Central Force Motion

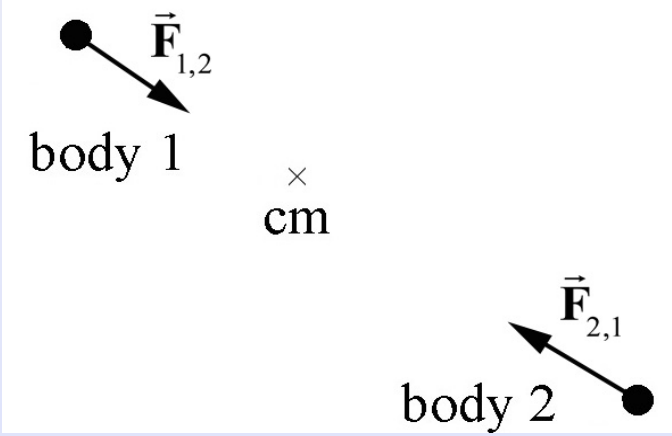
8.01

W14D1

Central Force Problem

Find the motion of two bodies interacting via a central force.

Examples:



Gravitational force (Kepler problem):

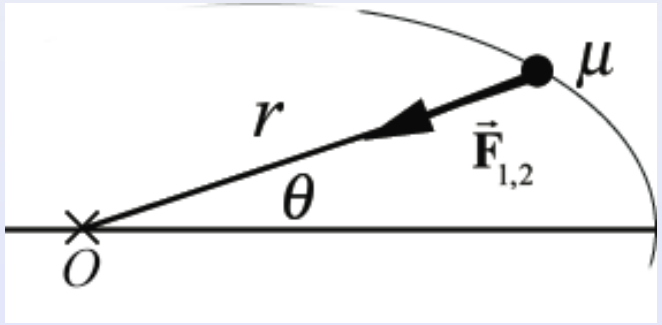
$$\vec{\mathbf{F}}_{1,2}(r) = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$$

Linear restoring force:

$$\vec{\mathbf{F}}_{1,2}(r) = -kr\,\hat{\mathbf{r}}$$

Reduction of Two Body Problem to One Body Problem

Reduce two body problem to one body of reduced mass μ moving about a central point *O* under the influence of the central force with ***position vector corresponding to the relative position vector*** from object 2 to object 1



Solving the problem means finding the distance from the origin $r(t)$ and angle $\theta(t)$ as functions of time

$$\vec{\mathbf{F}}_{1,2} = \mu \frac{d^2\vec{\mathbf{r}}}{dt^2}$$

Equivalently, finding $r(\theta)$ as a function of angle θ

Central Force Motion: Constants of the Motion

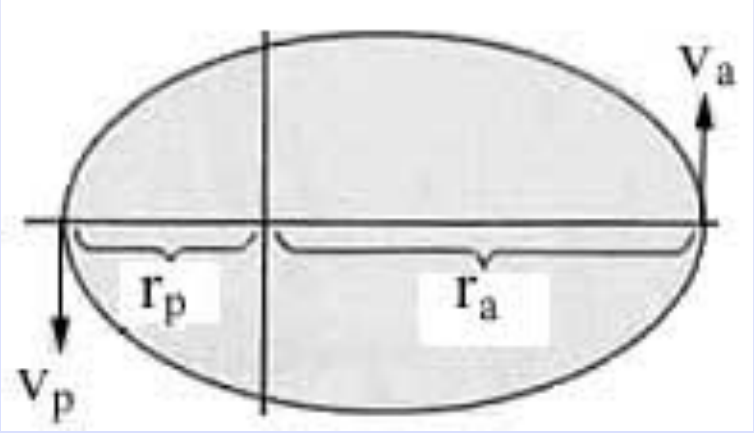
Total mechanical energy E is conserved because the force is radial and depends only on r and not on θ

Angular momentum L is constant because the torque about central point is zero

The force and the velocity vectors determine the plane of motion

Table Problem: Satellite in Elliptic Orbit

A satellite of mass m_s is in an elliptical orbit around a planet of mass m which is located at one focus of the ellipse. The satellite has a velocity v_a at the distance when it is furthest r_a from the planet. The distance of closest approach is r_p. What is the magnitude of the velocity v_p of the satellite when it is closest to the planet?



Equal Area Law and Conservation of Angular Momentum

Change in area

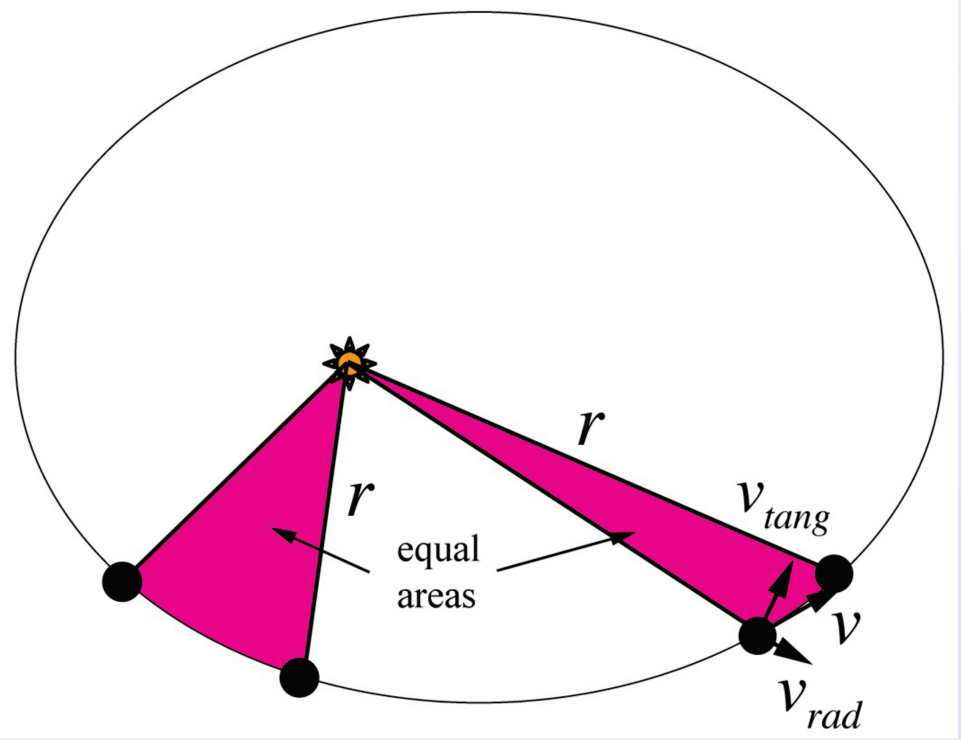
$$\Delta A = (1/2) r v_\theta \Delta t$$

per time

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} v_\theta r$$

Angular momentum

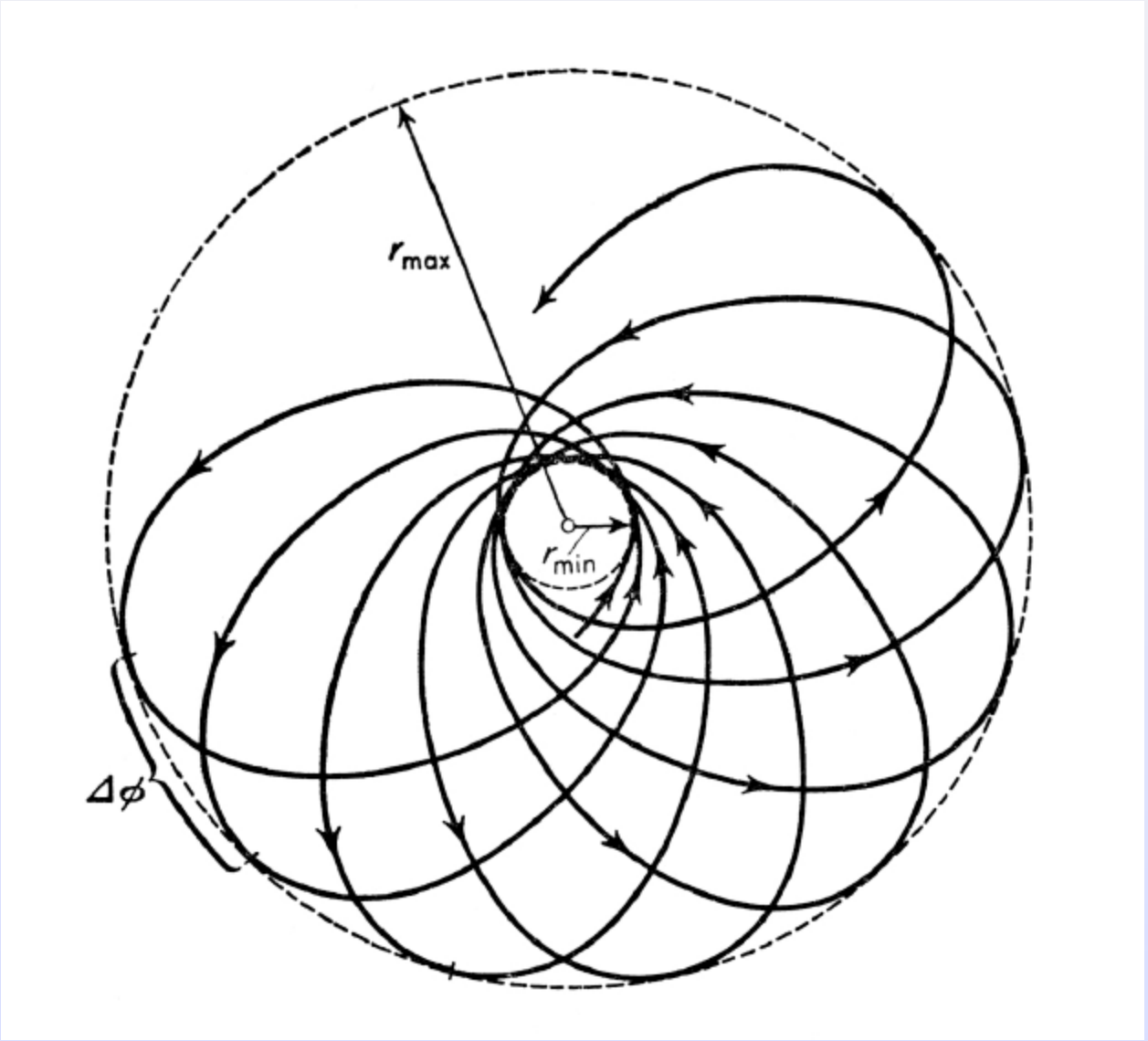
$$L = r \mu v_\theta$$



Equal area law

$$\frac{\Delta A}{\Delta t} = \frac{L}{2\mu}$$

For Most Force Laws the Most General Solution Does Not Involve Closed Orbits



Worked Example: Isotropic Harmonic Oscillator

A special solution of the equation of motion for a linear restoring force

$$\mu \frac{d^2\vec{\mathbf{r}}}{dt^2} = -kr\,\hat{\mathbf{r}}$$

is given by $\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}}$

with

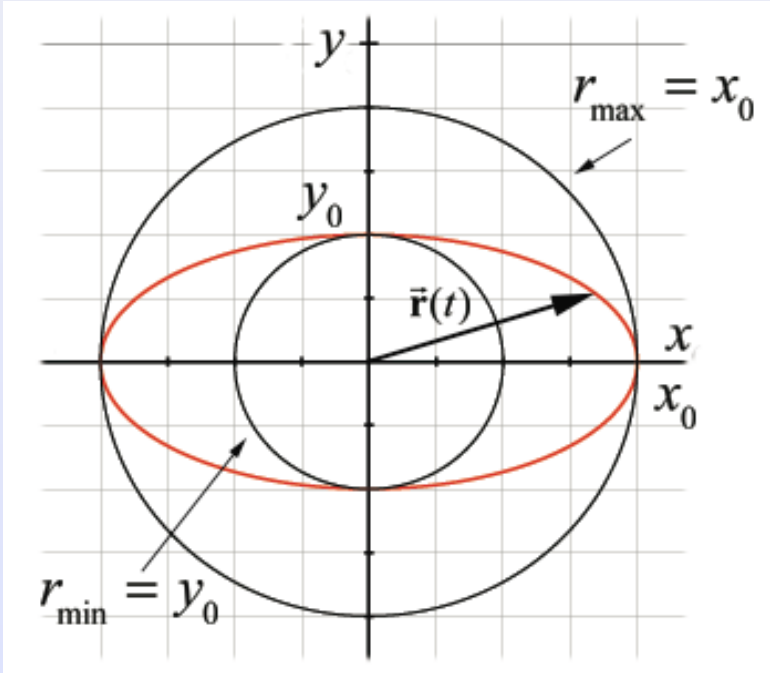
$$x(t) = x_0 \sin(\omega t)$$

$$y(t) = y_0 \cos(\omega t)$$

where for the case shown in the figure with

$$y_0 < x_0 \qquad r_{\min} = y_0 \qquad r_{\max} = x_0$$

The solution for $\vec{\mathbf{r}}(t)$ is an ellipse centered at the origin



Demo : Open Pendulum Orbits

A pendulum moves in a precessing orbit

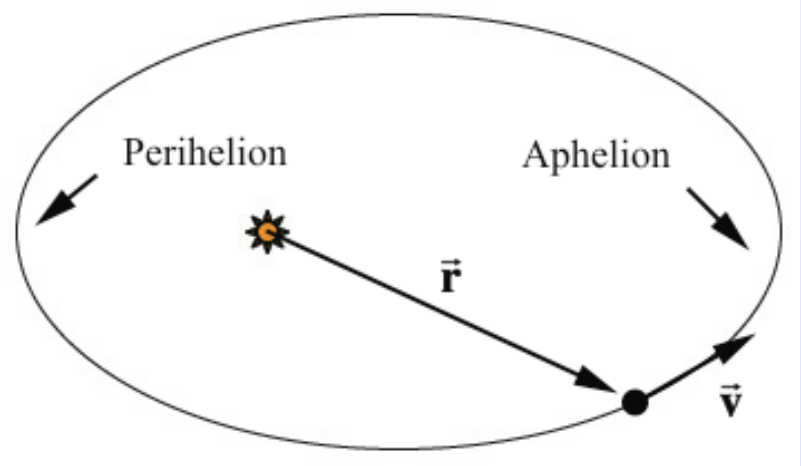
Concept Problem:

The orbit of the pendulum in the experiment precesses because

1) I pushed it in that direction.
2) The pendulum mass is spinning.
3) The Earth rotates.
4) The potential energy is not precisely cylindrically symmetric.
5) The potential energy is not precisely parabolic.

Kepler's Laws

1. Each planet moves in an elliptical orbit, with the sun at the focus of the ellipse.

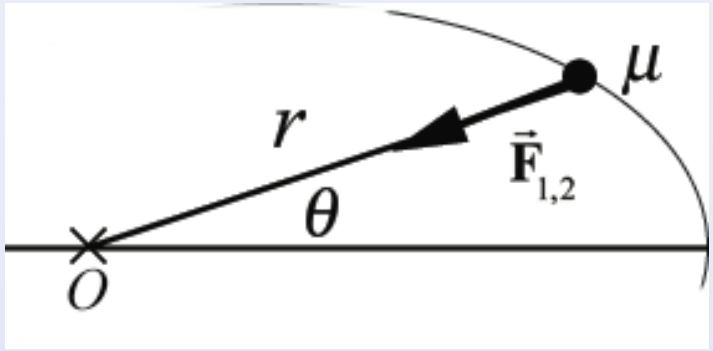


2. A line from the sun to a given planet sweeps out equal areas in equal times.

3. The periods of the planets are proportional to the 3/2 power of the semi-major axis lengths of their orbits.

Orbit Equation for Kepler Problem

Solution (See Course Notes)



$$r = \frac{r_0}{1 - \varepsilon \cos\theta}$$

Eccentricity

$$\varepsilon = \left(1 + \frac{2EL^2}{\mu\left(Gm_1m_2\right)^2}\right)^{1/2} = \left(1 - \frac{E}{E_{\min}}\right)^{1/2}$$

$$E_{\min} = -\frac{1}{2}\frac{\mu(Gm_1m_2)^2}{L^2} \qquad r_0 = \frac{L^2}{\mu G m_1 m_2}$$

Kepler's Laws: Period

Integral of Equal Area Law $$\int_{orbit} \frac{2\mu}{L} dA = \int_0^T dt$$

Period $$T = \frac{2\mu}{L} A$$

Period (*T)* squared is proportional to cube of the semi major axis (*a)* but depends on both masses

$$T^2 = \frac{4\pi^2 a^3}{G\left(m_1 + m_2\right)}$$

Stars Nearby Galactic Center

The UCLA Galactic Center Group, headed by Dr. Andrea Ghez, developed an animation of the orbitsof eight stars about the galactic center http://www.astro.ucla.edu/~jlu/gc/images/2004orbit_animfull_sm.gif

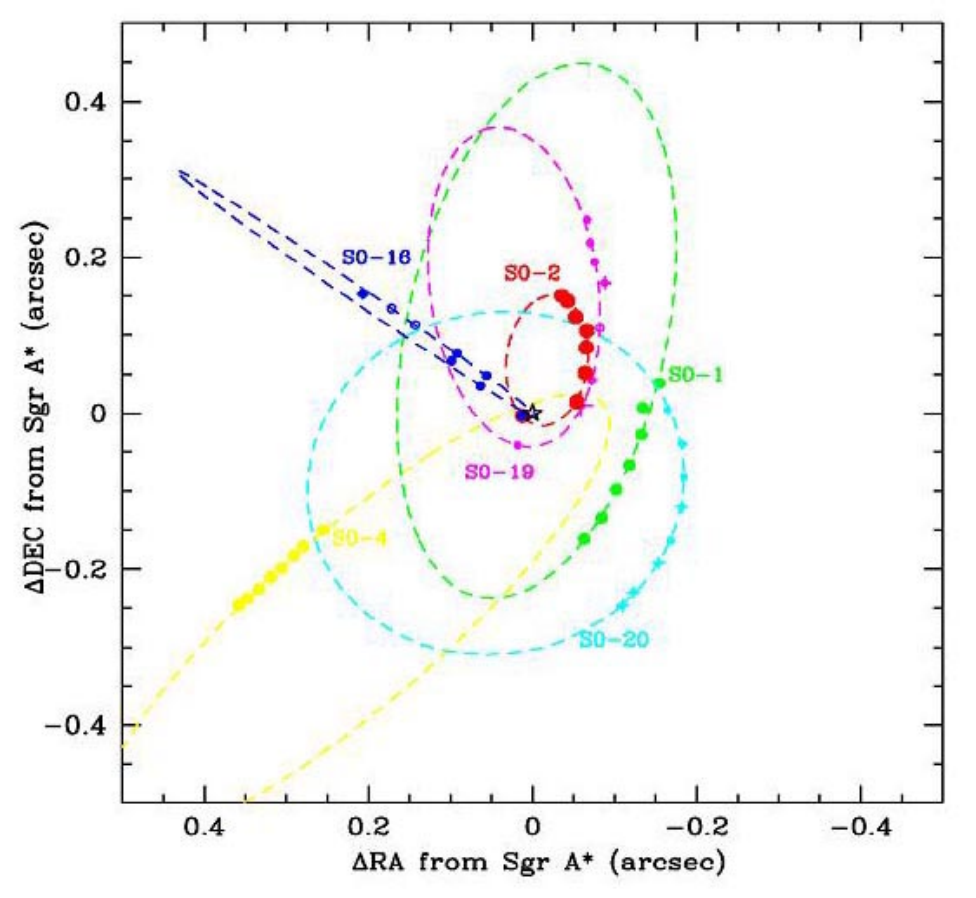


Table Problem

Find the velocity of an object of mass m in a circular orbit of radius R around a massive spherical object of mass M.

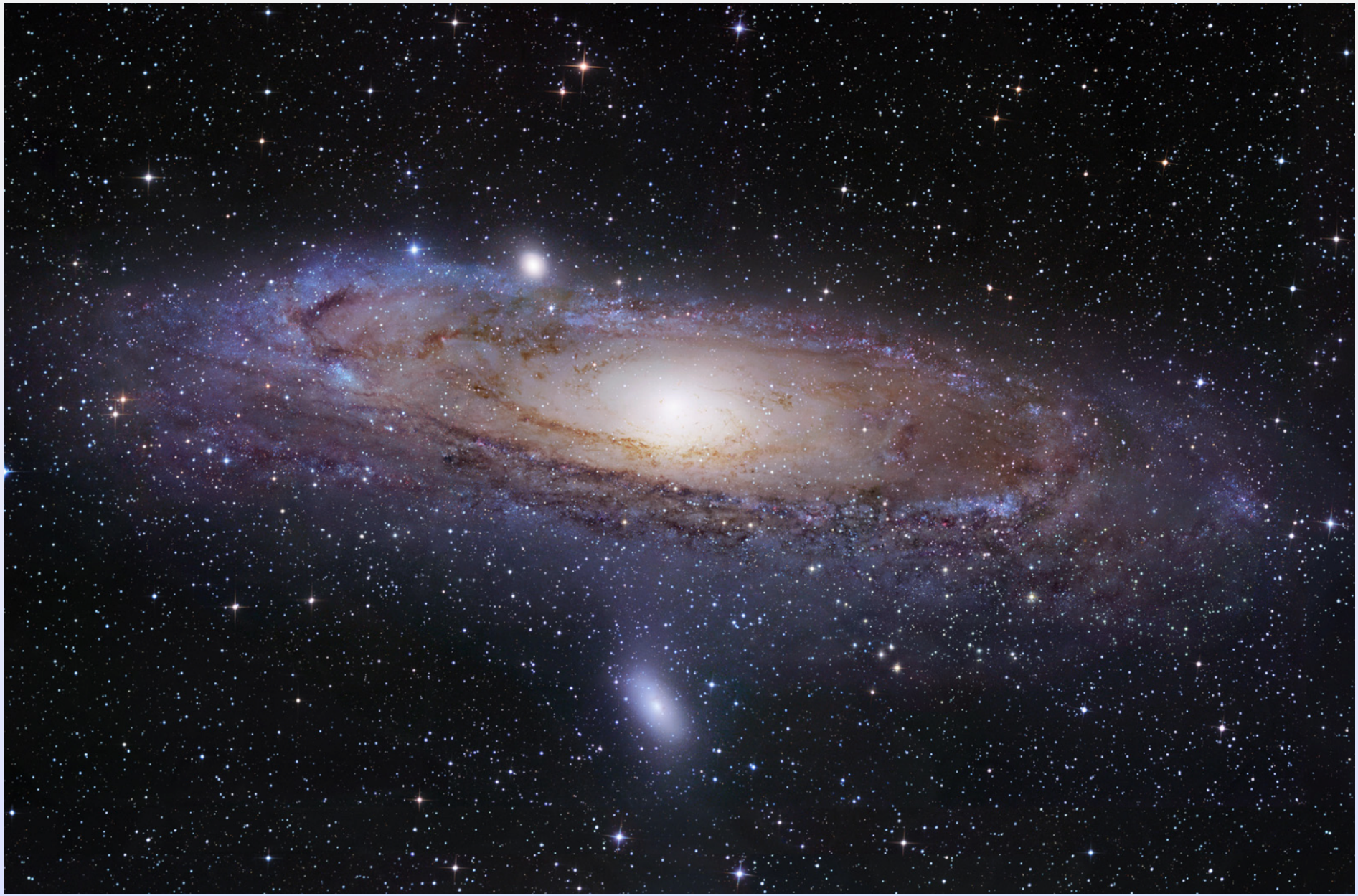
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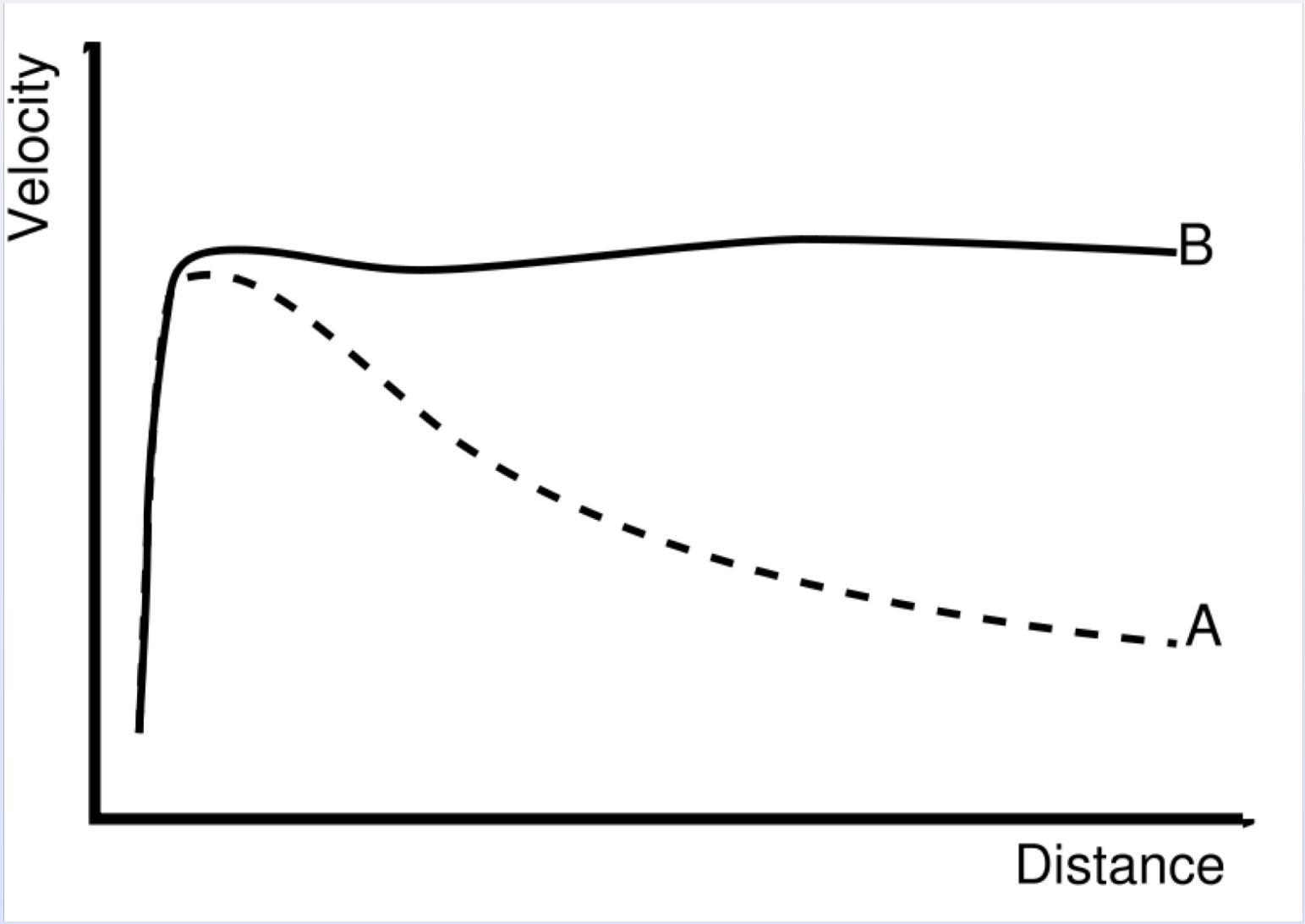
$$F = ma$$

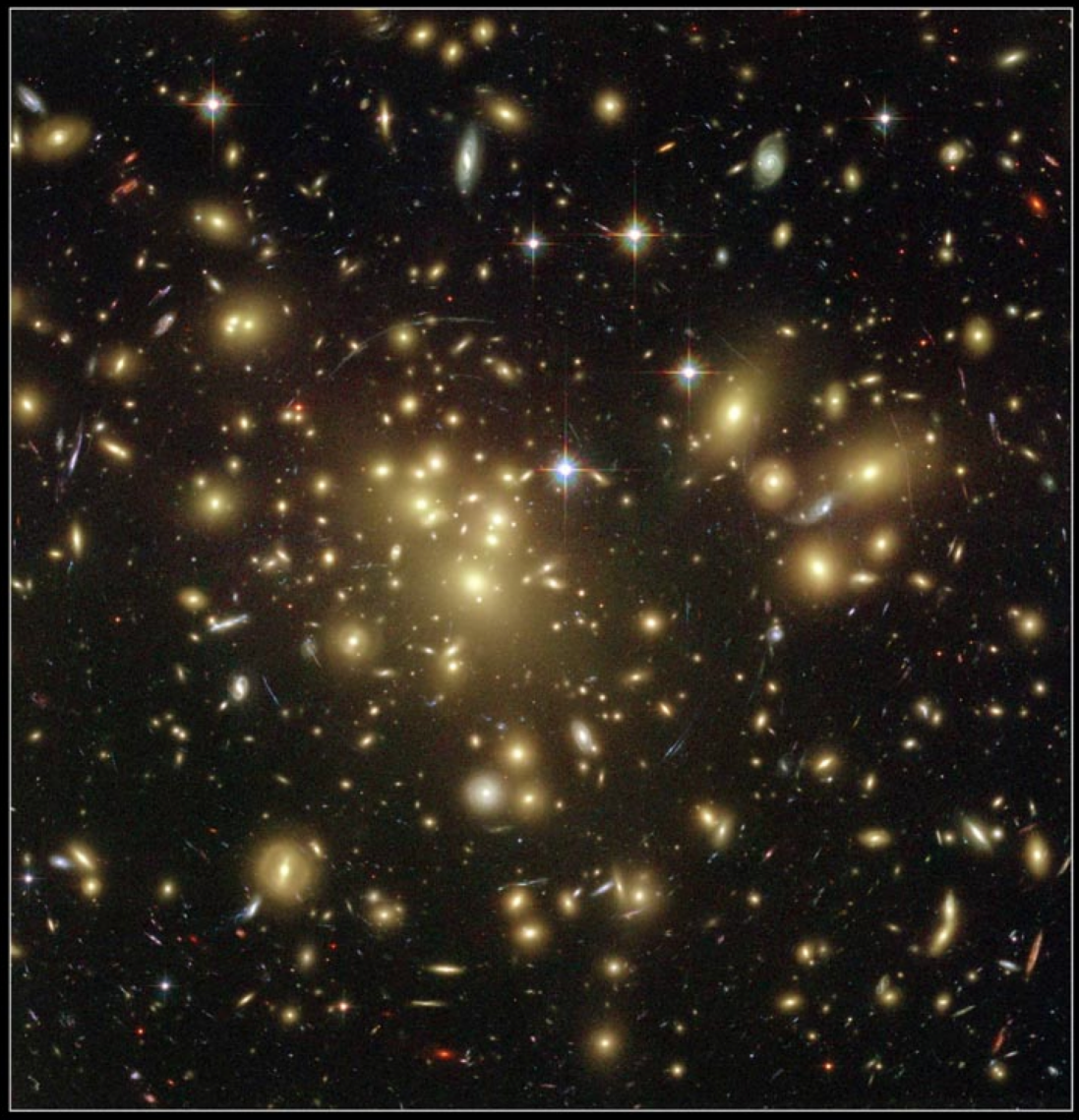
$$-\frac{GMm}{R^2} = -\frac{mv^2}{R}$$

$$\frac{GM}{R} = v^2$$

$$v = \sqrt{\frac{GM}{R}}$$




Velocity
B
A
Distance



Galaxy Cluster Abell 1689

Hubble Space Telescope • Advanced Camera for Surveys

NASA, N. Benitez (JHU), T. Broadhurst (The Hebrew University), H. Ford (JHU), M. Clampin(STScI), G. Hartig (STScI), G. Illingworth (UCO/Lick Observatory), the ACS Science Team and ESA
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