Conservation of Angular Momentum

8.01

W11D2

Rotational and Translational Comparison

Quantity	Rotation	Translation
Momentum		$p = mV_{cm}$
Ang Momentum	$L_{cm} = I_{cm}\omega$	
Force		$\vec{\mathbf{F}}^{ext} = d\vec{\mathbf{p}}^{sys}/dt = m^{total}\vec{\mathbf{A}}_{cm}$
Torque	$\vec{\boldsymbol{\tau}}_{cm} = d\vec{\mathbf{L}}_{cm}/dt$	
Kinetic Energy	$K_{rot} = (1/2)I_{cm}\omega^2$ $K_{rot} = L_{cm}^2/2I_{cm}$	$K_{trans} = (1/2)mV_{cm}^2$ $K_{trans} = p^2/2m$
Work	$W = \int_{\theta_0}^{\theta_f} \tau_S \, d\theta$	$W = \int_0^f \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$
Power	$P_{rot} = \tau_S \, \omega$	$P = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}$

Torque and the Time Derivative of Angular Momentum

Torque about a point S is equal to the time derivative of the angular momentum about S .

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt}$$

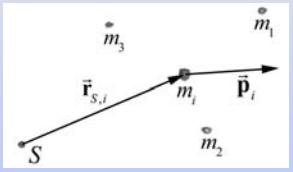
Angular Momentum for a System of Particles

Treat each particle separately

$$\vec{\mathbf{L}}_{S,i} = \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{p}}_i$$

Angular momentum for system about S

$$\vec{\mathbf{L}}_S^{sys} = \sum_{i=1}^{i=N} \vec{\mathbf{L}}_{S,i} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{p}}_i$$



Angular Momentum and Torque for a System of Particles

Change in total angular momentum about a point S equals the total torque about the point S

$$\frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} = \sum_{i=1}^{i=N} \vec{\mathbf{L}}_{S,i} = \sum_{i=1}^{i=N} \left(\frac{d\vec{\mathbf{r}}_{S,i}}{dt} \times \vec{\mathbf{p}}_i + \vec{\mathbf{r}}_{S,i} \times \frac{d\vec{\mathbf{p}}_i}{dt} \right)$$

$$\frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} = \sum_{i=1}^{i=N} \left(\vec{\mathbf{r}}_{S,i} \times \frac{d\vec{\mathbf{p}}_i}{dt} \right) = \sum_{i=1}^{i=N} \left(\vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_i \right) = \sum_{i=1}^{i=N} \vec{\boldsymbol{\tau}}_{S,i} = \vec{\boldsymbol{\tau}}_S^{\text{ total}}$$

$$\frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} = \vec{\boldsymbol{\tau}}_S^{\text{ total}}$$

Internal and External Torques

The total external torque is the sum of the torques due to the net external force acting on each element

$$\vec{\boldsymbol{\tau}}_S^{\text{ ext}} = \sum_{i=1}^{i=N} \vec{\boldsymbol{\tau}}_{S,i}^{\text{ ext}} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_i^{\text{ext}}$$

The total internal torque arise from the torques due to the internal forces acting between pairs of elements

$$\vec{\boldsymbol{\tau}}_S^{\text{ int}} = \sum_{j=1}^{N} \vec{\boldsymbol{\tau}}_{S,j}^{\text{ int}} = \sum_{j=1}^{i=N} \sum_{\substack{i=1 \\ j \neq i}}^{i=N} \vec{\boldsymbol{\tau}}_{S,i,j}^{\text{int}} = \sum_{i=1}^{i=N} \sum_{\substack{j=1 \\ j \neq i}}^{i=N} \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_{i,j}$$

The total torque about S is the sum of the external torques and the internal torques

$$\vec{\boldsymbol{\tau}}_S^{\text{ total}} = \vec{\boldsymbol{\tau}}_S^{\text{ ext}} + \vec{\boldsymbol{\tau}}_S^{\text{ int}}$$

Internal Torques

We know by Newton's Third Law that the internal forces cancel in pairs and hence the sum of the internal forces is zero

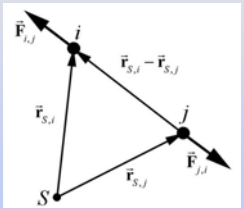
$$\vec{\mathbf{F}}_{i,j} = -\vec{\mathbf{F}}_{j,i} \qquad \vec{\mathbf{0}} = \sum_{i=1}^{i=N} \sum_{\substack{j=1 \\ j \neq i}}^{i=N} \vec{\mathbf{F}}_{i,j}$$

Does the same statement hold about pairs of internal torques?

$$\vec{\boldsymbol{\tau}}_{S,i,j}^{\text{int}} + \vec{\boldsymbol{\tau}}_{S,j,i}^{\text{int}} = \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_{i,j} + \vec{\mathbf{r}}_{S,j} \times \vec{\mathbf{F}}_{j,i}$$

By the Third Law this sum becomes

$$\vec{\boldsymbol{\tau}}_{S,i,j}^{\text{int}} + \vec{\boldsymbol{\tau}}_{S,j,i}^{\text{int}} = (\vec{\mathbf{r}}_{S,i} - \vec{\mathbf{r}}_{S,j}) \times \vec{\mathbf{F}}_{i,j}$$



The vector $\vec{\mathbf{r}}_{S,i} - \vec{\mathbf{r}}_{S,j}$ points from the j^{th} element to the i^{th} element.

Central Forces: Internal Torques Cancel in Pairs

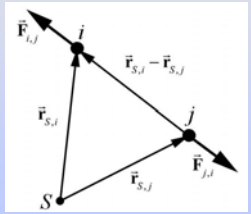
If the internal forces between a pair of particles are directed along the line joining the two particles then the torque due to the internal forces cancel in pairs.

$$\vec{\boldsymbol{\tau}}_{S,i,j}^{\text{int}} + \vec{\boldsymbol{\tau}}_{S,j,i}^{\text{int}} = (\vec{\mathbf{r}}_{S,i} - \vec{\mathbf{r}}_{S,j}) \times \vec{\mathbf{F}}_{i,j} = \vec{\mathbf{0}}$$

This is a stronger version of Newton's Third Law than we have so far used requiring that internal forces are *central forces*. With this assumption, the total torque is just due to the external forces

$$\vec{\boldsymbol{\tau}}_S^{\text{ ext}} = \frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt}$$

However, so far no isolated system has been encountered such that the angular momentum is not constant.



$$\vec{\mathbf{F}}_{i,j} \,||\, (\vec{\mathbf{r}}_{S,i} - \vec{\mathbf{r}}_{S,j})$$

Angular Impulse and Change in Angular Momentum

Angular impulse

$$\vec{\mathbf{J}} = (\vec{\boldsymbol{\tau}}_S^{\,\text{ext}})_{\text{ave}}\,\Delta t_{\text{int}} = \Delta\vec{\mathbf{L}}_S^{\,\text{sys}}$$

$$\vec{\mathbf{J}} = \int_{t_i}^{t_f} \vec{\boldsymbol{\tau}}_S^{\,\text{ext}}\,dt$$

Change in angular momentum

$$\Delta\vec{\mathbf{L}}_S^{\,\text{sys}} \equiv \left(\vec{\mathbf{L}}_S^{\,\text{sys}}\right)_f - \left(\vec{\mathbf{L}}_S^{\,\text{sys}}\right)_i$$

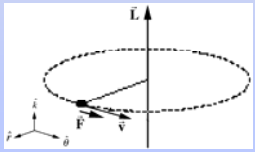
Rotational dynamics

$$\int_{t_i}^{t_f} \vec{\boldsymbol{\tau}}_S^{\,\text{ext}}\,dt = \left(\vec{\mathbf{L}}_S^{\,\text{sys}}\right)_f - \left(\vec{\mathbf{L}}_S^{\,\text{sys}}\right)_i$$

Concept Question: Change in Angular Momentum

A person spins a tennis ball on a string in a horizontal circle with velocity $\vec{\mathbf{v}}$ (so that the axis of rotation is vertical). At the point indicated below, the ball is given a sharp blow (force $\vec{\mathbf{F}}$) in the forward direction. This causes a change in angular momentum $\Delta\vec{\mathbf{L}}$ in the

1. $\hat{\mathbf{r}}$ direction
2. $\hat{\boldsymbol{\theta}}$ direction
3. $\hat{\mathbf{k}}$ direction



Conservation of Angular Momentum

Rotational dynamics

$$\vec{\boldsymbol{\tau}}_S^{\,\text{ext}} = \frac{d\vec{\mathbf{L}}_S^{\,\text{sys}}}{dt}$$

No external torques

$$\vec{\mathbf{0}} = \vec{\boldsymbol{\tau}}_S^{\,\text{ext}} = \frac{d\vec{\mathbf{L}}_S^{\,\text{sys}}}{dt}$$

Change in Angular momentum is zero

$$\Delta\vec{\mathbf{L}}_S^{\,\text{sys}} \equiv \left(\vec{\mathbf{L}}_S^{\,\text{sys}}\right)_f - \left(\vec{\mathbf{L}}_S^{\,\text{sys}}\right)_0 = \vec{\mathbf{0}}$$

Angular Momentum is conserved

$$\left(\vec{\mathbf{L}}_S^{\,\text{sys}}\right)_f = \left(\vec{\mathbf{L}}_S^{\,\text{sys}}\right)_0$$

So far no isolated system has been encountered such that the angular momentum is not constant.

Concept Question: Twirling Person

A woman,holding dumbbells in her arms, spins on a rotating stool. When she pulls the dumbbells inward, the moment of inertia changes and she spins faster. The magnitude of the angular momentum about the vertical axes passing through her center is

1. the same.
2. larger.
3. smaller.
4. not enough information is given to decide.

Concept Question: Figure Skater

A figure skater stands on one spot on the ice (assumed frictionless) and spins around with her arms extended. When she pulls in her arms, she reduces her rotational moment of inertia and her angular speed increases. Assume that her angular momentum is constant. Compared to her initial rotational kinetic energy, her rotational kinetic energy after she has pulled in her arms must be

1. the same.
2. Larger.
3. smaller.
4. not enough information is given to decide.

Table Problem : Rotating Chair and Wheel

A person is sitting on a chair that is initially not rotating and is holding a spinning wheel. The moment of inertia of the person and the chair about a vertical axis passing through the center of the stool is $I_{s,p}$, and the moment of inertia of the wheel about an axis, perpendicular to the plane of the wheel, passing through the center of mass of the wheel is $I_w = (1/4)I_{s,p}$. The mass of wheel is m_w. Suppose that the person holds the wheel as shown in the sketch such that the distance of an axis passing through the center of mass of the wheel to the axis of rotation of the stool is d and that $md^2 = (1/3)I_w$. Suppose the wheel is spinning initially at an angular speed ω_s. The person then turns the spinning wheel upside down. You may ignore any frictional torque in the bearings of the stool. **What is the angular speed of the person and stool after the spinning wheel is turned upside down?**



Demo: Rotating Wheel

Bicycle Wheel and Rotating Stool





Demo: Train

(1) At first the train is started without the track moving. The train and the track both move, one opposite the other.

(2) Then the track is held fixed and the train is started. When the track is let go it does not revolve until the train is stopped. Then the track moves in the direction the train was moving.

(3) Next try holding the train in place until the track comes up to normal speed (Its being driven by the train). When you let go the train remains in its stationary position while the track revolves. You shut the power off to the train and the train goes backwards. Put the power on and the train remains stationary.



A small gauge HO train is placed on a circular track that is free to rotate.

Constants of the Motion

When are the quantities, angular momentum about a point S, energy, and momentum constant for a system?

- No external torques about point S : angular momentum about S is constant

$$\vec{\mathbf{0}} = \vec{\boldsymbol{\tau}}_S^{\text{ ext}} = \frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt}$$

- No external work: mechanical energy constant

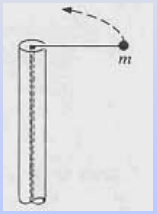
$$0 = W_{\text{ext}} = \Delta E_{\text{mechanical}}$$

- No external forces: momentum constant

$$\vec{\mathbf{F}}^{\text{ext}} = \frac{d\vec{\mathbf{p}}^{\text{sys}}}{dt}$$

Concept Question: Conservation Laws

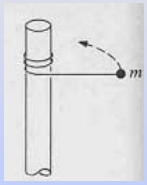
A tetherball of mass m is attached to a post of radius by a string. Initially it is a distance r_0 from the center of the post and it is moving tangentially with a speed v_0 . The string passes through a hole in the center of the post at the top. The string is gradually shortened by drawing it through the hole. Ignore gravity. Until the ball hits the post,



1. The energy and angular momentum about the center of the post are constant.
2. The energy of the ball is constant but the angular momentum about the center of the post changes.
3. Both the energy and the angular momentum about the center of the post, change.
4. The energy of the ball changes but the angular momentum about the center of the post is constant.

Concept Question: Conservation laws

A tetherball of mass m is attached to a post of radius R by a string. Initially it is a distance r_0 from the center of the post and it is moving tangentially with a speed v_0. The string wraps around the outside of the post. Ignore gravity. Until the ball hits the post,



1. The energy and angular momentum about the center of the post are constant.
2. The energy of the ball is constant but the angular momentum about the center of the post changes.
3. Both the energy of the ball and the angular momentum about the center of the post, change.
4. The energy of the ball changes but the angular momentum about the center of the post is constant.

Experiment 05: Moment of Inertia and Angular Collisions

Experiment 05: Goals

Measure the moment of inertia of a rigid body

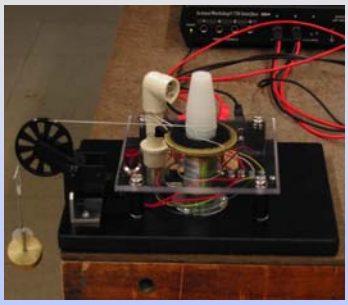
Investigate conservation of angular momentum and kinetic energy in rotational collisions.

Measure and calculate non-conservative work in an inelastic collision.

Keep a copy of your results for the homework problem.

Apparatus

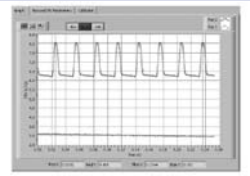
- Connect output of phototransistor to channel A of 750.
- Connect output of tachometer generator to channel B of 750.
- Connect power supply.
- Red button is pressed: Power is applied to motor.
- Red button is released: Rotor coasts: Read output voltage using LabVIEW program.



- Use black sticker or tape on white plastic rotor for generator calibration.

Calibrate Tachometer-generator

Spin motor up to full speed, let it coast. Measure and plot voltages for 0.25 s period. Sample Rate: 5000 Hz.



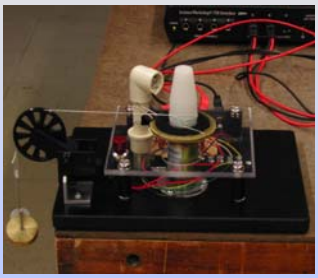
Count rotation periods to measure ω.

Program calculates average output voltage, angular velocity, and the calibration factor angular velocity per volt periods.



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Rotor Moment of Inertia



Plot only the generator voltage for rest of experiment.

Use a 55 gm weight to accelerate the rotor.

Analysis: Moment of Inertia

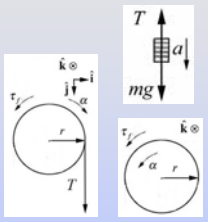
Force and rotational equations while weight is descending:

$$mg - T = ma_1$$

$$rT - \tau_f = I_R\alpha_1$$

Constraint: $a_1 = r\alpha_1$

Rotational equation while slowing down $-\tau_f = I_R\alpha_2$



Speeding up Slowing down

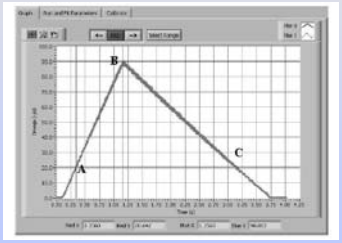
Solve for moment of inertia: $rm(g - r\alpha_1) + I_R\alpha_2 = I_R\alpha_1$

$$I_R = \frac{rm(g - r\alpha_1)}{(\alpha_1 - \alpha_2)}$$

Concept Question: Understand Graph Output to Measure I_R

The angular frequency along line A-B is increasing because

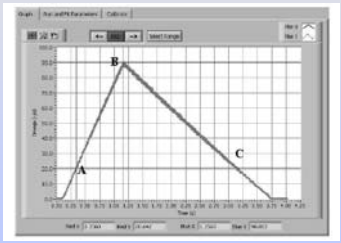
1. the weight has hit the floor and is tension in the string is no longer applying a torque.
2. the weight is descending and the tension in the string is applying a torque.
3. for some other reason.



Concept Question: Understand Graph Output to Measure I_R

The slope of the line B-C is equal to

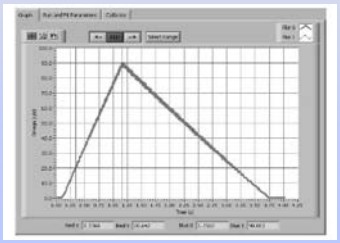
1. the angular acceleration after the weight has hit the floor.
2. angular acceleration before the weight has hit the floor.
3. Neither of the above.



Measure I_R: Results

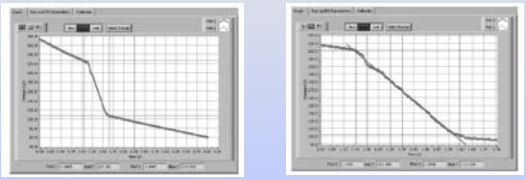
Reset sample rate: 500 Hz and 4 s, Switch from Tachometer to Moment of Inertia. Measure and fit best straight line to get α_1, α_2, and I_R:

$$I_R = \frac{rm(g - r\alpha_1)}{(\alpha_1 - \alpha_2)}$$



Slow Collision

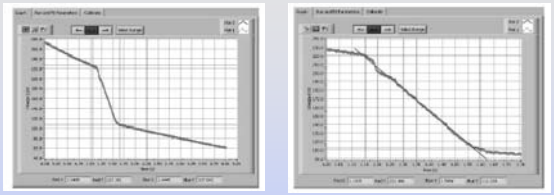
Sample Rate	Start Time
2000 Hz	4 sec



Find ω_1 and ω_2 , measure δt , fit to find α_1, α_R, α_2 . Calculate

$$I_{\text{washer}} = \tfrac{1}{2} m(r_{\text{outer}}^2 + r_{\text{inner}}^2) \qquad \alpha_w = \frac{\omega_2}{\delta t}$$

Slow Collision



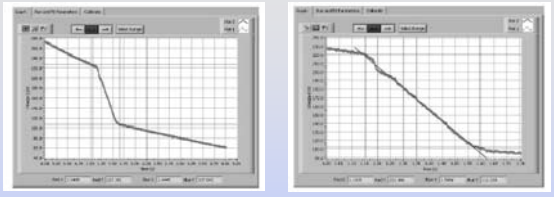
Angular momentum: Angular Impulse: $J_z = \tau_{\text{friction}}\,\delta t = I_R\,\alpha_1\,\delta t$

at t_1: $L_{z,1} = (I_R)\omega_1$

at $t_1 + \delta t$: $L_{z,2} = (I_R + I_w)\omega_2$

Change in Angular momentum: $I_R\,\alpha_1\,\delta t = (I_R + I_w)\omega_2 - (I_R)\omega_1$

Slow Collision



Angle rotated through by rotor:

$$\Delta\theta_{\text{rotor}} = \omega_1\delta t_1 + \frac{1}{2}\alpha_R\delta t_1^{\,2} = \omega_1\delta t_1 + \frac{1}{2}\left(\frac{\omega_2 - \omega_1}{\delta t_1}\right)\delta t_1^{\,2} = \frac{1}{2}(\omega_1 + \omega_2)\delta t_1$$

Angle rotated through by washer:

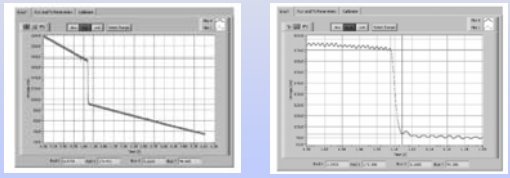
$$\Delta\theta_{\text{washer}} = \frac{1}{2}\alpha_w\,\delta t^2 = \frac{1}{2}\frac{\omega_2}{\delta t}\delta t^2 = \frac{1}{2}\omega_2\delta t$$

Angle washer slid along rotor:

$$\Delta\theta_{\text{rotor}} - \Delta\theta_{\text{washer}} = \frac{1}{2}\omega_1\,\delta t_1$$

Fast Collision

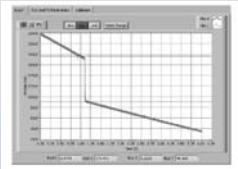
Sample Rate	Start Time
2000 Hz	4 sec

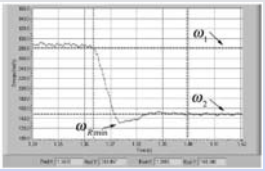


Find ω_1 (before) and ω_2 (after), estimate δt for collision.

Calculate $I_{\text{washer}} = \tfrac{1}{2} m(r_{\text{outer}}^2 + r_{\text{inner}}^2)$

Fast Collision





Angular momentum:

at t_1: $L_{z,1} = (I_R)\omega_1$

at $t_1 + \delta t$: $L_{z,2} = (I_R + I_w)\omega_2$

Kinetic energy at dip:

$$K = \frac{1}{2}(I_R + I_w)\omega_{R,\min}{}^2$$

Change in Angular momentum: $\Delta L_z = (I_R + I_w)\omega_2 - (I_R)\omega_1$

Change in Kinetic Energy: $\Delta K = \frac{1}{2}(I_R + I_w)\omega_2{}^2 - \frac{1}{2}(I_R)\omega_1{}^2$