Kinematics and One-Dimensional Motion: Non-Constant Acceleration

8.01
W01D3

Announcements

Familiarize Yourself with Website
https://lms.mitx.mit.edu/courses/MITx/8.01/2014_Fall/about

Buy or Download Textbook (Dourmashkin, Classical Mechanics:
MIT 8.01 Course Notes Revised Edition)

Downloadable links (require certificates):
https://lms.mitx.mit.edu/courses/MITx/8.01/2014_Fall/courseware/Intro/about:Resources/

Buy Clicker at MIT Coop

Sunday Tutoring in 26-152 from 1-5 pm

Average Velocity

The average velocity, $\vec{\mathbf{v}}_{ave}(t)$, is the displacement $\Delta\vec{\mathbf{r}}$ divided by the time interval Δt

$$\vec{\mathbf{v}}_{ave} \equiv \frac{\Delta\vec{\mathbf{r}}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{\mathbf{i}} = v_{ave,x}(t)\hat{\mathbf{i}}$$

The x-component of the average velocity is given by

$$v_{ave,x} = \frac{\Delta x}{\Delta t}$$

Instantaneous Velocity and Differentiation

For each time interval Δt, calculate the *x*-component of the average velocity

$$v_{ave,x}(t) = \Delta x / \Delta t$$

Take limit as $\Delta t \rightarrow 0$ sequence of the *x*-component average velocities

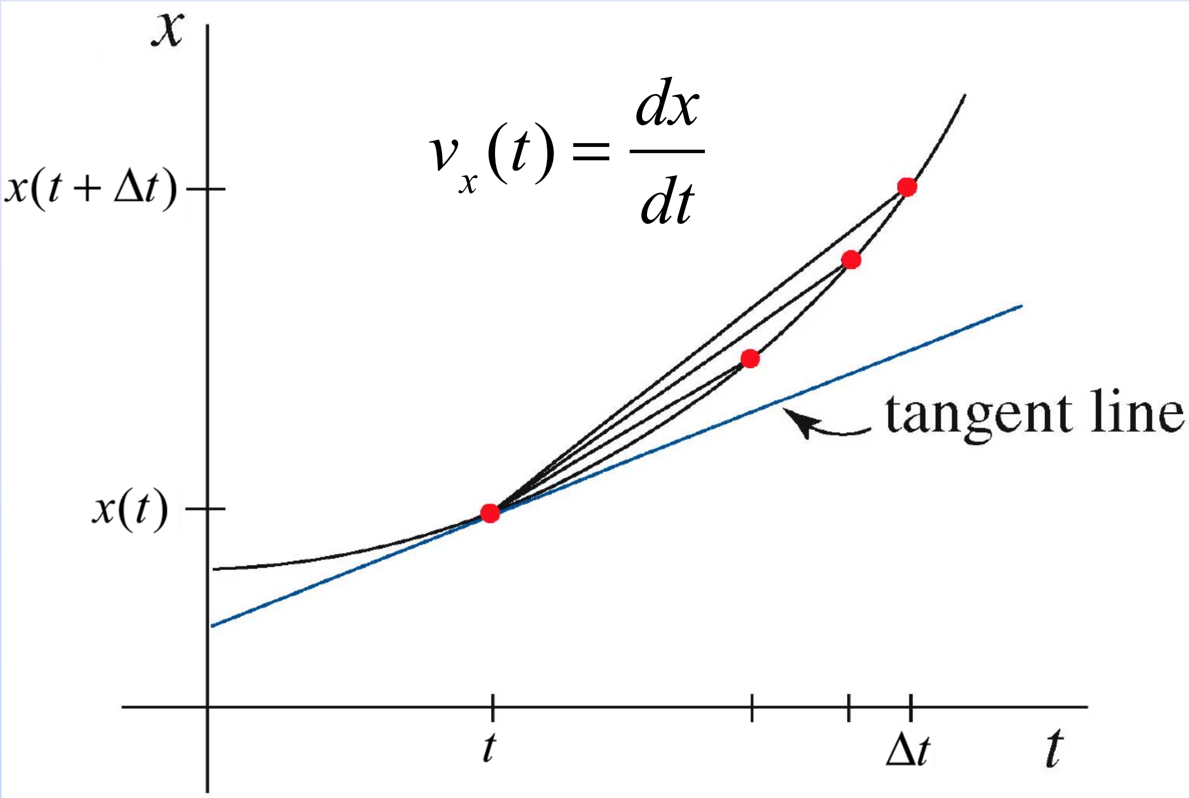
$$\lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} \equiv \frac{dx}{dt}$$

The limiting value of this sequence is *x*-component of the instantaneous velocity at the time t.

$$v_x(t) = dx / dt$$

Instantaneous Velocity

x-component of the velocity is equal to the slope of the tangent line of the graph of *x*-component of position vs. time at time *t*



Worked Example: Differentiation

$$x(t) = At^2$$

$$x(t+\Delta t) = A(t+\Delta t)^2 = At^2 + 2At\Delta t + A\Delta t^2$$

$$\frac{x(t+\Delta t) - x(t)}{\Delta t} = 2At + A\Delta t$$

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0} (2At + A\Delta t) = 2At$$

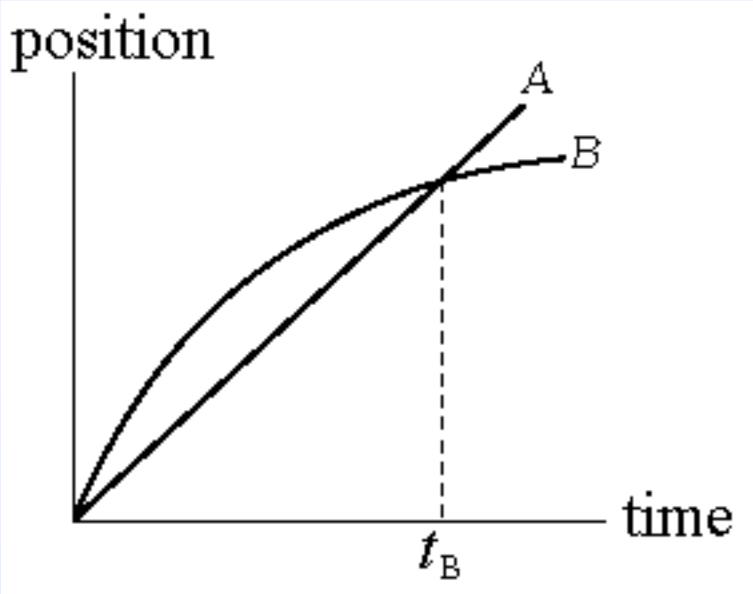
Generalization for Polynomials:

$$x(t) = At^n$$

$$\frac{dx}{dt} = nAt^{n-1}$$

Concept Question: Instantaneous Velocity

The graph shows the position as a function of time for two trains running on parallel tracks. For times greater than t = 0, which of the following is true:



1. At time t_B, both trains have the same velocity.
2. Both trains speed up all the time.
3. Both trains have the same velocity at some time before t_B, .
4. Somewhere on the graph, both trains have the same acceleration.

Average Acceleration

Change in instantaneous velocity divided by the time interval $\Delta t = t_2 - t_1$

$$\vec{\mathbf{a}}_{ave} \equiv \frac{\Delta \vec{\mathbf{v}}}{\Delta t} = \frac{\Delta v_x}{\Delta t}\hat{\mathbf{i}} = \frac{(v_{x,2} - v_{x,1})}{\Delta t}\hat{\mathbf{i}} = \frac{\Delta v_x}{\Delta t}\hat{\mathbf{i}} = a_{ave,x}\hat{\mathbf{i}}$$

The *x*-component of the average acceleration

$$a_{ave,x} = \frac{\Delta v_x}{\Delta t}$$

Instantaneous Acceleration and Differentiation

For each time interval Δt, calculate the *x*-component of the average acceleration

$$a_{ave,x}(t) = \Delta v_x / \Delta t$$

Take limit as $\Delta t \rightarrow 0$ sequence of the *x*-component average accelerations

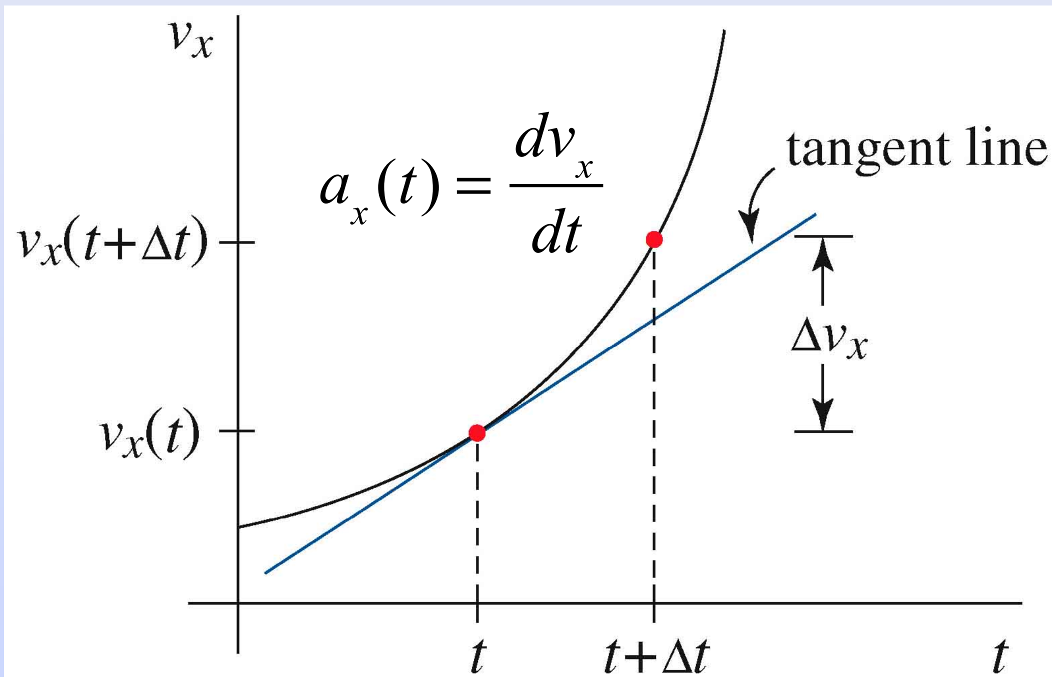
$$\lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \lim_{\Delta t \to 0} \frac{v_x(t + \Delta t) - v_x(t)}{\Delta t} \equiv \frac{dv_x}{dt}$$

The limiting value of this sequence is *x*-component of the instantaneous acceleration at the time t.

$$a_x(t) = dv_x / dt$$

Instantaneous Acceleration

The x-component of acceleration is equal to the slope of the tangent line of the graph of the x-component of the velocity vs. time at time t



Group Problem: Model Rocket

A person launches a home-built model rocket straight up into the air at $y = 0$ from rest at time $t = 0$. (The positive y-direction is upwards). The fuel burns out at $t = t_0$. The position of the rocket is given by

$$y = \begin{cases} \frac{1}{2}(a_0 - g)t^2 - \frac{a_0}{30}t^6 / t_0{}^4; & 0 < t < t_0 \end{cases}$$

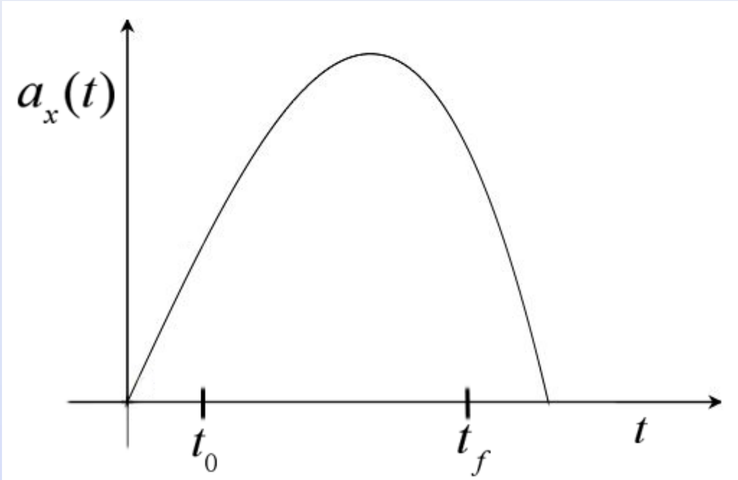
with a_0 and g are positive. Find the y-components of the velocity and acceleration of the rocket as a function of time. Graph a_y vs t for $0 < t < t_0$.

Non-Constant Acceleration and Integration

Change in Velocity: Integral of Acceleration

Consider some time *t* such that

$$t_0 < t < t_f$$



Then the change in the *x*-component of the velocity is the integral of the *x*-component acceleration (denote $v_{x,0} \equiv v_x(t_0)$).

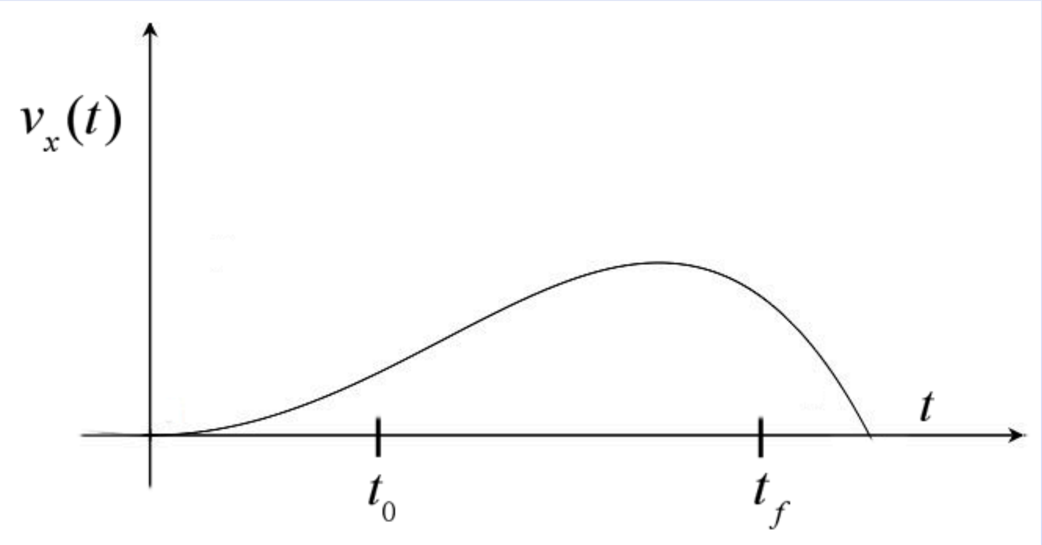
$$v_x(t) - v_{x,0} = \int_{t'=t_0}^{t'=t} a_x(t')\,dt'$$

"Integration is the inverse operation of differentiation"

Change in Position: Integral of Velocity

Area under the graph of *x*-component of the velocity vs. time is the displacement (denote $x_0 \equiv x(t_0)$).

$$x(t) - x_0 = \int_{t'=t_0}^{t'=t} v_x(t')\,dt'$$



Worked Example: Time-Dependent Acceleration

Acceleration is a non-constant function of time $a_x(t) = At^2$

with $t_0 = 0$, $v_{x,0} = 0$, and $x_0 = 0$.

Change in velocity:

$$v_x(t) - 0 = \int_{t'=0}^{t'=t} At'^2\, dt' = A\frac{t'^3}{3}\bigg|_{t'=0}^{t'=t} \Rightarrow v_x(t) = \frac{At^3}{3}$$

Change in position:

$$x(t) - 0 = \int_{t'=0}^{t'=t} A\left(\frac{t'^3}{3}\right) dt' \Rightarrow x(t) = \left(A\left(\frac{t'^4}{12}\right)\right)\bigg|_0^{t'=t} = \frac{At^4}{12}$$

Generalization for Polynomials:

$$\int_{t'=t_0}^{t'=t} t'^n\, dt' = \frac{t'^{n+1}}{n+1}\bigg|_{t'=t_0}^{t'=t} = \frac{t^{n+1}}{n+1} - \frac{t_0^{\;n+1}}{n+1}$$

Special Case: Constant Acceleration

Acceleration: $a_x = constant$

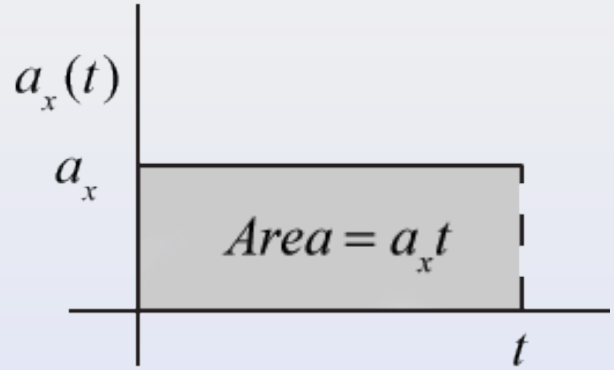
Velocity: $$v_x(t) - v_{x,0} = \int_{t'=0}^{t'=t} a_x\, dt' = a_x t \Rightarrow$$

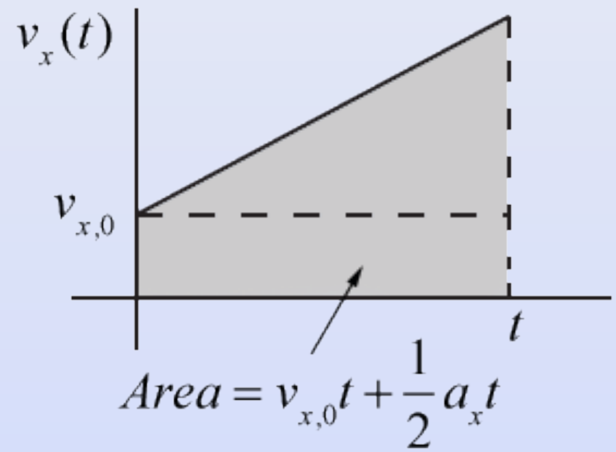
$$v_x(t) = v_{x,0} + a_x t$$

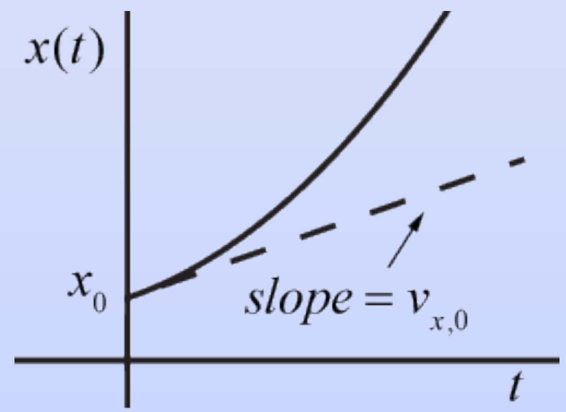
Position:

$$x(t) - x_0 = \int_{t'=t_0}^{t'=t} (v_{x,0} + a_x t')\, dt' \Rightarrow$$

$$x(t) = x_0 + v_{x,0} t + \frac{1}{2} a_x t^2$$







Concept Question: Integration

A particle, starting at rest at $t = 0$, experiences a non-constant acceleration $a_x(t)$. It’s change of position can be found by

1. Differentiating $a_x(t)$ twice.
2. Integrating $a_x(t)$ twice.
3. (1/2) $a_x(t)$ times t^2.
4. None of the above.
5. Two of the above.

Group Problem: Sports Car

At $t = 0$, a sports car starting at rest at $x = 0$ accelerates with an x-component of acceleration given by

$$a_x(t) = At - Bt^3, \text{for } 0 < t < (A/B)^{1/2}$$

and zero afterwards with $A, B > 0$

(1) Find expressions for the velocity and position vectors of the sports car as functions of time for $t > 0$.

(2) Sketch graphs of the x-component of the position, velocity and acceleration of the sports car as a function of time for $t > 0$

Appendix:
Integration and the Riemann Sum

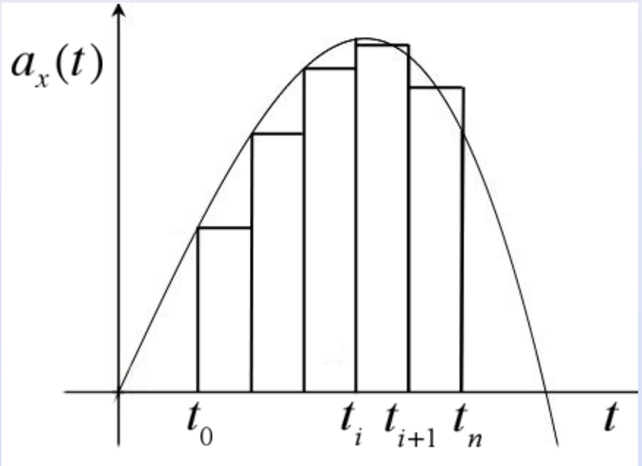
Change in Velocity: Area Under Curve of Acceleration vs. time

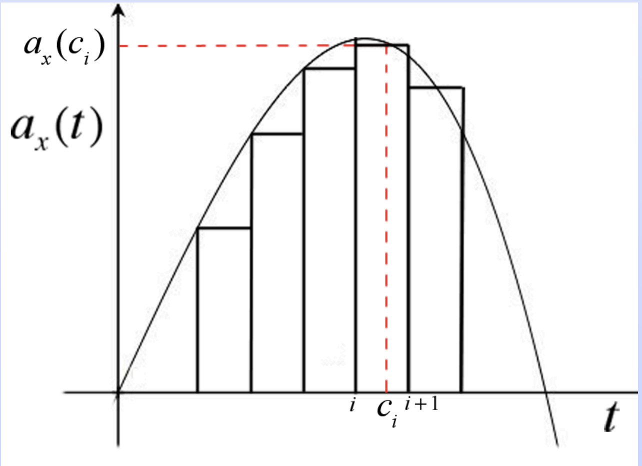
Mean Value Theorem: For each rectangle there exists a time

$$t_i < t = c_i < t_{i+1}$$

such that

$$v_x(t_{i+1}) - v_x(t_i) = \frac{dv_x}{dt}(c_i)\Delta t = a_x(c_i)\Delta t$$





Apply Mean Value Theorem

$$v_x(t_1) - v_x(t_0) = a_x(c_0)\Delta t$$

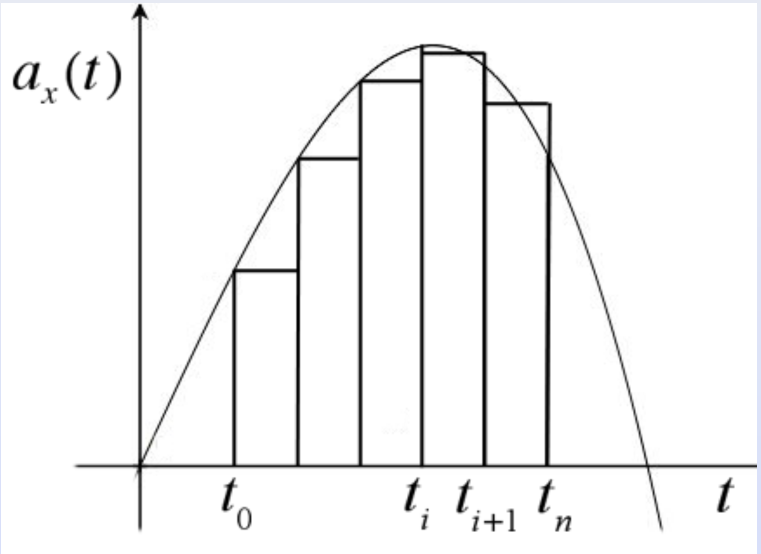
$$(v_x(t_2) - v_x(t_1)) = a_x(c_1)\Delta t$$

$$\cdots = \cdots$$

$$v_x(t_{i+1}) - v_x(t_i) = a_x(c_i)\Delta t$$

$$\cdots = \cdots$$

$$v_x(t_n) - v_x(t_{n-1}) = a_x(c_{n-1})\Delta t$$



We can add up the area of the rectangles and find

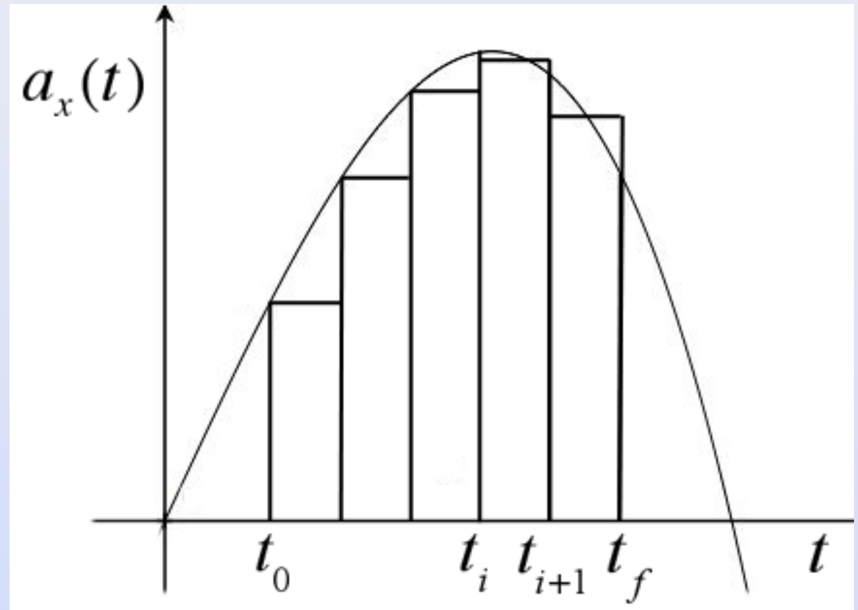
$$v_x(t_n) - v_x(t_0) = \sum_{i=0}^{i=n-1} ((a_x(c_i)\Delta t)$$

Change in Velocity: Integral of Acceleration

The area under the graph of the *x*-component of the acceleration vs. time is the change in velocity

$$v_x(t_f) - v_x(t_0) = \lim_{\Delta t_i \to 0} \sum_{i=1}^{i=N} a_x(t_i)\Delta t_i$$

$$v_x(t_f) - v_x(t_0) \equiv \int_{t=t_0}^{t=t_f} a_x(t)\,dt$$



http://giphy.com/gifs/math-mathematics-calculus-zTGUIIASZx83u