ESSENTIALS OF INTRODUCTORY CLASSICAL MECHANICS

Sixth Edition

CHAPTER 13

REVIEW

OVERVIEW

This chapter contains no new ideas. Instead, we shall take what we have learned in this book and use it to solve more complicated problems. Notice that in this chapter we shall not divide the problems up into sections dealing with specific topics; instead you will have to decide for yourself which of the physical principles you have learned are relevant to a given problem. Most of the problems will concentrate on the material of the last two chapters, but some problems included here relate to earlier material. Since this is the last chapter of the book we also include a Summary reviewing what we have covered and explaining how it fits into the wider context of physical science.

When you have completed this chapter you should:

- ✔ be able to extract the essential features of a problem and express them in mathematical equations;
- ✔ be capable of manipulating the equations relevant to a problem to obtain an expression for the required quantity, either symbolically or numerically;
- ✔ be able to analyse a hypothetical situation in terms of the physics presented in previous chapters, and explain that analysis clearly in non-mathematical terms;
- ✔ see how classical mechanics fits into the general structure of scientific knowledge.

CHAPTER 13

SUMMARY

WHAT HAVE WE ACHIEVED?

We have now completed our introduction to classical mechanics. At this point it may be useful to look back over the semester as a whole and see what we have achieved in this book and how the knowledge we have gained relates to the rest of physics and wider fields of science.

We began by studying the motion of a simple point particle—an idealized, simplified object which is easy to describe and manipulate mathematically and avoids the complications of real-world effects such as friction, internal energy, etc. We saw how to analyze and predict the motion of a point particle by means of Newton's laws and introduced the concepts of position, time, force, mass and energy as measurable quantities whose mathematical interrelations allow us to describe and predict the behavior of the particle.

We then extended our picture to systems of point particles, first simple two-particle systems and then those involving many particles. We found that the motion of a system of particles could be decomposed into an overall motion of the whole system, which obeyed the same laws as point particles, and internal motions of components of the system relative to the system center of mass. The first result indicates how our simple point-particle dynamics can be applied to real objects, while the second underlies such apparently unrelated concepts as rotation and temperature.

A system of many particles is described in exact classical mechanics by a large system of simultaneous differential equations, which very rapidly becomes impossible to handle analytically. However, there are many special cases in which the exact analytical approach is not necessary. We considered the case where all the components of a system have fixed relative positions (a rigid body, or idealized solid) and, at the opposite extreme, the case where the components of the system have completely random and uncorrelated motion (the kinetic theory of an ideal gas; also, with slightly different conditions, fluid mechanics and idealized liquids). The results of analyses of this type are not ***complete*** solutions to the motion of the system, but they are often complete predictions of the ***observable*** behavior of the system, and they have many extremely important practical applications.

We developed our theoretical structure of classical mechanics on a remarkably small foundation: the fact that mathematics can be used to describe natural phenomena, a few basic rules (Newton's Laws and the conservation of energy, momentum and angular momentum) and some fundamental concepts (position, time, mass, energy, the four fundamental forces of gravity, electromagnetism and the short-range strong and weak interactions). These building blocks, especially the conservation laws which are deeply rooted in the most basic properties of the universe, underpin the whole theoretical structure of physical science, although they appear in a particularly clear and straightforward way in classical mechanics. It is an amazing demonstration of the essential simplicity of our universe that so many disparate experimental findings and observations can be interpreted with such a small number of initial assumptions.

One should, however, stress that in concentrating on the logical construction of this mathematical structure we have seriously misrepresented its historical development. Physics, like all science, is based on observation, and in fact this structure began as a collection of independent experimental results, observations and partial theories which were only gradually unified into the elegant logical structure we have erected. Archimedes' Principle, for example, predates Newton's laws (from which we derived it) by more than 1500 years, and the ideal-gas law was first deduced from experiment and only subsequently understood in terms of kinetic theory.

We could increase the number of practical applications of our techniques by making some small improvements to our idealizations:

- Solids are not perfectly rigid, but deform and break if subjected to large forces. We have already modeled this behavior for the particular case of a spring, and we could improve our description of solids by introducing analogues of $F = -k\Delta x$ for the various types of deformation possible.
- Our discussion of ideal liquids neglected frictional forces, which are important for many liquids (compare water and treacle). This can be remedied by introducing the concept of ***viscosity***.
- Finally, the ideal gas model is actually an excellent description of most common gases, but it can be improved by allowing for the finite size of the gas molecules and the existence of weak intermolecular forces. The resulting ***van der Waals equation*** can be used for gases near their liquefaction point, such as gasoline vapor at room temperature.

Although these extensions of our models would improve, in some cases dramatically, our description of real objects, they would ***not*** introduce any new principles or concepts.

There are some physical systems which are described by the laws of mechanics we have developed here, but for which neither exact analytical solutions nor helpful idealized models exist. An example is the gravitational interactions of the members of the Solar System. Although the gravitational interactions of stars and planets are very simple, there is in fact no general exact analytical solution to even the second-simplest possible problem, namely the motion of three gravitationally interacting point particles, so a computational approach using numerical integration is the only hope. A common difficulty with such systems is that the result after a long period of numerical simulation may turn out to be extremely sensitive to the initial conditions (e.g. giving one planet an initial velocity of 10.0000000001 km/s rather than just 10 km/s may produce a completely different outcome after the numerical equivalent of 100 million years). This sensitivity is also common in numerical problems in fluid dynamics, the most familiar example being the inaccuracy of long-term weather forecasts. The behavior of such systems is generally extremely complicated; when graphed it looks random and disorganized, even though in fact it may be governed by very simple deterministic laws like those of gravity or fluid mechanics. Systems of this kind are common and important, but have long been neglected by physicists because there was no simple way to handle them mathematically. In recent years they have been attracting a great deal of interest from theoretical physicists and applied mathematicians, and the branch of science known as ***chaos theory*** has developed as a result.

WHERE DO WE GO FROM HERE?

Many features of classical mechanics are extremely typical of the scientific approach to understanding natural phenomena.

- The laws of mechanics are firmly based on ***observation and experiment***.

- We start with ***simple situations*** and make use of ***idealized models*** to extract the essential features of a phenomenon.
- We develop ***predictive theories*** which are then ***tested*** by more experimentation and/or observation.

Although the content and formalism of other branches of science may be very different from classical mechanics, you will see this underlying structure everywhere, from designing clinical trials in medicine to developing high temperature superconductors in solid-state physics.

Because the idea of objects in motion underlies so much of physical science, classical mechanics also forms a jumping-off point for many other areas of physics. Some, such as the study of periodic motion (waves and oscillations) resulting from small displacements from equilibrium, are alternative ways of studying the behavior of matter in bulk, while others, such as the study of material properties, make use of the molecular picture we developed in studying kinetic theory (although we neglected interactions between atoms, whereas such interactions are fundamental to understanding the physical and chemical properties of matter). We looked at the Newtonian theory of gravity, which is the first and simplest of the theories describing the fundamental forces. Soon you may study the far richer field of electricity and magnetism, the phenomena associated with the electromagnetic fundamental force.

We can also extend the core topic of our subject—mechanics itself. Classical mechanics has a wide field of applications, from the expansion of gases to the trajectories of space probes, but it does fail in the domains of the very large (where the geometry of spacetime may differ from our simple assumptions), the very small (where quantum phenomena are important) and the very fast (where our assumptions about the absolute flow of time break down). Studying mechanics in these conditions is like studying a pendulum which has a large amplitude (so that one cannot take $\sin\theta \approx \theta$): we must throw out our approximations and develop more exact theories. The more exact theories developed to date are relativity, which describes motion on very large scales and involving very high velocities, and quantum theory, which applies to very small scales.

The existence of these more precise theories does not imply that classical mechanics is wrong, any more than the analysis of a pendulum swinging with a small amplitude is wrong because we can also analyze the large amplitude case. If we use velocities and distances which are in the appropriate range, the mathematics of the more exact theories reduces, to a very high degree of accuracy, to that of classical mechanics. On a very large scale, we are once again using an idealized model—our theory is an approximation, but one which in its proper context gives results which are virtually indistinguishable from more exact calculations.

Furthermore, even relativity and quantum mechanics are ***not*** complete and finished descriptions of nature. Our current state of knowledge does not connect quantum mechanics (the theory of the very small) with General Relativity (the theory of the very large, especially gravity and the structure of spacetime), essentially because we do not understand the quantum mechanical structure of the fundamental force of gravity. Recent developments in ***superstring theory***, an exotic desciption of the elementary matter and force particles in terms of vibrating multidimensional strings, offer the prospect of describing quantum mechanics and gravity in a consistent framework, but as yet the mathematics of this field is not sufficiently well understood to make quantitative calculations and predictions possible. Physics is not simply a fixed body of knowledge, but a dynamic and evolving subject which will surely continue to grow and develop for many years to come.

PROBLEMS AND QUESTIONS

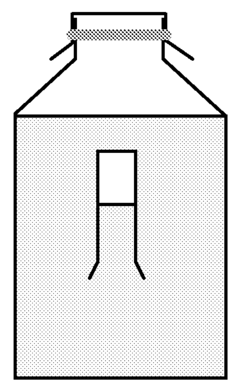
By the end of this chapter you should be able to answer or solve the types of questions or problems stated below.

At the end of the chapter there are answers to all the problems. In addition, for problems with an (H) or (S) after the number, there are respectively hints on how to solve the problems or completely worked-out solutions.

13.1 (H) Liquid water has a density of 1000 kg/m^3 and a molecular weight of 18 u (1 u = 1.66×10^{-27} kg). How many molecules are there in one liter (1000 cm^3) of liquid water? What average spacing does this imply between the centers of adjacent molecules? Repeat this calculation for an ideal gas at 10^5 Pa and 300 K. Discuss your answers in terms of the assumptions we made in constructing the ideal-gas model.

(Atoms and small molecules have diameters of the order of 10^{-10} or 10^{-9} m.)

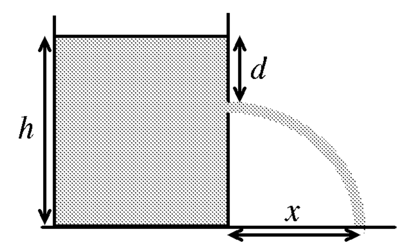
13.2 (H) The diagram on the right shows a ***Cartesian diver***. The large bottle is covered by a rubber sheet, so you can increase the surface pressure on the water by pushing down on the sheet. If at a certain pressure P the diver (i.e. the small bottle with the air bubble in it) is stationary, what will happen if you push down harder on the rubber sheet, and why?



13.3 (S) A uniform cuboidal block of wood, of dimensions $a \times b \times c$ where $a \gg b \gg c$, is dropped into a lake. In which orientation will it float, and why?

13.4 (S) A quantity of liquid of density ρ is contained in a U shaped tube of constant cross-section A. Initially one end of the tube is closed and the liquid in that end stands higher by an amount $2h$ than the liquid in the other end. The closed end of the tube is now opened so that the pressures at both ends are equal. What is the subsequent motion of the liquid in the tube?

13.5 A large open tank is filled with water to a height h. There is a small leak in the tank a distance d below the water surface. Calculate the speed at which the water emerges from this hole, and hence the distance from the tank at which it hits the ground, x. For a given height h, what is the value of d for which x is maximized? Neglect viscosity and air resistance and assume laminar flow.



13.6 A thin uniform wooden rod of mass M and length ℓ hangs vertically from one end on a frictionless pivot. A bullet of mass m is shot horizontally with speed v_i into the other end of the rod; it passes through and emerges with a lower speed v_f, but with no change of direction. The amount of mass lost by the rod due to splintering by the bullet is negligible.

(a) Derive an expression for the angular speed of the rod immediately after the bullet emerges from it.

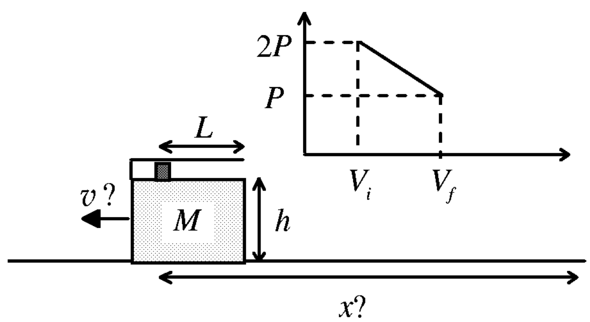
13.6, continued:

(b) In terms of the given quantities, what is the maximum angle $\theta_{\max}$ that the rod makes with the vertical?

(c) Describe the subsequent motion of the rod.

13.7 (H) You have a canoe made of aluminum (density 2700 kg/m^3) designed so that when empty it floats in water with its sides extending 20 cm above water and 30 cm below water. If it were possible to take the floating canoe and gradually increase the value of g from 9.8 m/s^2 to an infinitely large value, what would happen to the canoe as g increased, and why? Assume water and aluminum are both incompressible, i.e. the increase in g does not affect the density of either substance. [Note for science-fiction buffs: this problem was inspired by an episode in Hal Clement's classic novel ***Mission of Gravity***.]

13.8 (H) A mass M is initially stationary on a horizontal frictionless air table. Firmly attached to the mass is a tube of gas (of negligible mass) sealed with a cork of mass m as shown. The height of the cork above the table is h and it is initially located a distance L from the edge of the block.



The experimenter now heats the gas inside the tube. The inner edges of the tube have some static friction, such that the cork does not blow off until the pressure inside the tube is double the ambient pressure, but negligible kinetic friction. As the cork travels along the tube the pressure of the gas behind it decreases linearly as shown in the inset PV plot, reaching ambient pressure just as the cork exits the tube.

(a) With what speed is the block traveling immediately after the cork leaves the tube?

(b) Relative to its starting position, where is the block when the cork leaves the tube?

(c) Relative to ***its*** starting position, how far away does the cork land?

13.9 Hailstones falling with speed v at an angle θ to the vertical collide elastically with a vertical wall. If the density of the hail (in kg of hailstones per cubic meter of air) is ρ, calculate the pressure exerted on the wall.

13.10 Discuss, in as much detail as possible, the aerodynamics of a frisbee, particularly the application of Bernoulli's equation to the airflow over, under and around the frisbee in flight. You should assume that the airflow is essentially laminar and that the speeds involved are small enough that air behaves as an incompressible fluid. You should ***not*** assume that there is no friction between the air and the frisbee.

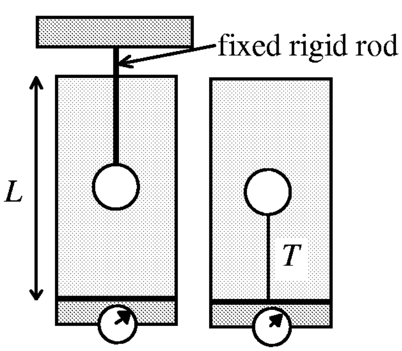
If you are right-handed, you normally throw a frisbee so that it spins clock-wise as seen from above. Compare the trajectory and spin of a frisbee thrown normally, with the same initial velocity, by a left-handed person. Explain.

13.11 (S) Mercury has a surface tension of 465 mN/m, a latent heat of vaporization of 272 kJ/kg, and a density of 13.6×10^3 kg/m^3. Using only these data, estimate the size and mass of an atom of mercury.

13.12 (H) If the pressure of water in the mains is 4.0×10^5 Pa, i.e. four times atmospheric pressure, what is (i) the speed, (ii) the volume flow rate, of water emerging from a fire hose with a nozzle diameter of 6.5 cm? Assume the diameter of the mains pipe is much larger than that of the fire hose, that the nozzle of the hose is 2 m above the level of the mains, and that viscosity and frictional effects can be neglected. What will happen to the speed and flow rate if the firefighter climbs 10 m up a ladder to fight a fire on the third floor, so that the nozzle is now 12 m above mains level? What is the maximum height to which any water can be delivered; does it matter if the firefighter climbs a ladder with the hose, or simply stands on the ground and directs the stream of water upwards?

> A little experimentation with a garden hose will convince you that the results you obtain in doing this problem are wrong—the experimental results don't agree with theoretical predictions. Since Bernoulli's equation follows directly from energy conservation, it must surely be correct (energy conservation is probably the single most trusted axiom in physics): we conclude that one of the assumptions we used in applying Bernoulli's equation to this problem must be unjustified. The trouble turns out to be that viscosity (a dissipative force) is ***not*** negligible in this situation: in fact it is the dominant effect. When viscous forces are important internal energy increases in the fluid and Bernoulli's equation is not valid. We should have used Poiseuille's law (which is beyond the scope of this book) to calculate the flow rate.

13.13 A cylindrical container of length L is full to the brim with a liquid which has mass density ρ. It is placed on a weigh-scale (which measures the downward force on the pan of the scale), and the scale reading is W. A light ball (which would float on the liquid if allowed to do so) of volume V and mass m is pushed gently down and held beneath the surface of the liquid with a rigid rod of negligible volume, as shown.



(a) What is the mass M of liquid which overflowed while the ball was being pushed into the container?

(b) What is the reading on the scale when the ball is fully immersed?

(c) If instead of being pushed down by a rod the ball is held in place by a fine string attached to the bottom of the container, what is the tension T in the string?

(d) In part (c), what is the reading on the scale?

13.14 A child is playing with a motorized toy airplane of mass M. The plane is attached to a string of length ℓ and is currently flying in horizontal circles directly above the child's hand, such that the string makes an angle θ to the vertical. The engine supplies a power P, and the plane flies at constant speed v. The direction of its motion is counterclockwise as seen from above, and its wings remain horizontal as it flies (it does not bank). Neither the motor nor the aerodynamic forces (the forces exerted on the plane by the air around it) contribute any net horizontal force perpendicular to the plane's direction of motion.

(a) Why is the engine necessary?

(b) Draw a force diagram for the plane, explaining the origin of all your forces.

(c) What is the tension in the string?

(d) What is the angular velocity vector of the plane?

(e) What is the angular momentum vector of the plane about the child's hand (considered as a stationary point anchoring the string)?

(f) About the same point, what torque acts on the plane?

(g) What force is responsible for this torque?

(h) An airliner in a holding pattern would bank as it flew in horizontal circles. Why?

13.15 (H) A level conveyor belt moves with constant speed u. At time $t = 0$, a bowling ball is placed on the conveyor belt. The bowling ball is a uniform sphere of mass M and radius R, and at $t = 0$ it has zero linear and angular velocity, $v(0) = 0$ and $\omega(0) = 0$. The coefficient of kinetic friction between the ball and the conveyor is μ_k, and the coefficient of static friction is μ_s, with $\mu_k < \mu_s$.

(a) At $t = 0$, what is the total force $\vec{\mathbf{F}}$ acting on the ball, and what is the total torque acting on the ball about its center? Assume that the conveyor is moving in the positive x direction, and that the y-axis points vertically upward.

(b) The ball will slip for some period after it is placed on the conveyor belt. What is its angular velocity $\omega(t)$ during this period of slipping?

(c) After a time t_1, the bowling ball stops slipping. Determine t_1. What is the (linear) velocity of the center of the ball at this point?

(d) What is the total work done by the force of friction on the bowling ball between $t = 0$ and $t = t_1$?

COMPLETE SOLUTIONS TO PROBLEMS WITH AN (S)

13.3 *A uniform cuboidal block of wood, of dimensions $a \times b \times c$ where $a \gg b \gg c$, is dropped into a lake. In which orientation will it float, and why?*

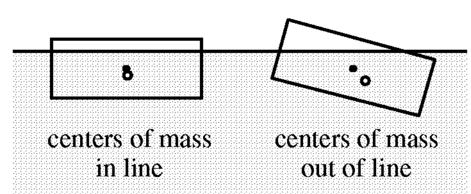
Conceptualize

In order for the wood to be stable, the following conditions must hold:

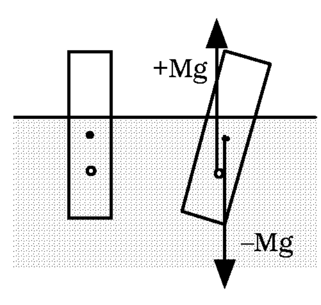
- there is no net ***force*** on the wood (this simply sets the depth at which it floats, i.e. the proportion of its volume not submerged);
- there is no net ***torque*** on the wood;
- if the wood is tilted slightly from its present position, the torque that results tends to ***reduce*** the amount of tilt.

The first two conditions produce equilibrium, while the third ensures that the equilibrium is stable.

The second condition implies that the center of mass of the wood (which is at the geometrical center of the block, since the wood is uniform) lies on the same vertical line as the center of mass of the displaced water. (Remember that the buoyant force acts through the same point as the weight of the water, ***not*** the block.) We can see from the symmetry of the situation that this means the block must float with one of its sides parallel to the water surface. (If this is not clear, look at the diagram, where the black blob represents the block's center of mass and the open circle the center of mass of the displaced water. When the block is tilted, more water is displaced on the side where it is lower, so the center of mass of the water is off to one side.)

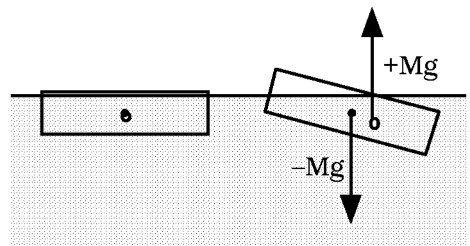


To deal with the stability question in the most general case requires some slightly tedious geometry to calculate the position of the center of mass of the displaced water. However, if the sides of the block are very different in length, we can see what happens diagrammatically. If the block floats with a long side vertical, the submerged volume is dominated by a region of rectangular cross-section whose center of mass is clearly to the ***left*** (as drawn) of the block's center of mass. The remaining small submerged region, of triangular cross-section, has its center of mass to the right of the block's, but its mass is much smaller. The overall torque is clockwise, increasing the tilt: this position is not stable.



13.3, continued:

If, on the other hand, the block floats with a short side vertical, the center of mass of the exposed triangular section is clearly to the left of the block center of mass, and so the center of mass of the displaced water must be to the right. (If the block were very light, we would have a submerged triangle, whose center of mass would likewise be to the right of that of the block.) In either case, the resulting torque reduces the tilt, and the position is stable.



Solve

We therefore conclude that our block will float so that the longest sides are parallel to the water surface. For a more symmetrical block the situation would be less clear, and we would have to do a full calculation of the water's center of mass to determine if a position with long side vertical was stable against ***small*** displacements. (For a large angle of tilt, a block with unequal sides will always settle with the shortest sides vertical, but the other positions could still be stable against small tilts. As an analogy, consider a cereal box: in its normal upright orientation, it is stable against small nudges, but if you drop it from a height it will come to rest on its side.)

Learn

The question of stability is an important one for boats, particularly large cargo vessels. Since a ship is continuously being subjected to tilting forces from wave and wind action, it is vital that the load distribution in the vessel be adjusted to ensure stability. A particular problem is cargo shifting: if the ship develops a sideways list and the cargo is inadequately secured, it will tend to slide towards the low side of the ship, moving the ship's center of mass that way and increasing the probability that the net torque will be in the wrong direction. For this reason cargo holds are subdivided by bulkheads, and ships carry ballast to lower their center of mass. The problem is especially acute for car ferries, which tend to have large unobstructed car decks full of unsecured cars.

13.4 *A quantity of liquid of density ρ is contained in a U shaped tube of constant cross-section A. Initially one end of the tube is closed and the liquid in that end stands higher by an amount $2h$ than the liquid in the other end. The closed end of the tube is now opened so that the pressures at both ends are equal. What is the subsequent motion of the liquid in the tube?*

Conceptualize

There are two possible approaches to this problem:

- When the tube is opened, the surface pressures are equalized. Since the height of the liquid is greater in the previously closed arm, the pressure at the base of this arm will be greater than that at the base of the other arm, by Pascal's law, and therefore there will be a net force. We can use this to set up the equations of motion using Newton's laws.

13.4, continued:

- When the tube is opened, the 'extra' liquid in the previously sealed arm clearly has positive gravitational potential energy compared to the equilibrium position where both columns of liquid have equal height. We can differentiate this to find the force.

The energy approach seems cleaner, because we don't have to worry about the exact shape of the base of the tube.

Formulate

When the tube is first unsealed, the level of liquid in the previously closed arm is $+h$, and the level in the other arm $-h$, compared to the equilibrium state where both levels are equal. The extra mass of liquid in the closed tube is $2Ah\rho$. Taking $-h$ as the reference level for the gravitational potential energy, it has

$$U(h) = (2Ah\rho)gh = 2A\rho gh^2,$$

since its center of mass is at $y = 0$ (where y is the vertical coordinate).

At equilibrium, this 'extra' liquid is distributed equally between the two arms. Its potential energy relative to $-h$ is now

$$U_0 = 2(Ah\rho)g(\tfrac{1}{2}h = A\rho gh^2,$$

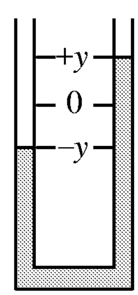
since each arm contributes a mass $Ah\rho$ of liquid whose center of mass is $h/2$ above the reference level.

The potential energy of the initial configuration compared to the equilibrium position is thus

$$U(h) - U_0 = A\rho gh^2.$$

Solve

To see how the motion develops, consider a time t when the right-hand column is at height $+y$ compared to the equilibrium position (where both columns have the same height) and the left-hand column is at height $-y$. The potential energy of the liquid at this point, relative to equilibrium, is $U(y) - U_0 = A\rho gy^2$. Differentiating, we have



$$F = M\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -2A\rho gy.$$

If H is the total length of liquid in the whole tube, the total mass $M = AH\rho$, and so we have the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -\frac{2g}{H}y.$$

13.4, continued:

This is the equation for simple harmonic motion, which we first met in Chapter 2. Our boundary conditions in this case are that the top of the right-hand column was at coordinate $+h$ at time $t = 0$, so the solution is

$$y = h \cos \sqrt{\frac{2g}{H}}\, t$$

(and the top of the left-hand column is always at position $-y$, since the total length is constant). This assumes that our liquid has zero viscosity; in practice, friction between the liquid and the tube would gradually reduce the amplitude of the oscillations.

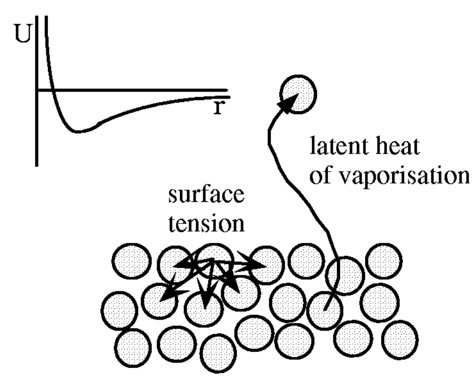
Scrutinize

$\sqrt{2g/H}$ clearly has the appropriate dimensions of $1/[\text{time}]$, and the whole solution is highly analogous to the simple pendulum, although in this case there is no requirement that h be small. By considering the net force at the lowest point in the tube, we can check the solution using the Pascal's law approach: the pressure difference between the bases of the right and left columns is $\rho g(2y)$, so the net force acting on mass $M = \rho AH$ is $-2\rho gyA$ (with a minus sign because the force acts to reduce y), giving an acceleration of $-2gy/H$ as above. The advantage of the energy approach is that we do not have to worry about the possible effects of contact forces where the tube bends.

13.11 *Mercury has a surface tension of* 465 mN/m*, a latent heat of vaporization of* 272 kJ/kg*, and a density of* 13.6×10^3 kg/m^3*. Using only these data, estimate the size and mass of an atom of mercury.*

Conceptualize

To do this problem, we need to find some relationship between the surface tension—the force which acts to minimize surface area—and the latent heat of vaporization—the energy required to transform unit mass of the substance from the liquid phase to the gaseous phase. To see what this relationship might be, we have to consider the origins of surface tension and the latent heat.



Surface tension arises because the constituent atoms or molecules of a liquid interact with one another via intermolecular forces: a schematic diagram of the relevant potential energy is shown in the diagram. An atom at the surface is in an asymmetric position, and a net force will act to pull it in towards the body of the liquid: it has a higher potential energy than an atom in the interior of the liquid.

The latent heat of vaporization is the energy one has to supply to move all the atoms or molecules away from the minimum of the intermolecular potential energy, out to $r \approx \infty$. Both surface tension and latent heat are therefore related to the intermolecular potential energy. If we express γ as energy divided by area,

$$\gamma = U/A \ ,$$

13.11, continued:

and the latent heat of vaporization as energy divided by mass,

$$L_v = U/M \ ,$$

we might reasonably hope that in each case U is essentially the average intermolecular potential energy.

Formulate

To quantify this argument, consider a spherical particle with diameter d. Its surface area is πd^2 and its mass is $\frac{1}{6}\pi d^3 \rho$, where ρ is its density. If we substitute these for A and M in the above equations, we have

$$U = \gamma A = \gamma \pi d^2$$

$$U = L_v M = \frac{1}{6} L_v \pi d^3 \rho$$

which gives

$$d = \frac{6\gamma}{\rho L_v} .$$

Solve

For mercury, we conclude that

$$d = \frac{6\gamma}{\rho L_v} = \frac{6 \times (0.465 \text{ N/m})}{(13.6 \times 10^3 \text{ kg/m}^3) \times (2.72 \times 10^5 \text{ J/kg})} = 7.5 \times 10^{-10} \text{ m} .$$

For the atomic mass, we use $M = \frac{1}{6}\pi \rho d^3$, giving

$$M = \left(\frac{\pi}{6}\right) \times (13.6 \times 10^3 \text{ kg/m}^3) \times (7.5 \times 10^{-10} \text{ m})^3 = 3.1 \times 10^{-24} \text{kg} .$$

Scrutinize

We can see from dimensional arguments that if there *is* any way to obtain the size of an atom from surface tension, latent heat and density, the only suitable combination is $\gamma/\rho L_v$, since it is the only combination with dimensions of length. The numerical factor cannot be deduced dimensionally, and in fact slightly different ways of formulating the relationship between L_v and γ give different numerical coefficients in d, which can change our estimate for m by up to a factor of 10. From modern tables, the atomic mass of mercury is 200.6 u $= 3.33 \times 10^{-25}$ kg. Our estimate is within a factor of 10 of this, which may seem unimpressive,but recall that we are using numbers derived from macroscopic measurements involving grams or kilograms of fluid to derive properties of single molecules!

Learn

This method of estimating molecular sizes was first used by Waterston in 1858. The accuracy of our results for mercury is fairly typical (for instance, using textbook data

13.11, continued:

for water gives the molecular mass within a factor of 5, and for ethanol within a factor of about 10). The most interesting feature of this technique is that the phenomena that provide our numbers (the force exerted by the surface of the liquid, and the heat required to vaporize it) are apparently unrelated until we interpret them in terms of the molecular model. Under the circumstances, the fact that our estimates are anywhere near the right value is remarkable, and provides a strong indication that our model of the molecular structure of a liquid is reasonable.

There is a somewhat more direct method of estimating molecular sizes using macroscopic measurements, which can be most easily applied to oily fluids which float on water. If a small volume V of such a substance is poured onto the surface of a large pool of water, it will form a roughly circular slick of radius r. Assuming that the slick is one molecule thick, the length d of one molecule is simply $d = V/(\pi r^2)$.

HINTS FOR PROBLEMS WITH AN (H)

The number of the hint refers to the number of the problem

13.1 If you imagine a substance in which each molecule lies at the center of a small cube of side d, what is the average distance between molecules, and how many molecules are there in a cube of side 1 meter? How are these two quantities related?

13.2 If the pressure at the surface of the water is increased, what happens to the pressure on the air in the bubble? What then happens to the volume of the bubble?

13.7 What is the condition for the canoe to float stably? Does this depend on g?

What about the net pressure on the sides of the canoe?

13.8 How much work is done by the gas in expanding?

What happens to the center of mass of the block-cork system?

13.12 Is the speed of the water the same if it is contained within a hose? Is the volume flow rate the same?

13.15 (c) What is the condition for the ball to roll without slipping?

(d) What is the total kinetic energy of the ball at time t_1?

ANSWERS TO HINTS

13.1 d; $1/d^3$; spacing is cube root of one over number density.

13.2 Pressure increases; volume decreases.

13.7 $M = \rho V$, mass of canoe equals density of water times volume displaced; no (except insofar as increased gravity may tend to slightly increase the mass of air within the canoe).

Net pressure $= \rho g d$, where d is depth of bottom of canoe below water surface; this does depend on g.

13.8 $\frac{1}{2}P(V_f - V_i)$; it remains stationary until cork leaves tube, then drops slightly (no external forces acting except gravity).

13.12 Yes; no.

13.15 (c) $R\omega(t) + v(t) = u$.

(d) $\frac{1}{2}Mv^2(t_1) + \frac{1}{2}I\omega^2(t_1) = \frac{1}{2}Mv^2(t_1) + \frac{1}{5}MR^2\omega^2(t_1)$.

ANSWERS TO ALL PROBLEMS

13.1 3.35×10^{25}; $\approx 3 \times 10^{-10}$ m;

2.4×10^{22}; $\approx 3.5 \times 10^{-9}$ m.

In liquid water the molecules are almost in contact, so surely the inter-molecular forces cannot be neglected; in an ideal gas they are separated by perhaps ten times their size, so there is some reason to expect that forces between them are small (note: this suggests that big molecules containing ten or more atoms are unlikely to behave like ideal gases under typical conditions of pressure and temperature).

13.2 The diver sinks, because the increased pressure compresses the air bubble inside it, reducing the volume it occupies and thus increasing the overall density of the diver.

13.3 See complete solution.

13.4 See complete solution.

13.5 $v = \sqrt{2gd}$; $x = 2\sqrt{d(h-d)}$; $d = h/2$.

13.6 (a) $\omega = \dfrac{3m(v_i - v_f)}{M\ell}$

(b) $\sqrt{\dfrac{6m^2(v_i - v_f)^2}{M^2 g\ell}}$

(c) It will oscillate back and forth in simple harmonic motion.

13.7 The canoe will continue to float undisturbed until some critical point, at which it will suddenly collapse. The reason is that the relative densities of canoe and water don't change (except for a slight effect caused by increase in air density), but the net pressure on the sides of the canoe increases, as the water pressure is proportional to g. Eventually the sides of the canoe will begin to bend inwards: this reduces the volume of the canoe, increasing its relative density and causing it to sink slightly, which in turn further increases the net pressure. The result is a sudden catastrophic collapse.

13.8 (a) $\sqrt{\dfrac{2K}{M + M^2/m}}$, where $K = \frac{1}{2}P(V_f - V_i)$.

(b) $\dfrac{L}{1 + \frac{M}{m}}$

(c) $\dfrac{L}{1 + \frac{m}{M}} + 2\sqrt{\dfrac{hK}{gm(1 + \frac{m}{M})}}$.

13.9 $2\rho v^2 \sin^2\theta$.

13.10 The path of the air deflected over the frisbee is longer than that of the undisturbed air passing underneath, so the air over the frisbee has a higher speed, and thus (by Bernoulli) a lower pressure. Hence there is a net upward force, so the frisbee will remain airborne longer than it would in the absence of this effect. Friction between the frisbee and the surrounding

13.10, continued:

air tends to drag the streamlines around the frisbee in the direction of its spin, producing a difference in pressure between the two sides of the frisbee and hence causing its path to curve sideways. The spin also provides stability due to angular momentum effects (Chapter 9).

Your frisbee tends to curve to the right as you look at it, your friend's curves left.

13.11 See complete solution.

13.12 24 m/s; 79 liters/s; speed reduces to 19 m/s, volume flow to 63 liters/s; 31 m above the level of the mains; yes: the maximum height and the speed of the water at any height will be the same in both cases, but the volume flow rate is higher if the fireman stands on the ground. This is because the equation of continuity forces the volume flow rate to be lower if the water is confined within the hose, whereas once it leaves the hose it can spread out over a wider area as it slows down.

13.13 (a) $V\rho$; (b) W; (c) $(M-m)g$; (d) $W - Mg + mg$.

13.14 (a) The motor supplies the forward force needed to counteract the backward force due to air drag.

(b) Forward force F_E from engine (more specifically, the engine supplies a torque which turns the propellor, and the motion of the propellor creates aerodynamic forces which act on the plane). Backward force F_D from air drag, related to the velocity of the plane relative to the air around it (probably of form $-kv^2\hat{\boldsymbol{v}}$). Mg is weight of plane, T is tension in string, and L is lift, created by the Bernoulli effect acting mainly on the plane's wings, as in Problem 12C.6: the wing cross-section is shaped so that the air traveling over the top has a longer path, and must therefore move more quickly, than the air beneath. (It would be reasonable to draw two lift forces, one on each wing, but there is no way to separate out the individual contributions in this problem.)

(c) $T = \dfrac{Mv^2}{\ell \sin\theta}$.

(d) $\vec{\boldsymbol{\omega}} = \left[0, 0, \dfrac{v}{\ell \sin\theta}\right]$ in a coordinate system where z points directly upwards.

(e) $\vec{\mathbf{L}} = M\ell v[-\cos\theta\cos\omega t, -\cos\theta\sin\omega t, \sin\theta]$, where $\omega = |\vec{\boldsymbol{\omega}}|$.

(f) $\vec{\boldsymbol{\tau}} = \dfrac{Mv^2}{\sin\theta}[\cos\theta\sin\omega t, -\cos\theta\cos\omega t, 0]$.

(g) The lift (more precisely, the difference between lift and weight).

(h) The lift acts perpendicular to the wing surfaces, so by banking the plane, the pilot can arrange to give the lift an inward component which supplies the centripetal force needed to maintain circular motion.

13.15 (a) $\vec{\mathbf{F}} = [\mu_k Mg, 0, 0]$; $\vec{\boldsymbol{\tau}} = [0, 0, \mu_k MgR]$. (As the axis of rotation of the bowling ball has a fixed orientation, always parallel to the z-axis, we will use the 'scalar' definitions of angular quantities in answers to the remainder of this problem. With the given coordinate system, these correspond to the z-components of the vector equivalents; the x- and y- components are zero.)

(b) $\omega(t) = \dfrac{5\tau t}{2MR^2} = \dfrac{5\mu_k g}{2R} t.$

(c) $t_1 = \dfrac{2u}{7\mu_k g}$; $v = \frac{2}{7}u$ in the positive x direction.

(d) $W = \frac{1}{7}Mu^2$.

(This should be evaluated using the work-energy theorem. Although it is also possible to use force times distance, one has to be particularly careful, for two reasons: first, since the ball is turning as it skids, the point on the surface of the ball to which the force is applied is constantly changing; secondly, the part of the ball which is in contact with the conveyor belt does not have the same velocity as the center of the ball. One must use a calculus-based approach: in an infinitesimal time Δt, the work done $\Delta W = F\Delta x$, where Δx is the displacement of the atoms to which the force is applied. The power $P = \Delta W/\Delta t = Fv_c$, where $v_c = \Delta x/\Delta t$ is the velocity of the atoms at the contact point, $v_c(t) = v(t) + R\omega(t)$. One can then integrate $P\,dt$ from $t = 0$ to $t = t_1$ to find the work done. Applying the work-energy theorem is much more straightforward—another example of the advantages of conservation-law approaches to problem solving.)