***axis***, which are effectively scalar quantities, and torque and angular momentum about a ***point***, which are vectors. Although our two calculations are perfectly self-consistent, they represent quite different ways of looking at the situation. In particular, notice that the connection between torque and angular acceleration is no longer apparent for vector torques: our conical pendulum is still moving at constant angular velocity in case (b), but this does not mean that it is not subject to a net torque. The relation between torque and angular ***momentum***, however, still holds, and is the more fundamental concept (in a similar way, if we extend our consideration to objects moving near the speed of light, the $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ formulation of the second law for linear motion likewise fails, but $\vec{\mathbf{F}} = \mathrm{d}\vec{\mathbf{p}}/\mathrm{d}t$ is still relevant).

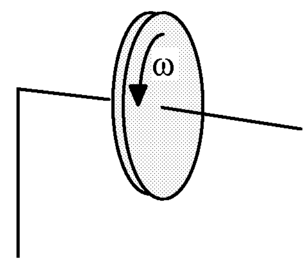
9D.4 *A gyroscope is a massive rapidly spinning wheel mounted on an axle of negligible mass. If such a device is held with the axle horizontal and supported at one end by a vertical post, what will happen when it is released? Assume the wheel has mass m and moment*

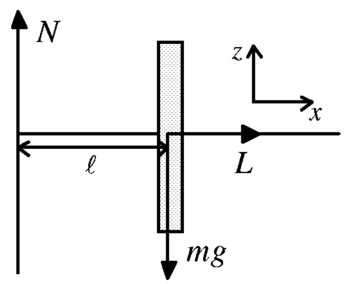
9D.4, continued:

of inertia I, that it is spinning with angular velocity ω, and that the distance between the wheel and the supported end of the axle is ℓ.

Conceptualize

Let us define the x-axis as the initial direction of the axle. Since the gyroscope is symmetric about its axis of rotation, the rapid spin of the gyroscope about its axle will result in an angular momentum along its axis of rotation (see Problem 9B.1), which initially is along the x-axis. We will soon see that the actual motion is slightly more complicated than this, but if the gyroscope is spinning very rapidly, then the angular momentum will be dominated by the contribution from the rotation about the axle, and all other contributions to the angular momentum can be ignored. The forces acting on it are gravity, which acts through the center of mass, and the contact force from the supporting post, which acts at the end of the axle. Since these two points are not coincident, there ***must*** be a net torque on the gyro.





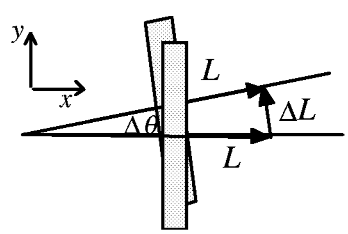
Formulate

It is clearly appropriate to define the torque about the point of support (it's the only point which has any reason to remain fixed), and so only the gravitational force contributes. The net torque is therefore

$$\vec{\tau} = [\ell, 0, 0] \times [0, 0, -mg] = [0, mg\ell, 0]\ .$$

Solve

In a short time Δt, the angular momentum thus changes by an amount $\Delta L_y = mg\ell\Delta t$, in the y direction. Since this is perpendicular to the angular momentum, the change in the magnitude of the angular momentum would be second order in Δt, and therefore vanishes in the limit of very small Δt. (That is, the rate at which the magnitude of the angular momentum is changing would be



$$\frac{\sqrt{L_x^2 + m^2 g^2 \Delta t^2} - L_x}{\Delta t}\ ,$$

and this quantity vanishes in the limit $\Delta t \to 0$.) The direction of the angular momentum, however, does change by an amount proportional to Δt:

$$\Delta\theta = \frac{mg\ell\Delta t}{I\omega}\ .$$

9D.4, continued:

If we use again our assumption that the only relevant contribution to the angular momentum is a vector of magnitude $I\omega$ pointing along the axle of the gyroscope, then the direction of the gyroscope must change by this same angle, $\Delta\theta$. Since the wheel's orientation with respect to its axle is fixed, the entire gyroscope must turn through an angle $\Delta\theta$—hence the gyroscope rotates in the horizontal plane with an angular velocity

$$\Omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{mg\ell}{I\omega} \; .$$

If we consider Ω as a vector, it is oriented along the z direction, since this is the axis about which the axle of the gyroscope is rotating.

Scrutinize and Learn

This phenomenon is called ***precession***, and the velocity Ω is the ***precessional velocity***. Notice that the faster the gyro is spinning, the more slowly it will precess. The axis of the spin does not have to be horizontal for precession to occur: anything other than absolutely vertical will do (having it horizontal just simplifies the algebra).

Our description is in fact somewhat oversimplified: the development of the precessional motion produces a small angular momentum component in the z-direction which was not originally present. This is illegal because we don't have a torque acting in the z-direction to produce such a component of angular momentum. The result is that the axle of the gyroscope actually ends up pointing slightly below the horizontal (relative to the point of support), so that the spin angular momentum has a small negative z-component to balance that from precession. Before settling down to this steady state, the gyroscope will oscillate for a while about the 'right' position—this superimposes an additional wobble, called ***nutation***, on the steady precession. These effects are most clearly seen if the gyro is spinning fairly slowly, so that the precessional velocity (and hence the precessional angular momentum) is larger.

Precessional effects are actually quite common in everyday life, although we do not usually think of them as such—for example, the way in which a spinning coin comes to rest is a consequence of precessional motion.

9D.8 *Show that the equation relating the velocity of a point on a rigid body to the body's angular velocity*

$$\vec{\mathbf{v}} = \vec{\mathbf{v}}_P + \vec{\omega} \times (\vec{\mathbf{r}} - \vec{\mathbf{r}}_P),$$

where P is a reference point on the body, is indeed independent of the choice of reference point P, the origin of the coordinate system, and the velocity of the coordinate system, as stated in the text.

Conceptualize

If the formula is indeed independent of the choice of P, then choosing another point Q should give the same result. We can see if this is so by obtaining expression for the position and velocity ***of point P*** using Q as origin, and then substituting these for $\vec{\mathbf{r}}_P$ and $\vec{\mathbf{v}}_P$. A similar approach should work for the origin and velocity of the coordinate system.

9D.8, continued:

Formulate and Solve

The velocity of our test point relative to reference point P is

$$\vec{\mathbf{v}} - \vec{\mathbf{v}}_P = \vec{\boldsymbol{\omega}} \times (\vec{\mathbf{r}} - \vec{\mathbf{r}}_P).$$

Relative to reference point Q, it is

$$\vec{\mathbf{v}} - \vec{\mathbf{v}}_Q = \vec{\omega}' \times (\vec{\mathbf{r}} - \vec{\mathbf{r}}_Q),$$

where, to avoid assuming what we are trying to prove, we have taken the angular velocity vector to be $\vec{\omega}'$. Since this equation holds for any point on the body, it must hold for P:

$$\vec{\mathbf{v}}_P - \vec{\mathbf{v}}_Q = \vec{\omega}' \times (\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q).$$

Now, subtracting this equation from the previous one gives

$$\begin{aligned}\vec{\mathbf{v}} - \vec{\mathbf{v}}_P &= \vec{\omega}' \times (\vec{\mathbf{r}} - \vec{\mathbf{r}}_Q) - \vec{\omega}' \times (\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q)\\ &= \vec{\omega}' \times (\vec{\mathbf{r}} - \vec{\mathbf{r}}_Q - \vec{\mathbf{r}}_P + \vec{\mathbf{r}}_Q)\\ &= \vec{\omega}' \times (\vec{\mathbf{r}} - \vec{\mathbf{r}}_P)\,,\end{aligned}$$

using the distributive law for cross products.

This formula is identical to the one we started with if and only if $\vec{\omega}' = \vec{\omega}$. Therefore the angular velocity vector is unchanged by the change of reference point.

If we change the origin of the coordinate system, such that the distance from the initial origin O to the new origin O' is $\vec{\mathbf{r}}_0$, the position vectors of our test particle and P become $\vec{\mathbf{r}} - \vec{\mathbf{r}}_0$ and $\vec{\mathbf{r}}_P - \vec{\mathbf{r}}_0$ respectively. It is clear that $\vec{\mathbf{r}} - \vec{\mathbf{r}}_P$ is unchanged by this transformation. Similarly, if our new coordinate system has velocity $\vec{\mathbf{v}}_0$ relative to our original system, $\vec{\mathbf{v}}$ becomes $\vec{\mathbf{v}} - \vec{\mathbf{v}}_0$, $\vec{\mathbf{v}}_P$ becomes $\vec{\mathbf{v}}_P - \vec{\mathbf{v}}_0$, and $\vec{\mathbf{v}} - \vec{\mathbf{v}}_P$ is unchanged. Therefore the angular velocity vector is unchanged by changes in the origin or velocity of the coordinate system.

HINTS FOR PROBLEMS WITH AN (H)

The number of the hint refers to the number of the problem

9A.6 Under what conditions will a rigid body be stationary (and nonrotating)? Draw a force diagram for the rod, being careful to draw in all forces at the correct points. What are the equations for the net force and the net torque about some appropriate point?

9A.7 Draw a force diagram, taking care to show all forces acting at the right place. What are the general conditions for a rigid body to be stationary? If you still don't know what to do, review the solution to problems 9A.4 and 9A.5.

9A.9 Can you treat this problem as a rotation about a single axis? How many equations does the condition of zero net ***torque*** give you? What about zero net ***force***? If you're really stuck, look at the solution to problem 9A.8.

9B.4 Try using conservation of energy and conservation of angular momentum. What, in terms of its unknown speed u, is its total energy immediately after the decelerating burst of the engines? What is its angular momentum about the Sun? If at closest approach it just hits the Sun, what then is its total energy and its angular momentum, in terms of its new unknown speed v? You should now be able to solve for u and v.

If you are having difficulty, study the solution to problem 9B.3.

Challenge Problem hint: consider what happens if you start by firing the engines forwards for a short burst.

9C.1 What are the components of $\vec{\mathbf{F}}$ in a coordinate system where the x-axis points along $\vec{\mathbf{R}} \equiv [x, y, 0]$, and the z-axis is unchanged? What, therefore, is the component of $\vec{\mathbf{F}}$ perpendicular to $\vec{\mathbf{R}}$?

9D.3 From Chapter 5, what is the expression for the net force on a rigid body (i.e. a system of particles)? Under what circumstances will a net force produce no torque?

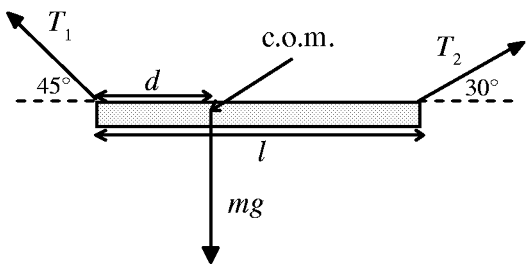
9D.5 What quantities are conserved in an isolated system? What is the direction of the angular momentum of the wheel before and after it is turned? What are the possible sources of torques applied (a) to the wheel, (b) to the whole wheel + person + chair system? What are the directions of these torques?

9D.6 Draw a force diagram for the wheel at the instant of touchdown. What force contributes a torque about the axle? What is the net horizontal force?

When the wheels stop sliding, what is the relation between the linear speed of the plane and the angular velocity of the wheels?

ANSWERS TO HINTS

9A.6 No net force and no net torque.

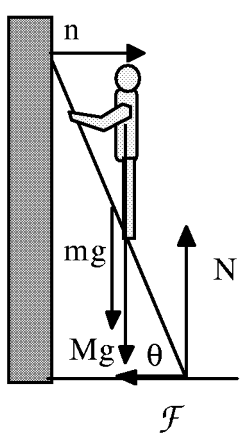


Taking the mass of the rod to be m and other symbols as in the diagram,

$$F_x = T_2 \cos 30^\circ - T_1 \cos 45^\circ;$$
$$F_y = T_2 \sin 30^\circ + T_1 \sin 45^\circ - mg;$$
$$\tau = mgd - T_2 \ell \sin 30^\circ,$$

where the torque is taken about the left-hand end of the rod, the x-axis is along the rod and the y-axis is vertical. (Other suitable points would be the right-hand end of the rod, or its center of mass. These would produce different torque equations.)

9A.7



No net force and no net torque.

9A.9 No. The slab is in principle free to rotate in three dimensions (in practice, two, because none of the forces involved acts horizontally, so there is no torque about any vertical axis). Two (about any two mutually perpendicular axes in the plane of the slab—easiest to calculate would be parallel to long and short sides). One (all forces act vertically).

9B.4 At Earth orbit:

$$\tfrac{1}{2}mu^2 - \frac{GMm}{d} = m(\tfrac{1}{2}u^2 - 8.9 \times 10^8\ \mathrm{m^2/s^2})$$

and $mud = (1.5 \times 10^{11}\ \mathrm{m})mu$.

At radius of Sun:

$$\tfrac{1}{2}mv^2 - \frac{GMm}{r} = m(\tfrac{1}{2}v^2 - 1.9 \times 10^{11}\ \mathrm{m^2/s^2})$$

and $mvr = (7 \times 10^8\ \mathrm{m})mv$.

9C.1 $[F_x \cos\theta \ + \ F_y \sin\theta,\ F_y \cos\theta \ - \ F_x \sin\theta,\ 0]$ where $\cos\theta = x/R$ and $\sin\theta = y/R$; $(F_y x - F_x y)/R$.

9D.3 $\vec{\mathbf{F}}^{\mathrm{net}} = M\vec{\mathbf{a}}_{\mathrm{cm}}$; if it acts through the axis of rotation (or, equivalently, if several forces produce torques which act in opposite directions and so cancel).

9D.5 Momentum and angular momentum. Along axis of wheel, so horizontal before turning, vertical afterwards.

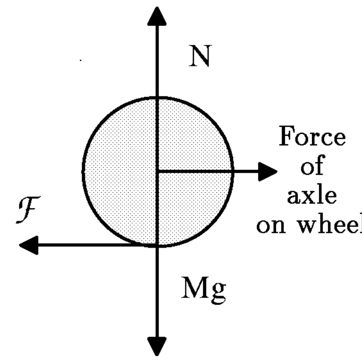
(a) forces exerted by person holding wheel (any direction);

(b) gravity, if center of mass of system not under suspension point (horizontal).

Torque taken about center of wheel in case (a) and about suspension point in case (b).

9D.6 Diagram at right.

Kinetic friction; $-\mu_k Mg$; $v = R\omega$.

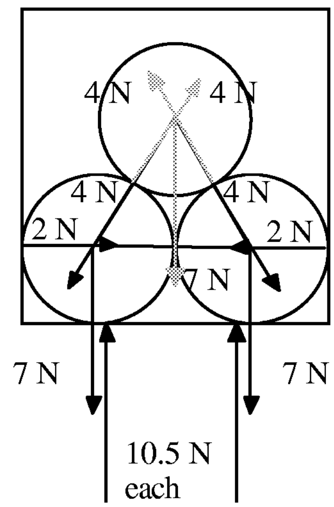


ANSWERS TO ALL PROBLEMS

9A.1 c; b.

9A.2 b.

9A.3 See diagram at right. (Weight of each bottle 7 N; contact forces between bottles 4 N along lines joining bottle centers; horizontal contact forces from walls 2 N; vertical contact forces from base of box 10.5 N. Forces acting on top bottle are shown in gray and vertical contact forces are displaced from their point of action for clarity.)



9A.4 See complete solution.

9A.5 See complete solution.

9A.6 3.7 cm from the left-hand (45°) end.

9A.7 $\mu_s = \dfrac{\frac{1}{2}m + Md/\ell}{m+M}\cot\theta$

Yes; yes, minimum required coefficient of friction decreases to 0.16.

9A.8 See complete solution.

9A.9 160, 240 and 400 N respectively;
third person always exerts 400 N, first exerts $160 \cdot (d/1\text{ m})$ N. The second exerts

$$160 \cdot \left(\frac{2.5\text{ m} - d}{1\text{ m}}\right)\ \text{N}\ ,$$

where d is the distance between the third person and the corner across from number 1.

9B.1 See complete solution.

9B.2 See complete solution.

9B.3 See complete solution.

9B.4 The maximum speed at the Earth's orbit which would lead to a minimum radius (perihelion) inside the Sun is 3 km/s, compared with the Earth's orbital speed of 30 km/s. For comparison, the speed needed to escape from the Solar System altogether starting from the Earth's orbit is 42 km/s. It is thus somewhat more than twice as hard, in terms of required momentum change, to hit the Sun by this method as it is to escape into interstellar space.

Challenge Problem answer: One can first fire the rockets forward, applying an impulse just short of what would be needed to leave the solar system altogether. The maximum radius (aphelion) would then be arbitrarily large, and, conserving angular momentum, the orbital velocity at aphelion would be arbitrarily small. At aphelion the rocket then fires a feeble burst, just canceling this small orbital velocity, and then falls slowly straight into the Sun.

9B.5 (a) $mv\ell\sin\theta$

(b) perpendicularly into plane of page, along $[0, 0, -1]$.

9C.1 Unit vectors along $\vec{\mathbf{R}} = [x, y, 0]$ and perpendicular to $\vec{\mathbf{R}}$ are

$$\hat{\mathbf{R}} = \frac{1}{R}[x, y, 0]$$

and

$$\hat{\mathbf{S}} = \frac{1}{R}[-y, x, 0]$$

respectively.

In terms of these unit vectors the force $\mathbf{F}$ is given by

$$\mathbf{F} = \frac{1}{R}[(xF_x + yF_y)\hat{\mathbf{R}} + (xF_y + yF_x)\hat{\mathbf{S}}] \ .$$

The component of this perpendicular to $\vec{\mathbf{R}}$ is by definition the $\hat{\mathbf{S}}$ term, so the torque is

$$\tau = F_\perp R = xF_y - yF_x$$

as required.

9C.2 See complete solution.

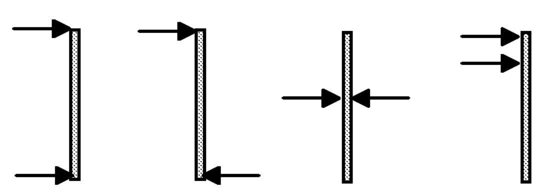
9C.3 An acceptable answer would be:

"Torque about an axis is effectively a scalar quantity, producing clockwise or counterclockwise (negative or positive, respectively) angular acceleration about that axis. Torque about a point is a vector, with no simple relation to angular acceleration (e.g. if an object rotates with constant angular velocity $\vec{\omega}$, the net torque about an axis in the direction of $\vec{\omega}$ must be zero, but the net torque about any point on that axis will not in general be zero). The x-component of the torque vector about a point corresponds to torque about an axis in the x direction, and so on."

9D.1 See complete solution.

9D.2 See complete solution.

9D.3 There are of course infinitely many correct answers to parts (a)–(d)—the diagrams below are just examples!



(a) move across table retaining same orientation; (b) rotate around fixed center of mass; (c) remain stationary; (d) move across table with changing orientation.

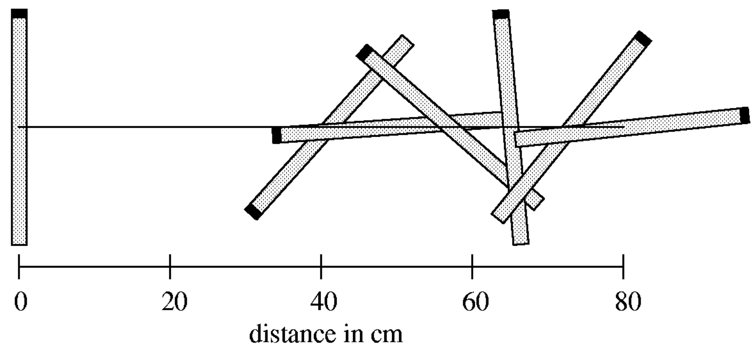
Case (d)—net force and net torque.

4 cm/s; 0.39 rad/s.

Your scale drawing should have the following features:

distance of center of mass from start: 0, 40, 48, 56, 64, 72, 80 cm;

angle swept through: 0°, 221°, 265°, 309°, 353°, 398°, 442°.



9D.4 See complete solution.

9D.5 Almost certainly, the first thing that happens is that the hapless victim tries to shift the axis to vertical by pushing one end up and the other down. If we define a coordinate system such that z is up and x points along the initial axis of the wheel, we now have a force in the z direction applied at a position on the axis of the wheel (i.e. $\vec{\mathbf{r}}$ is in the x direction). This produces a torque in the y direction: the axis of the wheel rotates ***sideways***, not vertically. Once our visitor has worked out which direction to push to get the axis to turn vertically, he or she will end up rotating in the opposite direction to the spin of the wheel.

In terms of conservation of angular momentum, we see that there is no external torque on the person + chair + gyro system in the z (vertical) direction. The change in the z-component of the gyro's angular momentum must therefore be balanced by an opposite change in that of the chair + occupant, hence the rotation. The change in the x-component of angular momentum would make the chair rotate around a horizontal axis, but when it does so its center of mass is no longer directly under its suspension point, so gravity provides an external torque. Hence angular momentum in the horizontal direction is not conserved.

In terms of torques, to make the gyro's axis vertical the person has to exert a torque in the z direction, which he/she does by pushing sideways on the axis of the gyro. The force exerted on the gyro is balanced by an equal and opposite force exerted by the gyro on the person (Newton's third law), so the person must develop an opposite angular momentum.

9D.6 $v = \dfrac{MR^2 v_0}{I + MR^2}$.

9D.7 (a) and (b): In both cases, the comparatively large angular momentum of the spinning projectile means that a small accidental torque (e.g. due to a gust of wind) will not produce a significant change in the orientation of the bullet or frisbee. The result is that the projectile maintains the desired attitude (point first, for a bullet; disk horizontal, for a frisbee), minimizing air resistance and thus improving range and accuracy.

(c) When the helicopter pilot changes the speed of the main rotor, one part of the helicopter system (the engine) is exerting an internal torque on another part (the rotor). We would expect the angular momentum of the whole system to remain unchanged by this internal torque, so the main body of the helicopter should start to counter-rotate against the change in the rotor speed. The tail rotor, which is a propeller (i.e. its blades are angled to produce a thrust) can exert a torque to balance that produced by the change in the rotor speed. Even in steady flight, there is an external torque on the helicopter from air friction with the main rotor, tending to reduce the overall angular momentum of the helicopter system; again, this can be counterbalanced by the torque from the tail rotor.

(d) Given the frictionless mount, there can be no net torque on the gyro, which will therefore maintain the same absolute orientation. Checking the orientation of the mount (i.e. the aircraft) relative to the axis of the gyro then gives you the orientation of the craft in that plane.

(e) As in (a) and (b), such a gyroscope stabilizes the orientation of the ship against accidental torques—i.e. it reduces roll and pitch (and hence seasickness among the passengers, or that's the idea).

9D.8 See complete solution.

9D.9 (a) $\vec{\mathbf{v}} = [0, \omega_z x_0, 0]$;

(b) $L_z = M\omega_z x_0^2$ about z-axis; $\vec{\mathbf{L}} = [-M\omega_z x_0 z_0,\ 0,\ M\omega_z x_0^2]$ about center of mass.

(c) As the meteoroid is in empty space, no external torques act on it, and therefore the orientation (and magnitude) of its angular momentum vector must remain fixed. But if we consider the frame in which the center of mass of the meteoroid is stationary, and require that $\vec{\mathbf{L}}$ remain fixed, clearly $\vec{\omega}$ cannot also remain fixed as the meteoroid rotates (either $\vec{\mathbf{L}}$ rotates around $\vec{\omega}$, which it can't, or $\vec{\omega}$ rotates around $\vec{\mathbf{L}}$). Therefore the cameras on the space probe will see the meteoroid ***tumble***: its rotation axis will precess around its angular momentum vector. (This is torque-free precession: see the solution to problem 9B.1.)