Final Exam Summary 8.01
Classical Mechanics
Fall 2023

Two-Dimensional Rotational Kinematics

- 8.01
- W08D1

Rigid Bodies

A ***rigid body*** is an extended object in which the distance between any two points in the object is constant in time.

Springs or human bodies are non-rigid bodies.

Translational Motion of the Center of Mass

Total momentum of system of particles

$$\vec{\mathbf{p}}_{sys} = m_{sys}\vec{\mathbf{V}}_{\mathrm{cm}}$$

External force and acceleration of center of mass

$$\vec{\mathbf{F}}_{\mathrm{ext}}^{\mathrm{total}} = \frac{d\vec{\mathbf{p}}_{sys}}{dt} = m_{sys}\frac{d\vec{\mathbf{V}}_{\mathrm{cm}}}{dt} = m_{sys}\vec{\mathbf{A}}_{\mathrm{cm}}$$

Goal: Fixed Axis Rotational Equation of Motion (about Center of Mass)

Torque about center-of-mass produces angular acceleration about center of mass

$$\vec{\tau}_{\text{cm}} = I_{\text{cm}} \vec{\alpha}_{\text{cm}}$$

$\vec{\tau}_{\text{cm}}$ is the torque about the center of mass

I_{cm} is the moment of inertia about the center of mass

$\vec{\alpha}_{\text{cm}}$ is the angular acceleration about center of mass

Analogous to Newton's Second Law for Linear Motion

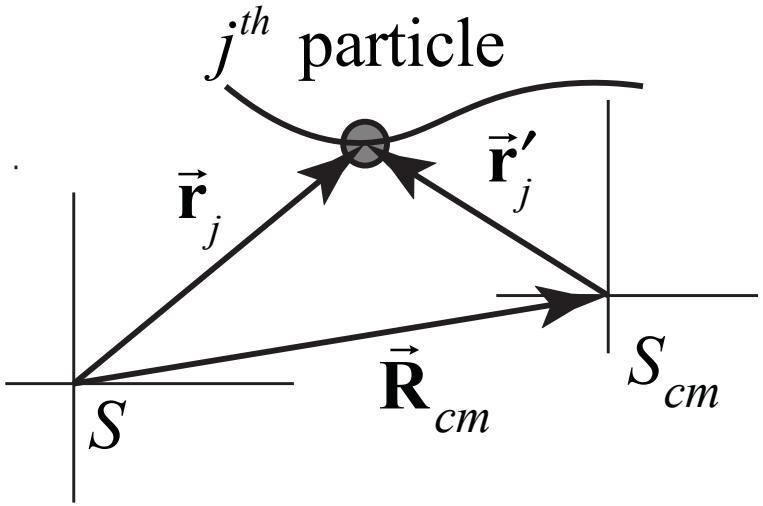
$$\vec{\mathbf{F}}_{ext} = m\vec{\mathbf{a}}_{\text{cm}}$$

Center of Mass Reference Frame

$$\vec{\mathbf{R}}_{cm} = \frac{1}{m_{\text{sys}}} \sum_{j=1}^{j=N} m_j \vec{\mathbf{r}}_j \rightarrow \frac{1}{m_{\text{sys}}} \int_{body} dm \vec{\mathbf{r}}$$

$$\vec{\mathbf{V}}_{cm} = \frac{d\vec{\mathbf{R}}_{cm}}{dt}$$

$$\vec{\mathbf{V}}_{cm} = \frac{1}{m_{\text{sys}}} \sum_{j=1}^{j=N} m_j \vec{\mathbf{v}}_j \rightarrow \frac{1}{m_{\text{sys}}} \int_{body} dm\, \vec{\mathbf{v}}$$



Fixed Axis Rotation: Angular Velocity

Angular velocity: $\vec{\omega} \equiv \omega_z \, \hat{\mathbf{k}} \equiv \dfrac{d\theta}{dt} \hat{\mathbf{k}}$

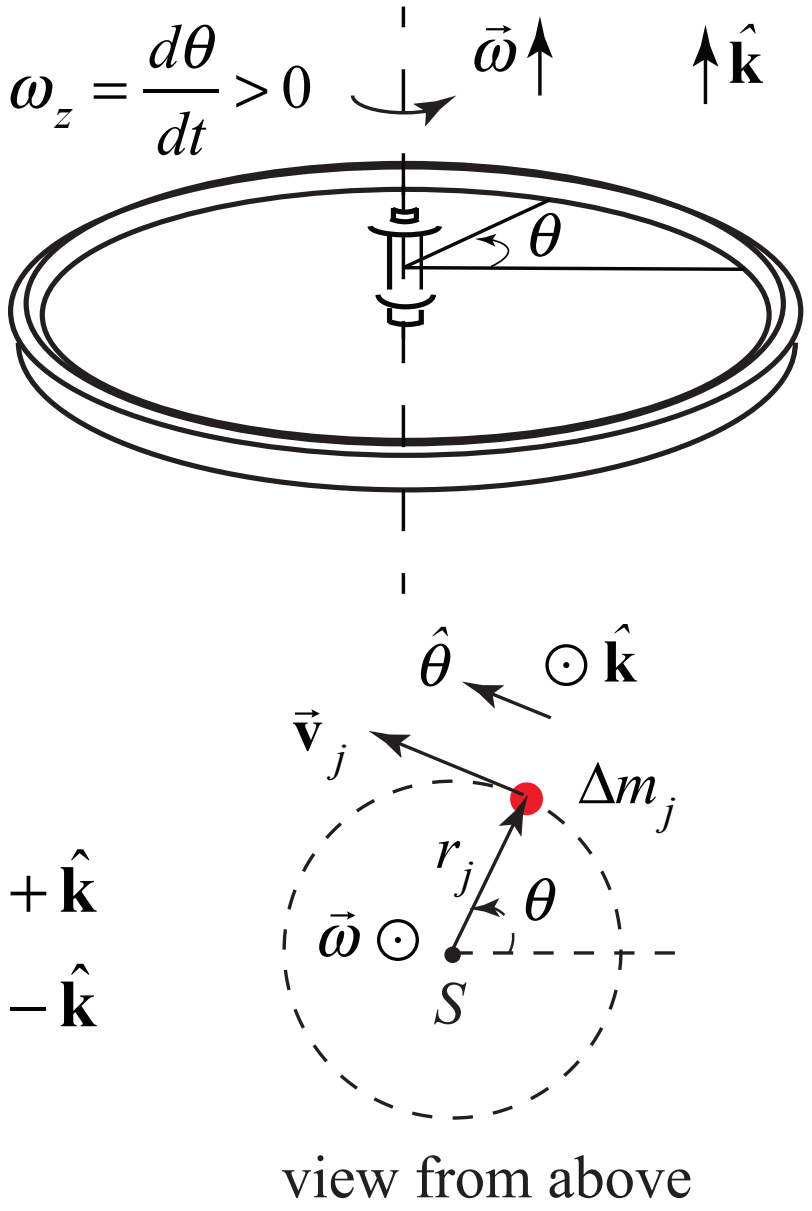
Vector Component: $\omega_z \equiv \dfrac{d\theta}{dt}$

Magnitude: $\omega \equiv \left|\omega_z\right| \equiv \left|\dfrac{d\theta}{dt}\right|$

Direction:

$\omega_z \equiv d\theta / dt > 0$, direction $+\hat{\mathbf{k}}$

$\omega_z \equiv d\theta / dt < 0$, direction $-\hat{\mathbf{k}}$

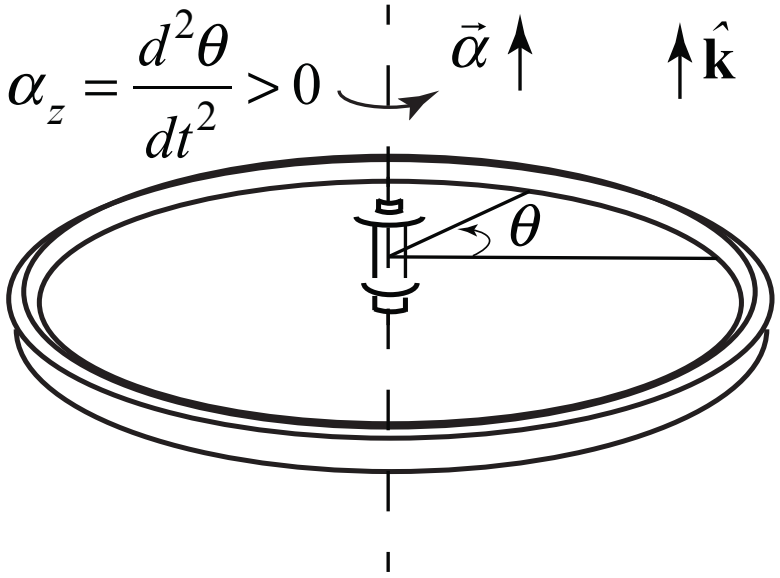


Fixed Axis Rotation: Angular Acceleration

Angular acceleration: $\vec{\alpha} \equiv \alpha_z \, \hat{\mathbf{k}} \equiv \dfrac{d^2\theta}{dt^2} \hat{\mathbf{k}}$

Vector Component: $\alpha_z \equiv \dfrac{d^2\theta}{dt^2} \equiv \dfrac{d\omega_z}{dt}$

Magnitude: $\alpha \equiv \left|\alpha_z\right| \equiv \left|\dfrac{d\omega_z}{dt}\right|$



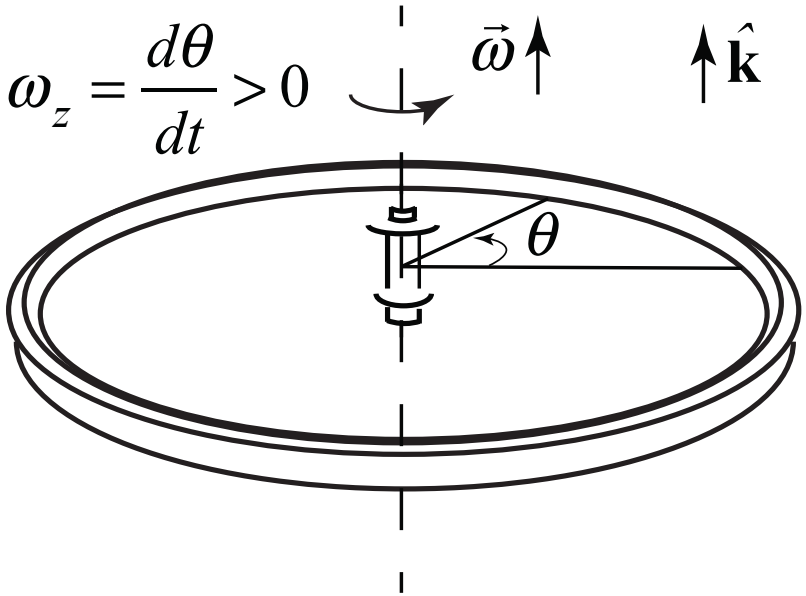
Direction:

$$\alpha_z \equiv \frac{d\omega_z}{dt} > 0, \text{ direction } +\hat{\mathbf{k}}$$

$$\alpha_z \equiv \frac{d\omega_z}{dt} < 0, \text{ direction } -\hat{\mathbf{k}}$$

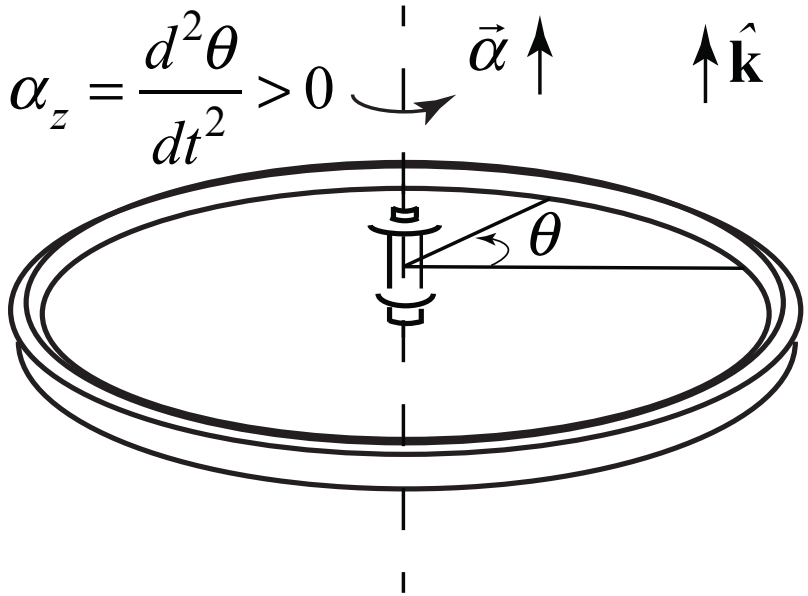
Summary: Rotational Kinematics for Fixed Axis Rotation

Bicycle wheel undergoing fixed axis rotation at a non-uniform angular speed has non-zero



(1) angular velocity vector,

$$\vec{\boldsymbol{\omega}} \equiv \omega_z \, \hat{\mathbf{k}} \equiv \frac{d\theta}{dt} \hat{\mathbf{k}}$$



(2) angular acceleration vector

$$\vec{\boldsymbol{\alpha}} \equiv \alpha_z \, \hat{\mathbf{k}} \equiv \frac{d^2\theta}{dt^2} \hat{\mathbf{k}}$$

Rotational Kinematics: Integral Relations

The angular quantities $\theta, \omega_z, \text{and } \alpha_z$

are exactly analogous to the quantities $x, v_x, \text{and } a_x$

for one-dimensional motion, and obey the same type of integral relations

$$\omega_z(t) - \omega_z(t_0) = \int_{t'=t_0}^{t'=t} \alpha_z(t')\,dt' ,$$

$$\theta(t) - \theta(t_0) = \int_{t'=t_0}^{t'=t} \omega_z(t')\,dt' .$$

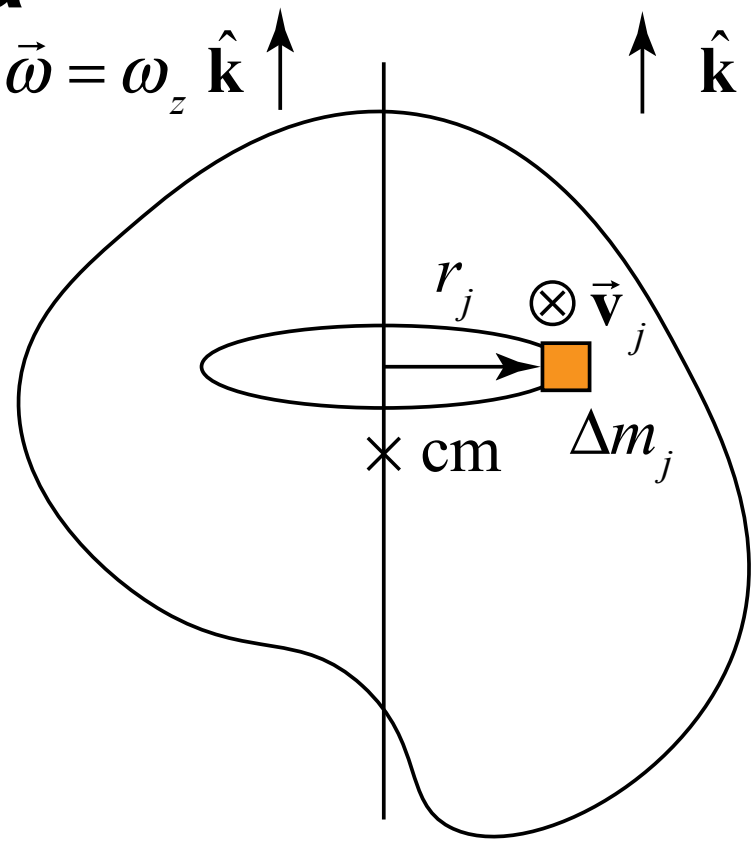
Rigid Body Kinematics for Fixed Axis Rotation Kinetic Energy and Moment of Inertia

Rotational Kinetic Energy and Moment of Inertia

Rotational kinetic energy about axis passing through center of mass

$$K_{\text{rot},j} = \frac{1}{2}\Delta m_j v_j^{\,2} = \frac{1}{2}\Delta m_j (r_j \omega)^2$$

$$K_{\text{rot},j} = \frac{1}{2}\Delta m_j r_j^2 \omega^2$$



Rotational Kinetic Energy:

$$K_{\text{rot}} = \sum_j K_{\text{rot},j} = \left(\sum_j \frac{1}{2}\Delta m_j r_j^{\,2} \right) \omega^2 = \frac{1}{2} I_{\text{cm}} \omega^2$$

Moment of Inertia about axis passing through center of mass:

$$I_{\text{cm}} \equiv \sum_j \Delta m_j r_j^{\,2}$$

$$\boxed{K_{\text{rot}} = \frac{1}{2} I_{\text{cm}} \omega^2}$$

Moment of Inertia: Continuous Body

Moment of Inertia about axle: $$I_{\text{cm}} = \sum_{j=1}^{N} \Delta m_j (r_j)^2$$

SI Unit: $$\left[\text{kg} \cdot \text{m}^2\right]$$

Continuous body: $$\Delta m_j \to dm, \quad r_j \to r_{dm}, \quad \sum_{j=1}^{N} \to \int_{body}$$

Moment of Inertia about center of mass: $$I_{\text{cm}} = \int_{\text{body}} dm\,(r_{dm})^2$$

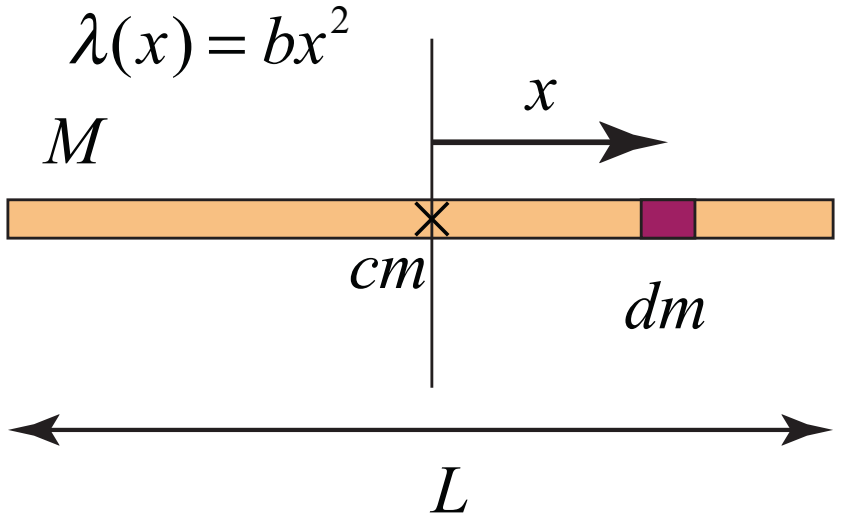
Moment of Inertia of Rod

a) The mass of the object:

$$M = \int_{\text{body}} dm = \int_{\text{body}} \lambda(x)\,dx$$

$$= \int_{-L/2}^{L/2} bx^2\,dx = \frac{bx^3}{3}\Big|_{-L/2}^{L/2} = \frac{bL^3}{12} \Rightarrow$$

$$b = 12M / L^3$$



b) The moment of inertia about an axis that passes perpendicular to the rod through its center of mass of the rod.

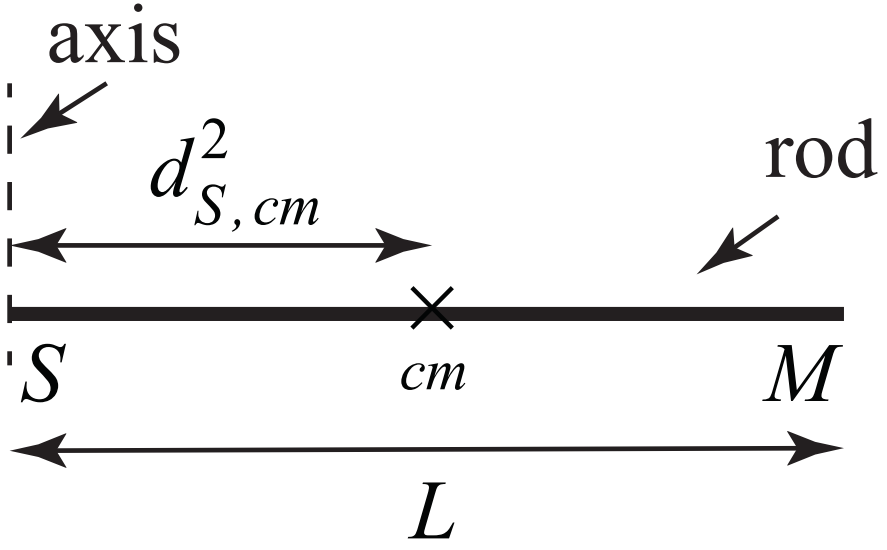
$$I_{\text{cm}} = \int_{\text{body}} dm\,(r_{dm})^2 = \int_{\text{body}} \lambda(x)\,dx\,x^2 = \int_{-L/2}^{L/2} bx^4\,dx = \frac{bx^5}{5}\Big|_{-L/2}^{L/2} = \frac{bL^5}{80}$$

$$= \frac{3}{20} ML^2$$

Parallel Axis Theorem

Rigid body of mass M.

Moment of inertia I_{cm} about axis through center of mass of the body.



Moment of inertia I_S about parallel axis through point S in body.

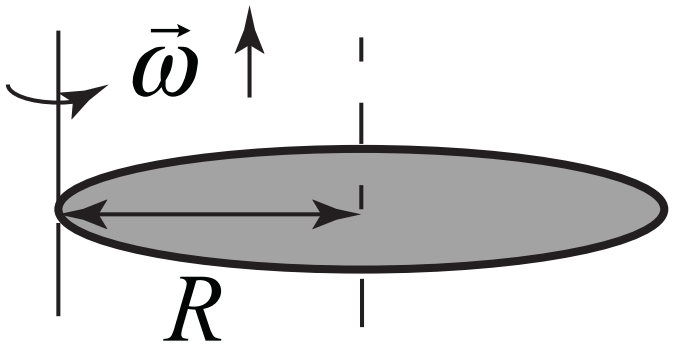
$$I_S = I_{\text{cm}} + Md^2_{S,\text{cm}}$$

$d_{S,\text{cm}}$ perpendicular distance between two parallel axes.

Kinetic Energy

The parallel axis theorem states the moment of inertia about an axis passing perpendicular to the plane of the disc and passing through a point *S* on the edge of the disc is equal to

$$I_S = I_{cm} + mR^2$$



Therefore

$$I_{cm} = (1/2)mR^2 \rightarrow$$

$$I_S = I_{cm} + mR^2 = (3/2)mR^2$$

The kinetic energy is then

$$K = (1/2)I_{edge}\omega^2 = (3/4)mR^2\omega^2$$

Two-Dimensional Rotational Dynamics

8.01

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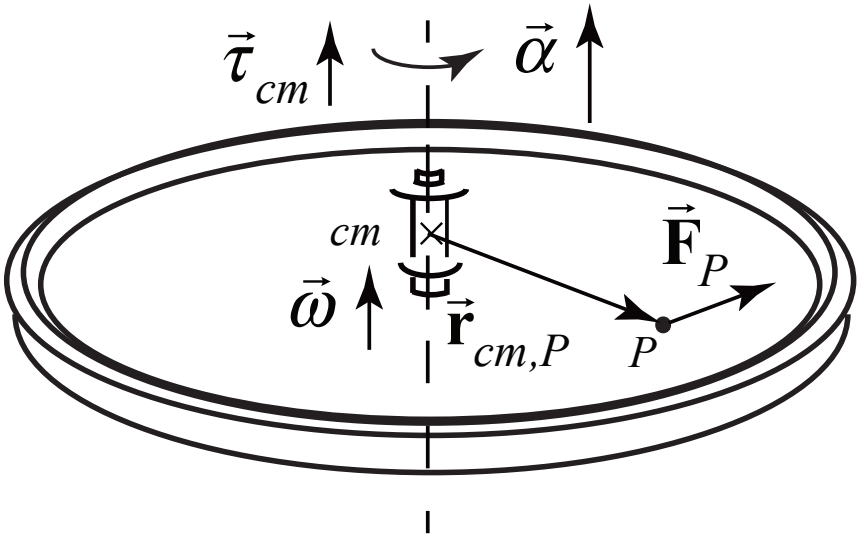
Torque on a Bicycle Wheel



A force $\vec{\mathbf{F}}_P$ is applied at the point *P* along the spoke. $\vec{\mathbf{r}}_{cm,P}$ is the vector from the center of mass located at point *S* to the point *P* where the force is applied

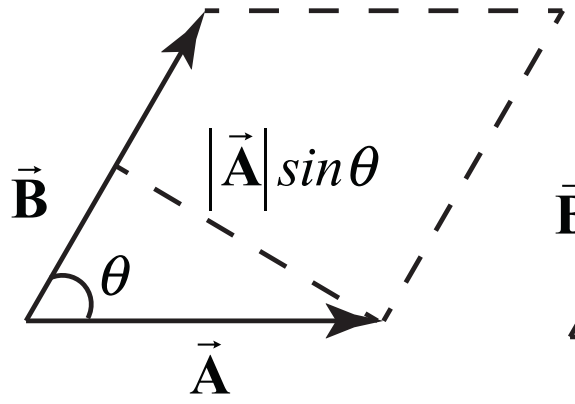
$$\vec{\tau}_{cm} = \vec{\mathbf{r}}_{cm,P} \times \vec{\mathbf{F}}_P = I_{\mathrm{cm}} \vec{\boldsymbol{\alpha}}_{\mathrm{cm}}$$

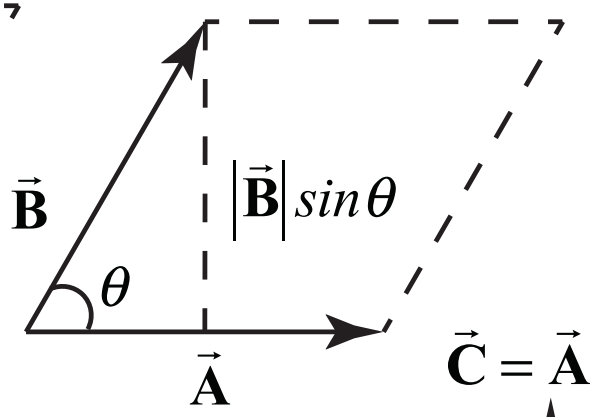
For fixed axis rotation, the direction of the torque is in the same direction as the angular acceleration

Vector Cross Product

Magnitude: equal to the area of the parallelogram defined by the two vectors

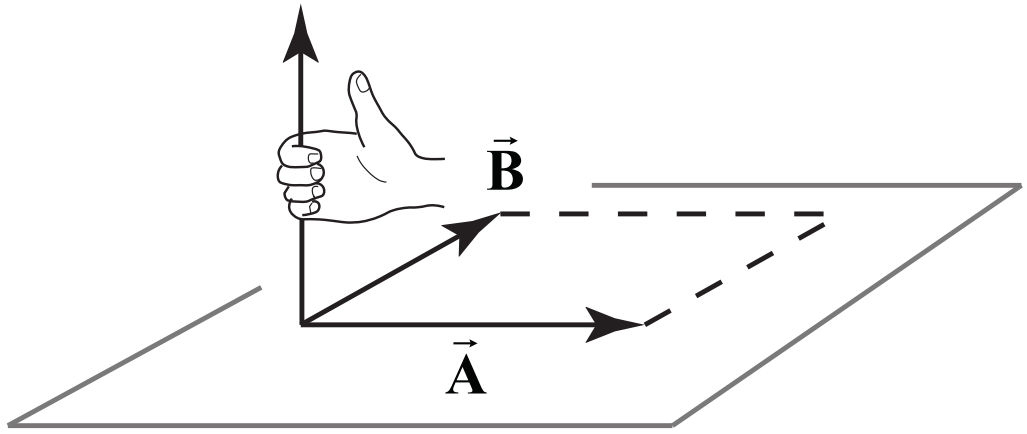
$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|\sin\theta = \left|\vec{\mathbf{A}}\right|\left(\left|\vec{\mathbf{B}}\right|\sin\theta\right) = \left(\left|\vec{\mathbf{A}}\right|\sin\theta\right)\left|\vec{\mathbf{B}}\right| \quad (0 \le \theta \le \pi)$$





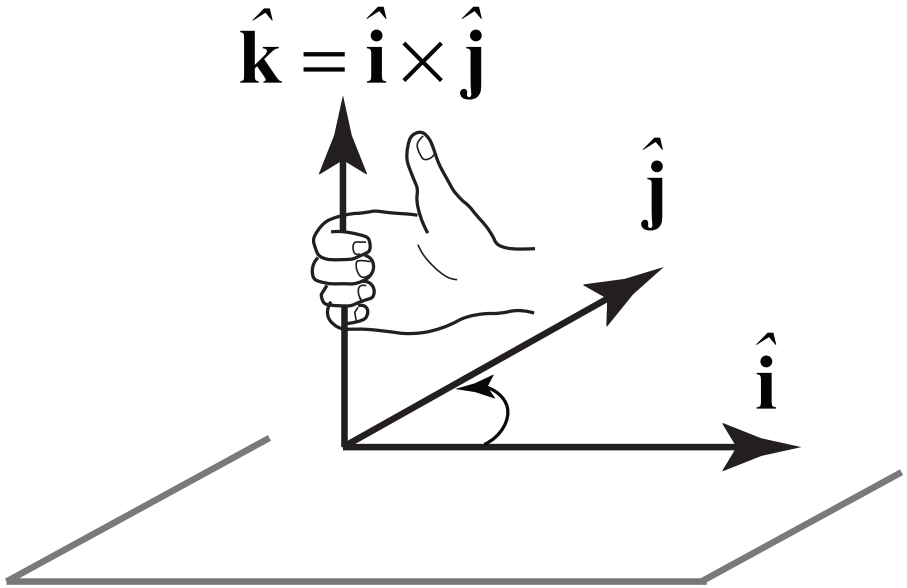
$\vec{\mathbf{C}} = \vec{\mathbf{A}}\times\vec{\mathbf{B}}$

Direction: determined by the Right-Hand-Rule



Vector Product of Unit Vectors

Unit vectors in Cartesian coordinates



$$\left|\hat{\mathbf{i}} \times \hat{\mathbf{j}}\right| = |\hat{\mathbf{i}}||\hat{\mathbf{j}}|\sin(\pi/2) = 1$$

$$\left|\hat{\mathbf{i}} \times \hat{\mathbf{i}}\right| = |\hat{\mathbf{i}}||\hat{\mathbf{i}}|\sin(0) = 0$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \quad \hat{\mathbf{i}} \times \hat{\mathbf{i}} = \vec{\mathbf{0}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} \quad \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \vec{\mathbf{0}}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \quad \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \vec{\mathbf{0}}$$

Components of Cross Product

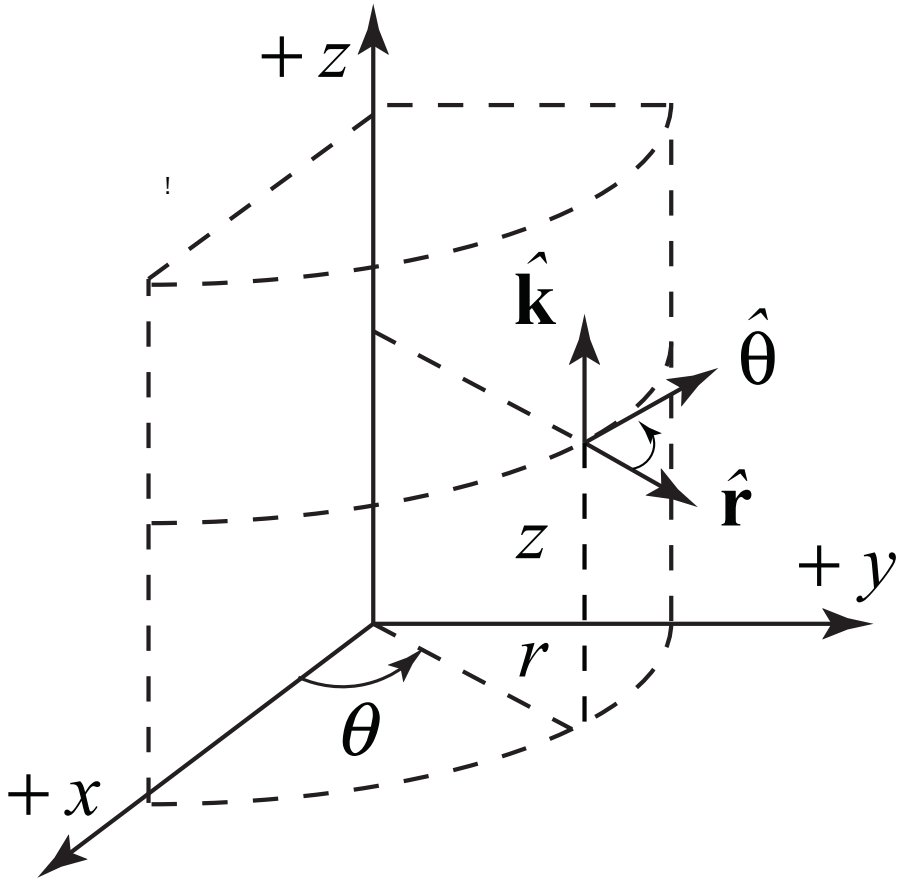
$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}, \quad \vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$$

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_yB_z - A_zB_y)\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\hat{\mathbf{k}}$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Vector Product of Cylindrical Unit Vectors

Unit vectors in Cylindrical coordinates



$$\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}=\hat{\mathbf{k}}, \qquad \hat{\mathbf{r}}\times\hat{\mathbf{r}}=\vec{\mathbf{0}}$$

$$\hat{\mathbf{k}}\times\hat{\mathbf{r}}=\hat{\boldsymbol{\theta}}, \qquad \hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\theta}}=\vec{\mathbf{0}}$$

$$\hat{\boldsymbol{\theta}}\times\hat{\mathbf{k}}=\hat{\mathbf{r}}, \qquad \hat{\mathbf{k}}\times\hat{\mathbf{k}}=\vec{\mathbf{0}}$$

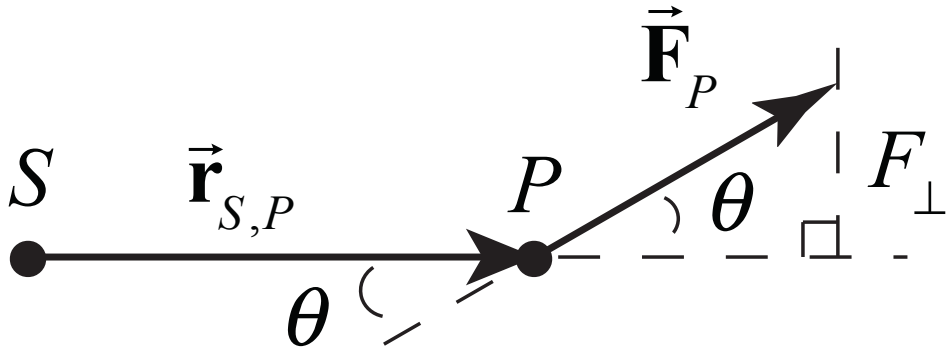
Torque: Method 1 Magnitude and Direction

Magnitude of torque about a point S due to force acting at point P.

$$\vec{\tau}_S = \vec{\mathbf{r}}_{S,P} \times \vec{\mathbf{F}}_P$$

$$\left|\vec{\tau}_S\right| = rF_\perp = rF\sin\theta$$

$$0 \le \theta \le \pi$$



Direction of torque:

Perpendicular to the plane formed by $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$, and determined by the right-hand-rule.

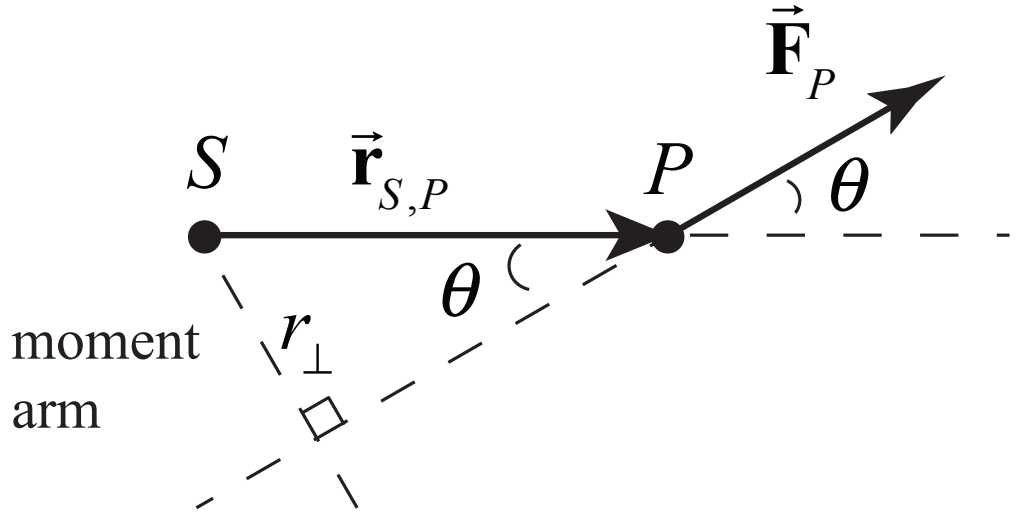
Torque: Method 2 Magnitude and Direction

Magnitude of torque about a point S due to force acting at point P.

$$\vec{\tau}_S = \vec{\mathbf{r}}_{S,P} \times \vec{\mathbf{F}}_P$$

$$\left|\vec{\tau}_S\right| = r_\perp F = rF\sin\theta$$

$$0 \le \theta \le \pi$$

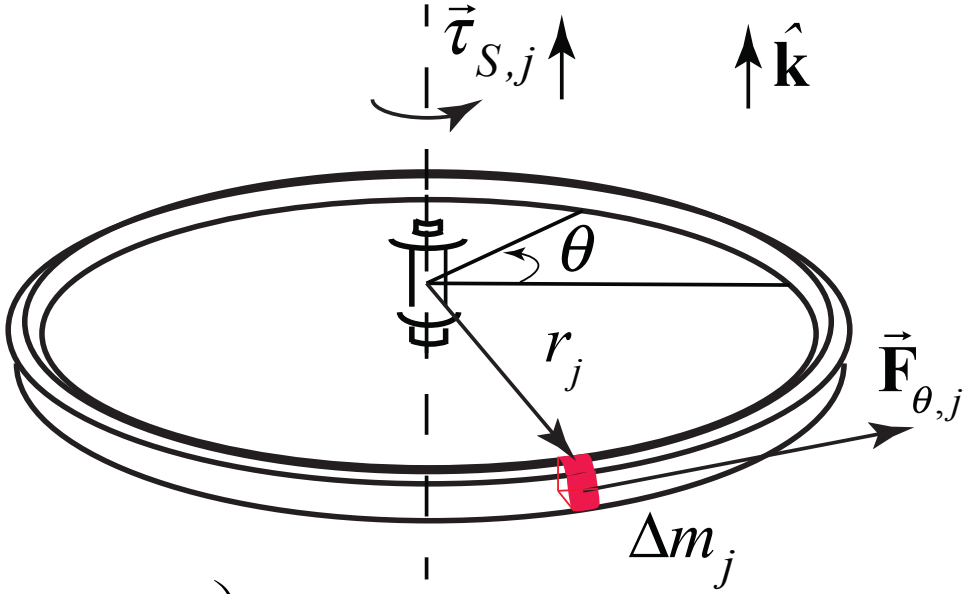


Direction of torque:

Perpendicular to the plane formed by $\vec{\mathbf{r}}_{S,P}$ and $\vec{\mathbf{F}}_P$, and determined by the right-hand-rule.

Torque, Moment of Inertia and Angular Acceleration

z-component of the total torque about an axis passing through S is the sum over all elements Recall: Moment of Inertia about and axis passing through S:



$$\tau_{S,z}^{ext} = \left(\lim_{N\to\infty} \sum_{j=1}^{j=N} \Delta m_j r_j^{\,2} \right) \alpha_z = \left(\int_{body} dm\, r_{S,dm}^2 \right) \alpha_z = I_S \alpha_z$$

Torque Law for Fixed Axis Rotation:

$$\vec{\tau}_S^{ext} = I_S \vec{\alpha}$$

Simple Pendulum by the Torque Method

Calculate torque about pivot point *S*

$$\vec{\tau}_S = \vec{\mathbf{r}}_{S,P} \times \vec{\mathbf{F}}_P$$

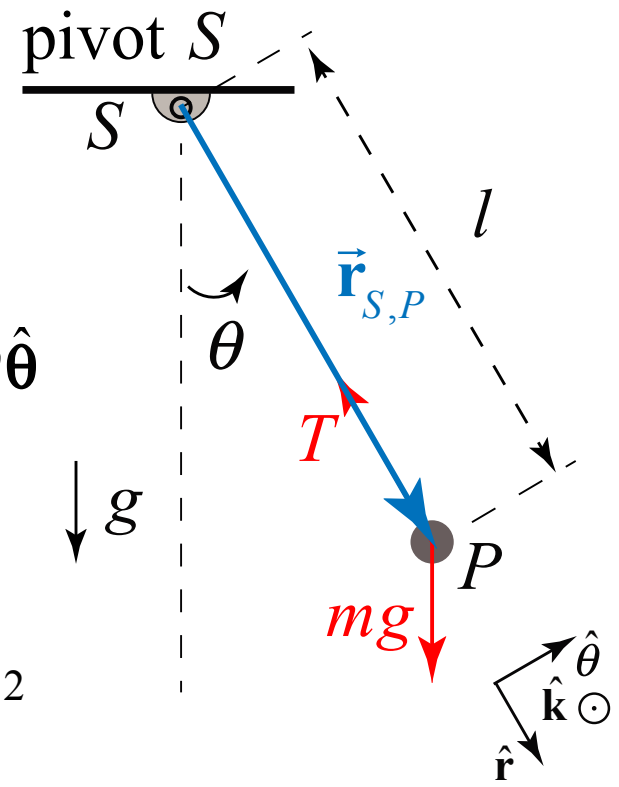
$$\vec{\mathbf{r}}_{S,P} = l\hat{\mathbf{r}} \quad \text{and} \quad \vec{\mathbf{F}}_P = (-T + mg\cos\theta)\hat{\mathbf{r}} - mg\sin\theta\hat{\boldsymbol{\theta}}$$

$$\vec{\tau}_S = l\hat{\mathbf{r}} \times ((-T + mg\cos\theta)\hat{\mathbf{r}} - mg\sin\theta\hat{\boldsymbol{\theta}})$$

$$= -mgl\sin\theta\hat{\mathbf{k}}$$

Moment of inertia about pivot point: $I_S = ml^2$

Angular acceleration: $\vec{\alpha} = \dfrac{d^2\theta}{dt^2}\hat{\mathbf{k}}$

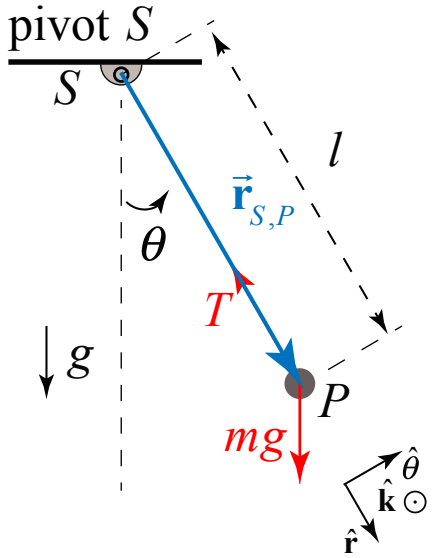


Simple Pendulum Torque Method

Torque Law: $\vec{\tau}_S = I_S \vec{\alpha}$

Equation of Motion: $-mgl\sin\theta\hat{\mathbf{k}} = ml^2\frac{d^2\theta}{dt^2}\hat{\mathbf{k}} \Rightarrow \frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$

Small angle (simple harmonic motion): $\sin\theta << 1\,;\ \sin\theta \approx \theta$



Simple harmonic oscillator: $\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta \approx 0$

Angular frequency: $\omega_0 = \sqrt{g/l}$

Angle and angular velocity equations of motion: $\theta(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

$$\frac{d\theta}{dt}(t) = -\omega_0 A\sin(\omega_0 t) + B\omega_0\cos(\omega_0 t)$$

Initial conditions: $\theta(t=0) = \theta_i = A;\quad \frac{d\theta}{dt}(t=0) = \omega_{z,i} = B\omega_0 \Rightarrow B = \omega_{z,i}\sqrt{l/g}$

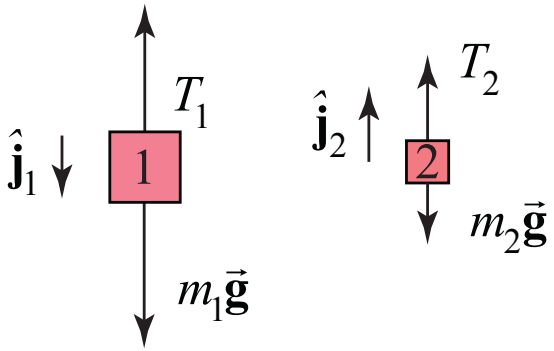
Equations of motion:

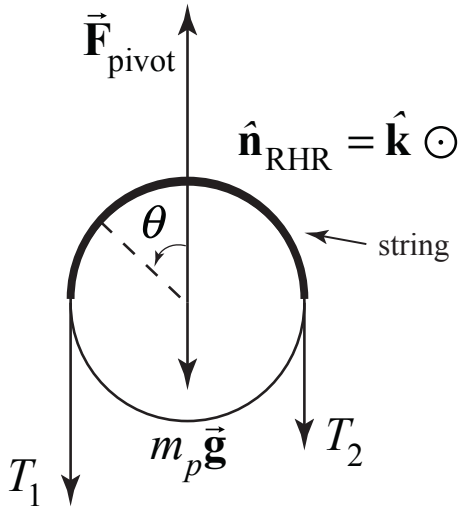
$$\theta(t) = \theta_i\cos(\sqrt{g/l}\,t) + \omega_{z,i}\sqrt{l/g}\sin(\sqrt{g/l}\,t)$$

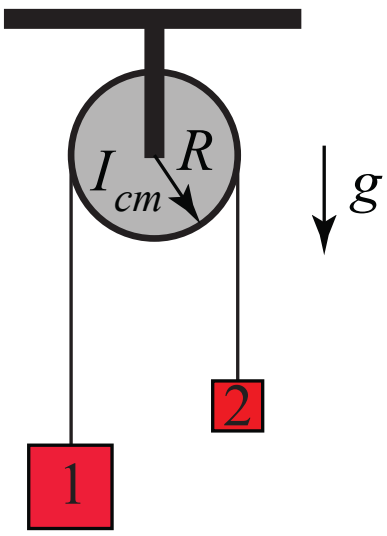
$$\frac{d\theta}{dt}(t) = -\theta_i\sqrt{g/l}\sin(\sqrt{g/l}\,t) + \omega_{z,i}\cos(\sqrt{g/l}\,t)$$

Atwood Machine

a) Coordinate systems for the blocks and pulleys:







b) Constraint condition between the accelerations of the blocks and the angular acceleration of the pulley: $a \equiv a_{y,1} = a_{x,2} = R\alpha_z$

c) Newton's Second Law for block 1: $m_1 g - T_1 = m_1 a$

d) Newton's Second Law for block 2: $T_2 - m_2 g = m_2 a$

e) Torque Law on pulley: $T_1 R - T_2 R = I_{cm}\alpha_z = I_{cm} a / R$

f) Acceleration of blocks: $a = \dfrac{(m_1 - m_2)}{(I_{cm} / R^2) + m_1 + m_2} g$

Torque Due to Uniform Gravitational Force

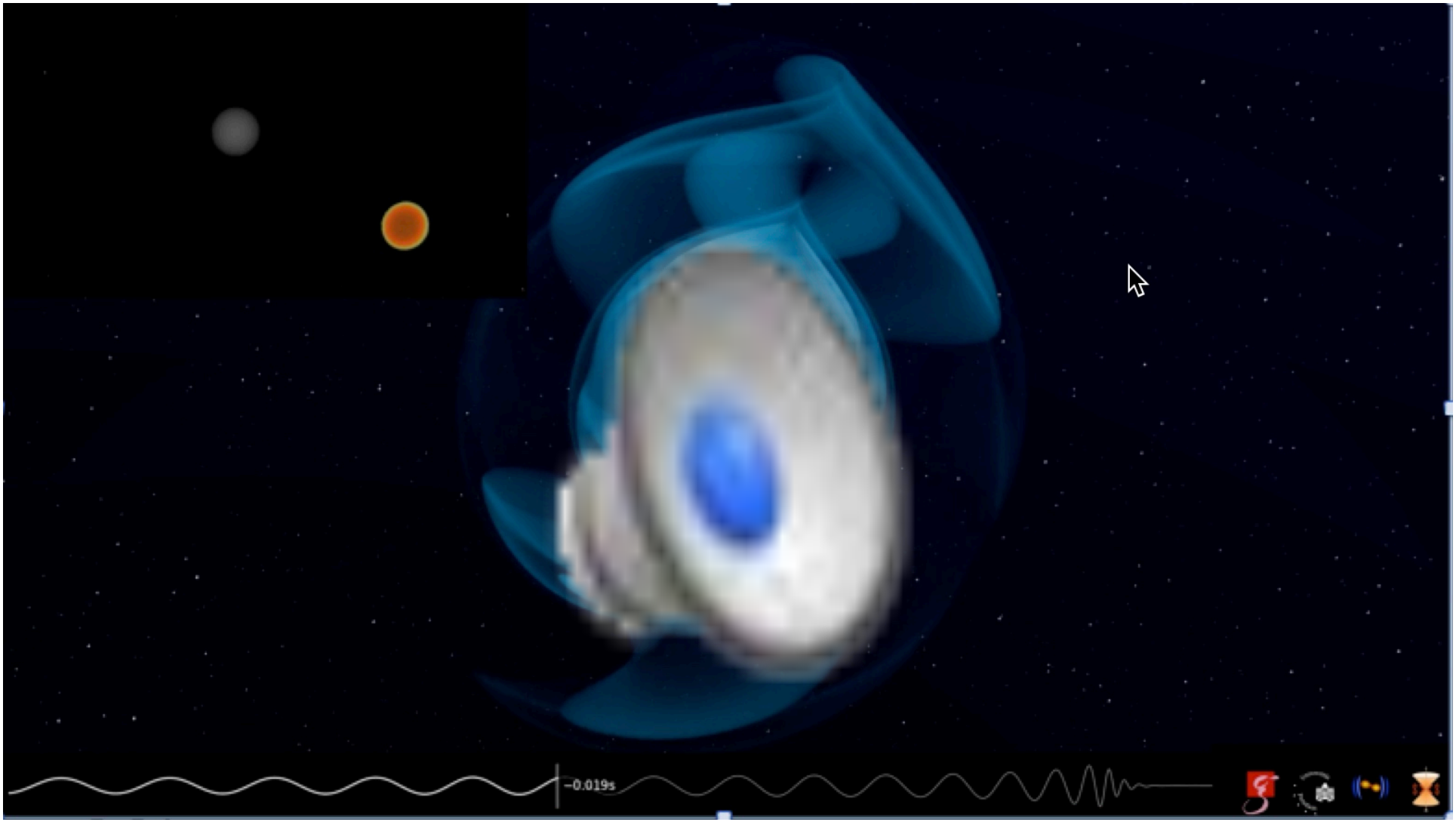
The total torque on a rigid body due to the gravitational force can be determined by placing all the gravitational force at the center-of-mass of the object.

$$\vec{\boldsymbol{\tau}}_S^g = \sum_i (\vec{\mathbf{r}}_{S,i} \times m_i \vec{\mathbf{g}}) = \left(\sum_i m_i \vec{\mathbf{r}}_{S,i} \right) \times \vec{\mathbf{g}}$$

$$= \left(\frac{1}{m_{total}} \sum_i m_i \vec{\mathbf{r}}_{S,i} \right) \times m_{total} \vec{\mathbf{g}} = \vec{\mathbf{R}}_{S,cm} \times m_{total} \vec{\mathbf{g}}$$

This is why, on free-body diagrams we always draw the gravitational force acting at the center-of-mass.

Angular Momentum

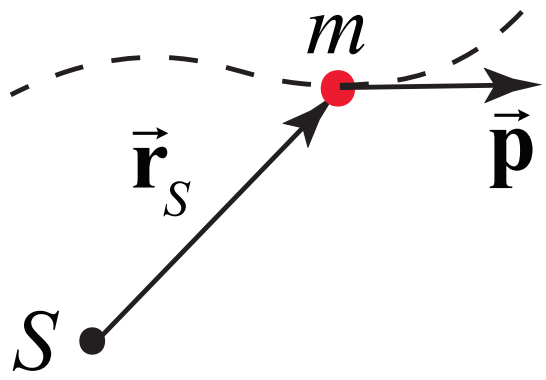


- **8.01 W10D1 Simulation of an NSBH coalescence consistent with GW200115**

Angular Momentum of a Point Particle

Point particle of mass m moving with a velocity $\vec{\mathbf{v}}$

Momentum $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$



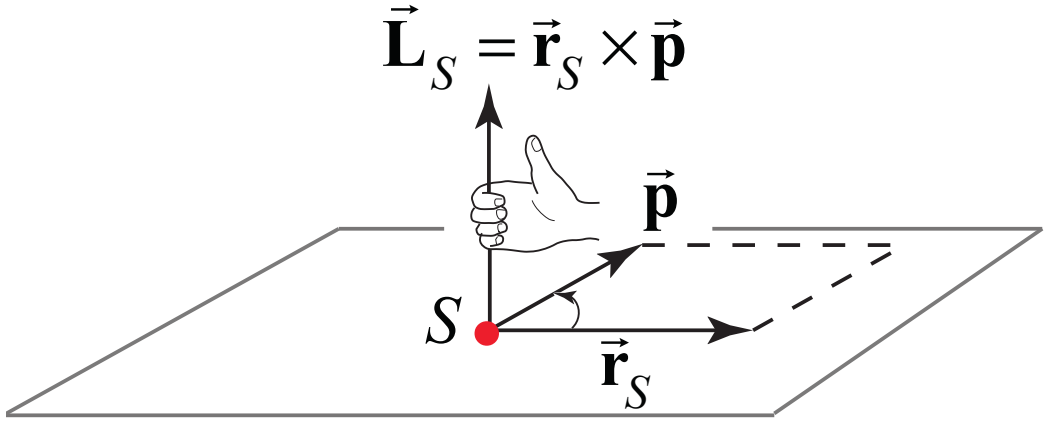
Fix a point S

Vector $\vec{\mathbf{r}}_S$ from the point S to the location of the object

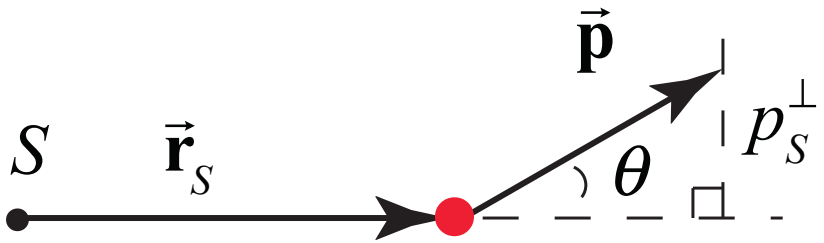
Angular momentum about the point S $\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}}$

SI Unit $[\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-1}]$

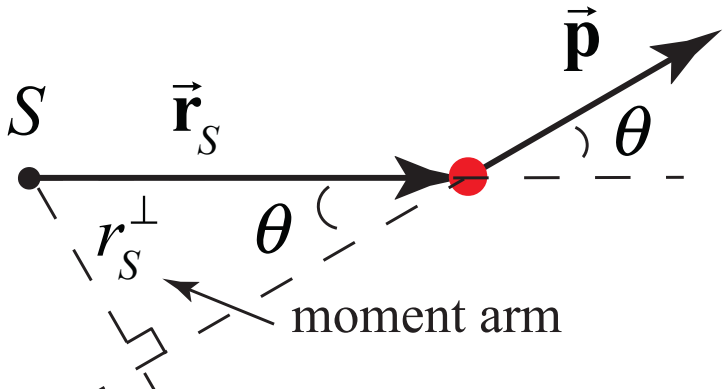
Angular Momentum of a Point Particle



- Right Hand Rule



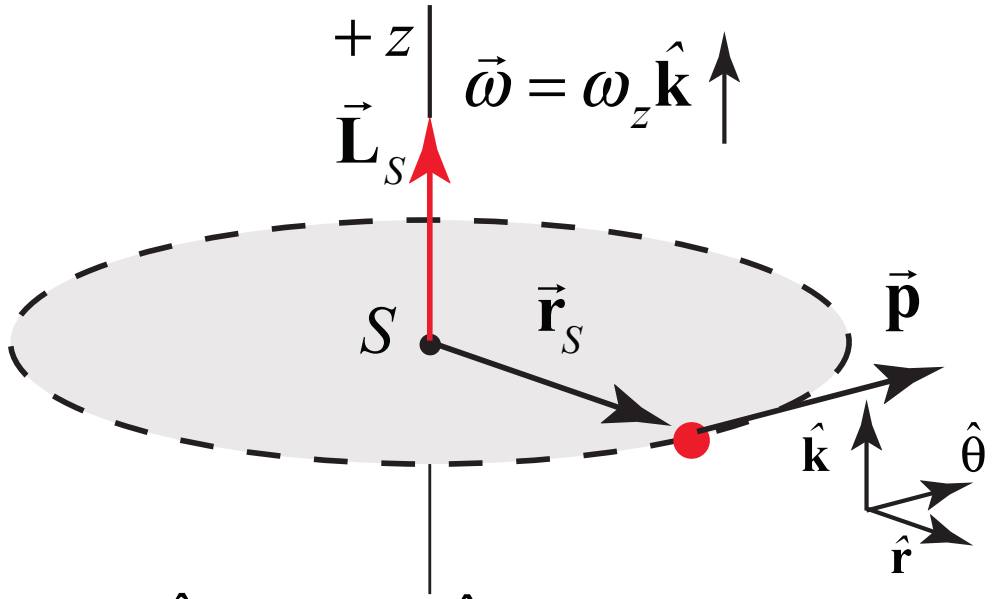
$$\left|\vec{\mathbf{L}}_S\right| = \left|\vec{\mathbf{r}}_S\right|\left|\vec{\mathbf{p}}\right|\sin\theta = \left|\vec{\mathbf{r}}_S\right| p_S^{\perp}; \quad 0 \le \theta \le \pi$$



$$\left|\vec{\mathbf{L}}_S\right| = \left|\vec{\mathbf{r}}_S\right|\left|\vec{\mathbf{p}}\right|\sin\theta = \left|r_S^{\perp}\right|\left|\vec{\mathbf{p}}\right|; \quad 0 \le \theta \le \pi$$

Angular Momentum of a Point Particle in Circular Motion

Fixed axis of rotation: *z*-axis



Moment vector: $\vec{\mathbf{r}}_S = R\hat{\mathbf{r}}$

Momentum: $\vec{\mathbf{p}} = mv\hat{\boldsymbol{\theta}}$

Angular momentum about the point S

$$\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}} = \vec{\mathbf{r}}_S \times m\vec{\mathbf{v}} = R\hat{\mathbf{r}} \times mR\omega_z\,\hat{\theta} = mR^2\omega_z\,\hat{\mathbf{k}}$$

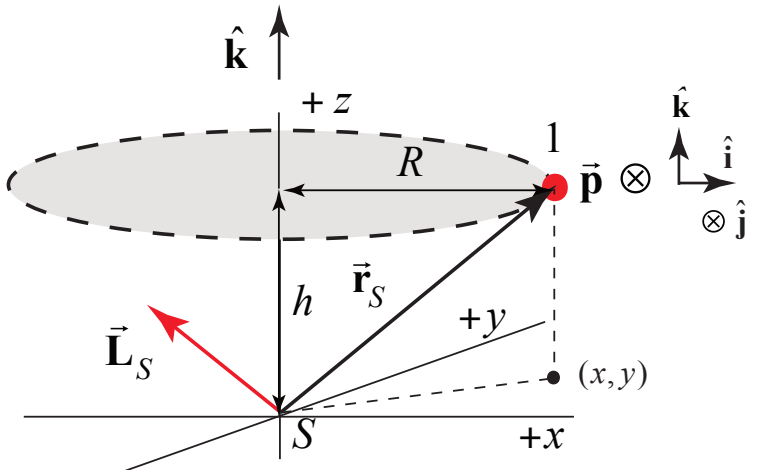
Angular velocity: $\vec{\omega} = \omega_z\,\hat{\mathbf{k}} = (v/R)\hat{\mathbf{k}}$

Second form for angular momentum: $\vec{\mathbf{L}}_S = mRv\hat{\mathbf{k}} = mR^2\omega_z\,\hat{\mathbf{k}} = I_S\vec{\boldsymbol{\omega}}$

Ang. Mom. About Origin

Moment vector: $\vec{\mathbf{r}}_S = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + h\hat{\mathbf{k}}$

Momentum vector: $\vec{\mathbf{p}} = m(v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}})$



Angular momentum about *S*:

$$\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + h\hat{\mathbf{k}}) \times (m(v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}))$$

$$\vec{\mathbf{L}}_S = m(xv_y - yv_x)\hat{\mathbf{k}} + (mv_x h\hat{\mathbf{j}} - mv_y\hat{\mathbf{i}})$$

Coordinate transformations:

$$x = R\cos\theta;\ \ y = R\sin\theta;$$

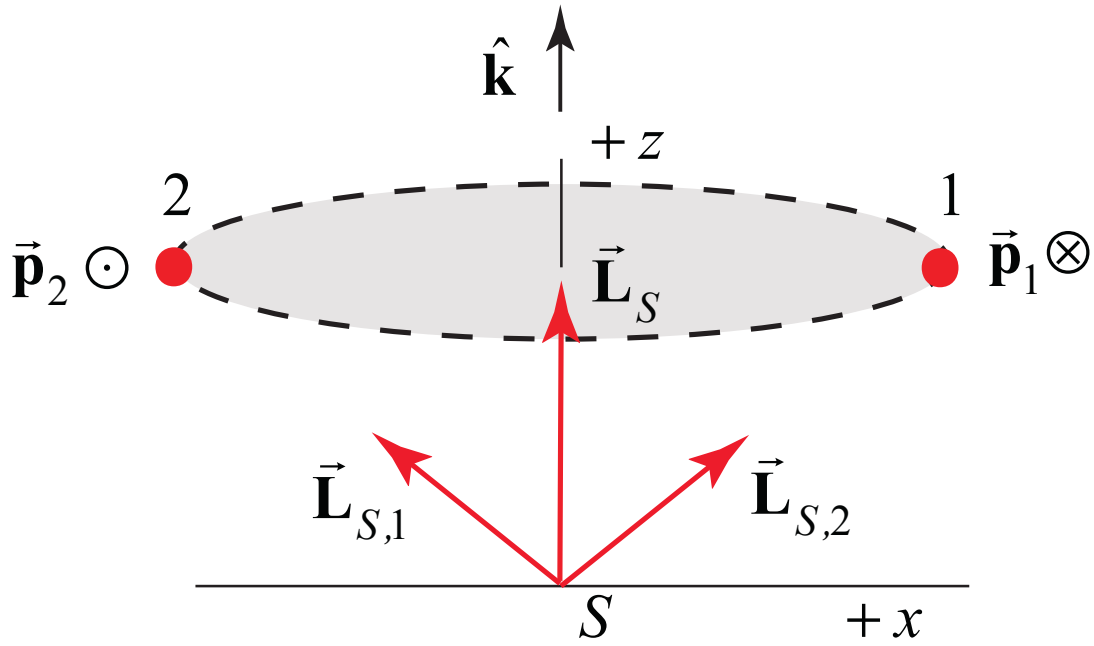
$$v_x = -v\sin\theta;\ \ v_y = v\cos\theta$$

Angular momentum about *S*:

$$\vec{\mathbf{L}}_S = mv(R(\cos^2\theta + \sin^2\theta)\hat{\mathbf{k}} + mvh(-\sin\theta\hat{\mathbf{j}} - \cos\theta\hat{\mathbf{i}})$$

$$\vec{\mathbf{L}}_S = mR^2\omega_z\hat{\mathbf{k}} - mR\omega_z h\hat{\mathbf{r}}$$

Angular Momentum of Two Particles:



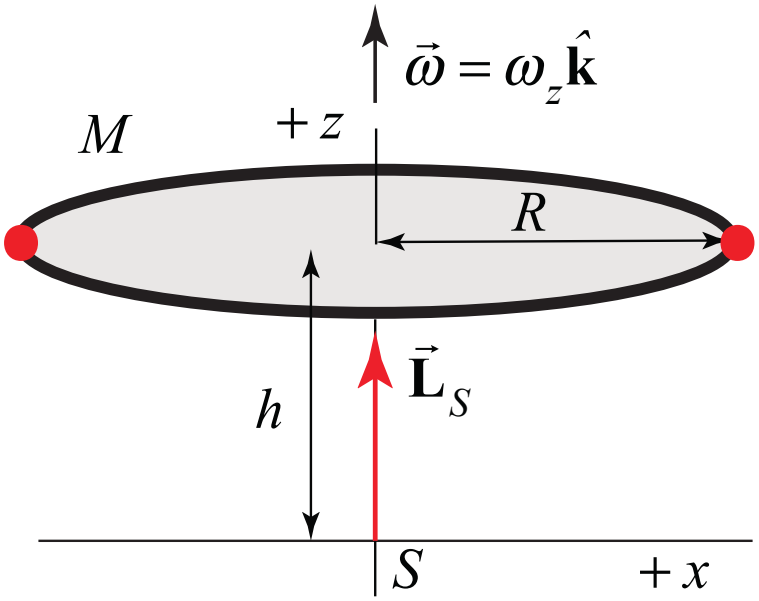
$$\vec{\mathbf{L}}_{S,1} = mR^2\omega_z\hat{\mathbf{k}} - hmR\omega_z\hat{\mathbf{r}}_1; \quad \vec{\mathbf{L}}_{S,2} = mR^2\omega_z\hat{\mathbf{k}} - hmR\omega_z\hat{\mathbf{r}}_2$$

$$\hat{\mathbf{r}}_1 = -\hat{\mathbf{r}}_2$$

$$\vec{\mathbf{L}}_S = \vec{\mathbf{L}}_{S,1} + \vec{\mathbf{L}}_{S,2} = 2mR^2\omega_z\hat{\mathbf{k}} = I_{cm}\omega_z\hat{\mathbf{k}}$$

Angular Momentum of a Ring

A circular ring of radius R and mass M rotates at an angular speed ω_z about the z-axis in a plane parallel to but a distance h above the x-y plane. Divide the ring into pairs of small objects with angular momentum about S



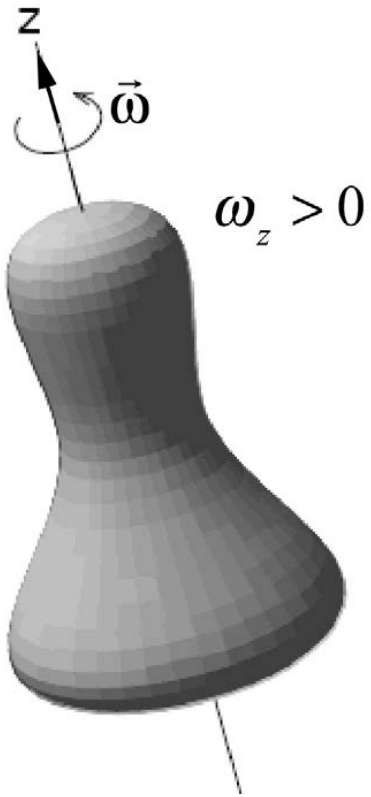
$$\vec{\mathbf{L}}_{S,1} + \vec{\mathbf{L}}_{S,2} = m_{pair} R^2 \omega_z \hat{\mathbf{k}}$$

Now sum over all pairs

$$\vec{\mathbf{L}}_S = \sum_{pairs} m_{pair} R^2 \omega_z \hat{\mathbf{k}} = MR^2 \omega_z \hat{\mathbf{k}} = I_{cm} \omega_z \hat{\mathbf{k}}$$

Angular momentum is independent of the location of S on the axis of symmetry

Angular Momentum of Cylindrically Symmetric Body



A cylindrically symmetric rigid body rotating about its symmetry axis at a constant angular velocity $\vec{\boldsymbol{\omega}} = \omega_z \hat{\mathbf{k}}$ with $\omega_z > 0$ has angular momentum about any point on its axis

$$\vec{\mathbf{L}}_z = \int_{\text{body}} d\vec{\mathbf{L}}_z = \int_{\text{body}} dm\, r_{dm}^2 \omega_z \hat{\mathbf{k}}$$

$$= \left(\int_{\text{body}} dm\, r_{dm}^2 \right) \omega_z \hat{\mathbf{k}}$$

$$= I_z \omega_z \hat{\mathbf{k}}$$

Torque Law for a Point Particle

Time derivative of the angular momentum about S:

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \frac{d}{dt}\left(\vec{\mathbf{r}}_S \times \vec{\mathbf{p}}\right)$$

Product rule

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \frac{d}{dt}\left(\vec{\mathbf{r}}_S \times \vec{\mathbf{p}}\right) = \frac{d\vec{\mathbf{r}}_S}{dt} \times \vec{\mathbf{p}} + \vec{\mathbf{r}}_S \times \frac{d}{dt}\vec{\mathbf{p}}$$

Key Fact:

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}_S}{dt} \Rightarrow \frac{d\vec{\mathbf{r}}_S}{dt} \times m\vec{\mathbf{v}} = \vec{\mathbf{v}} \times m\vec{\mathbf{v}} = \vec{\mathbf{0}}$$

Result:

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \vec{\mathbf{r}}_S \times \frac{d}{dt}\vec{\mathbf{p}} = \vec{\mathbf{r}}_S \times \vec{\mathbf{F}} = \vec{\boldsymbol{\tau}}_S$$

Summary: Torque and the Time Derivative of Angular Momentum: Point Particle

Torque about a point S is equal to the time derivative of the angular momentum about S,

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt}$$

generalizes to an extended rigid body.

Conservation of Angular Momentum:

Rotational dynamics: $$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt}$$

No torque: $$\vec{\mathbf{0}} = \vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt}$$

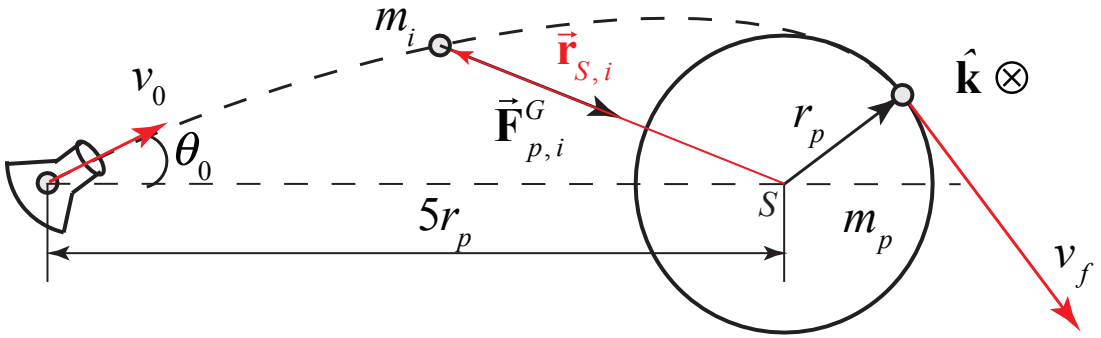
Change in Angular momentum is zero:

$$\Delta\vec{\mathbf{L}}_S \equiv \vec{\mathbf{L}}_{S,f} - \vec{\mathbf{L}}_{S,i} = \vec{\mathbf{0}}$$

Angular Momentum is constant:

$$\vec{\mathbf{L}}_{S,f} = \vec{\mathbf{L}}_{S,i}$$

Planetary Exploration



Gravitational Torque about center of planet is zero: $\vec{\tau}_S = \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}^G_{p,i} = \vec{\mathbf{0}}$

Angular momentum about center of planet is constant:

$$\vec{\mathbf{L}}_{S,0} = \vec{\mathbf{r}}_{S,0} \times m_i \vec{\mathbf{v}}_0 = 5r_p \hat{\mathbf{r}} \times (-m_i v_0 \cos\theta_0 \ \hat{\mathbf{r}} + m_i v_0 \sin\theta_0 \ \hat{\theta}) = 5r_p m_i v_0 \sin\theta_i \hat{\mathbf{k}}$$

$$\vec{\mathbf{L}}_{S,f} = \vec{\mathbf{r}}_{S,f} \times m_i \vec{\mathbf{v}}_f = r_p m_i v_f \hat{\mathbf{k}}$$

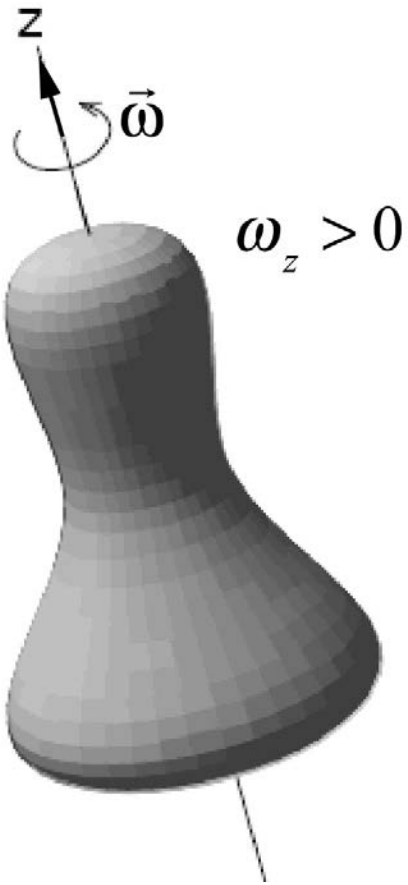
Conservation of angular momentum: $5r_p m_i v_0 \sin\theta_0 = r_p m_i v_f$

Conservation of energy: $$\frac{1}{2} m_i v_0^{\,2} - \frac{G m_i m_p}{5r_p} = \frac{1}{2} m_i v_f^{\,2} - \frac{G m_i m_p}{r_p}$$

Initial angle solution:

$$\frac{1}{2} m_i v_0^{\,2} - \frac{G m_i m_p}{5r_p} = \frac{1}{2} m_i (5v_0 \sin\theta_0)^2 - \frac{G m_i m_p}{r_p} \Rightarrow \theta_0 = \sin^{-1}\left(\frac{1}{5} \sqrt{\frac{8Gm_p}{5r_p v_0^{\,2}} + 1} \right)$$

Kinetic Energy of Cylindrically Symmetric Body



A cylindrically symmetric rigid body with moment of inertia I_z rotating about its symmetry axis at a constant angular velocity

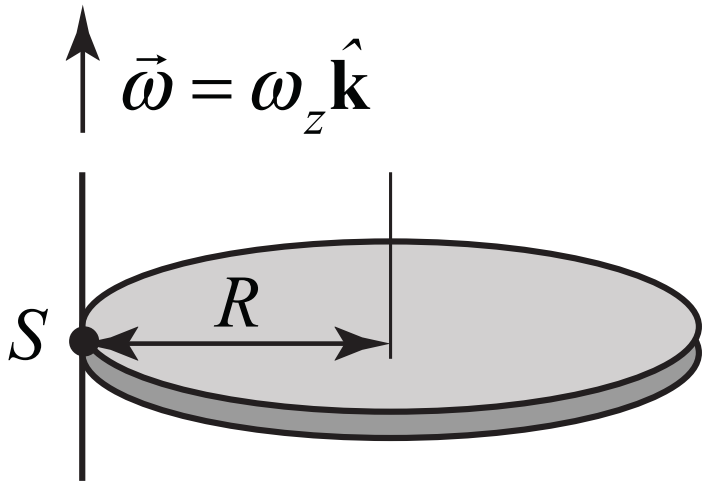
$$\vec{\boldsymbol{\omega}} = \omega_z \, \hat{\mathbf{k}}, \; \omega_z > 0$$

Angular Momentum: $L_z = I_z \omega_z$

Kinetic Energy: $K_{rot} = \frac{1}{2} I_z \omega_z^2 = \frac{L_z^2}{2 I_z}$

Ang. Mom. of Disk about point on rim

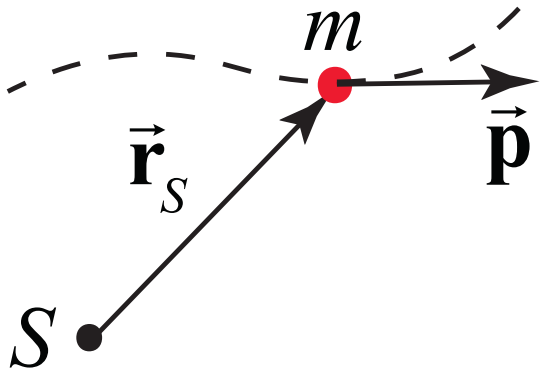
The moment of inertia of the disk about an axis that passes through the rim of the disk perpendicular to its plane is $I = I_{cm} + MR^2 = (3/2)MR^2$. So the magnitude of its angular momentum about S is $L = (3/2)MR^2 \omega$.



Angular Momentum Summary

Angular momentum of point particle about the point S

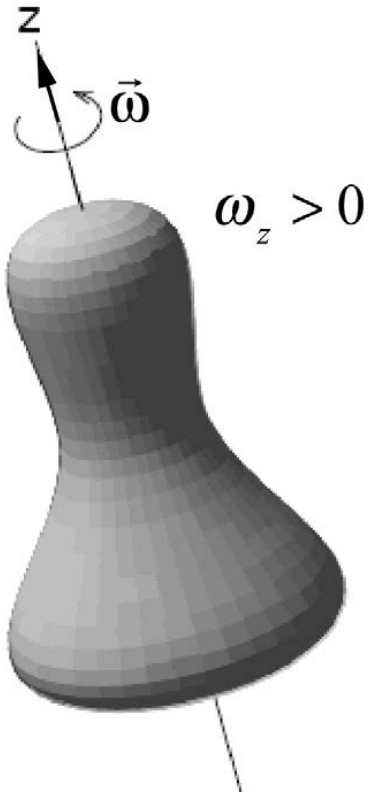
$$\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}}$$



For a rigid body rotating about the z-axis

The *z*-component of the angular angular momentum about the *z*-axis is given by

$$L_z = I_z \omega_z$$



For a symmetric body rotating about *z*-axis

$$\vec{\mathbf{L}}_{axis} = I_z \vec{\omega}$$

Torque and Angular Momentum

Angular momentum about a point *S* is defined as

$$\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}}$$

Torque about a point *S* is equal to the time derivative of the angular momentum about *S* .

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt}$$

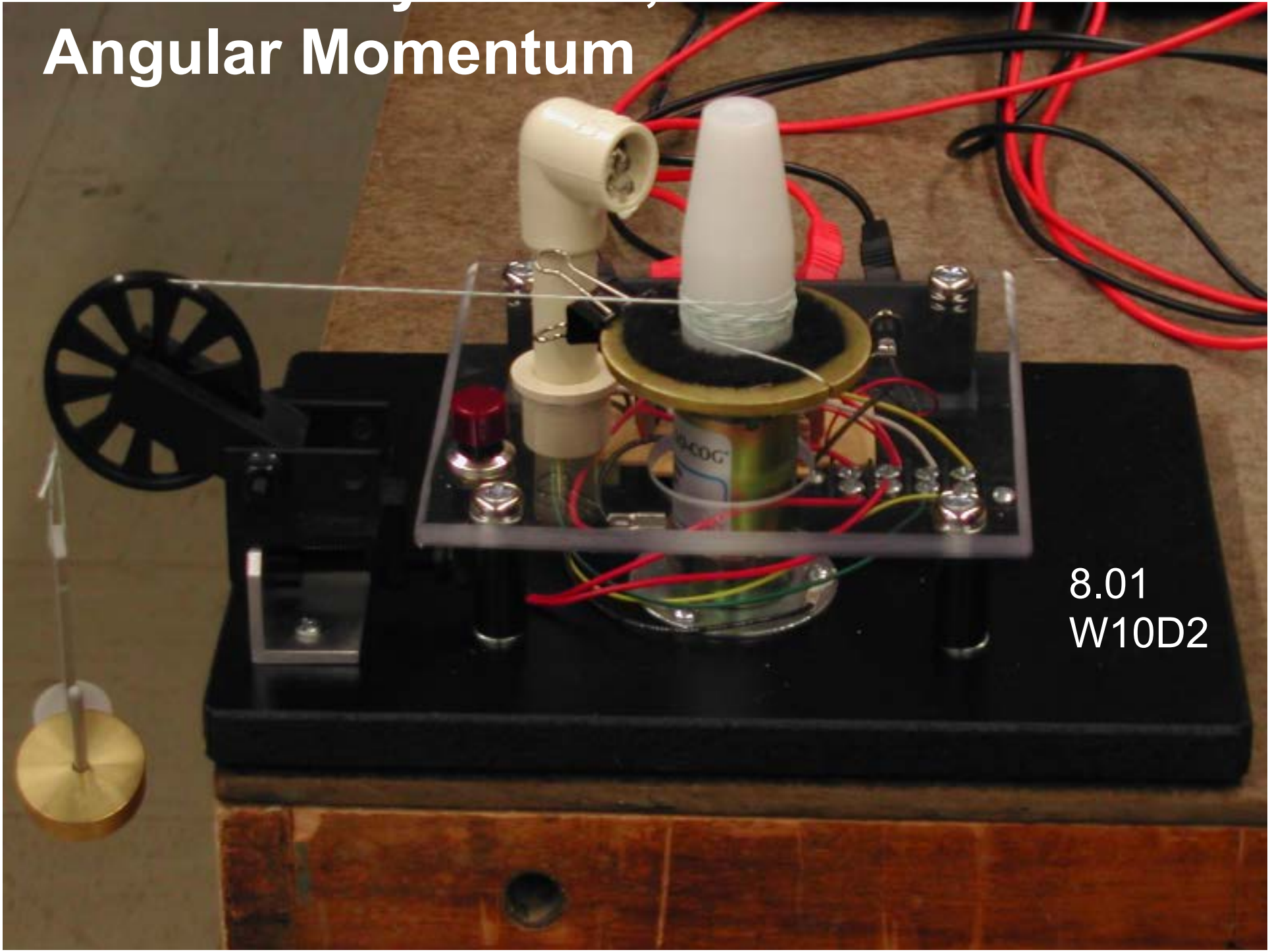
Angular Impulse-angular Momentum

$$\int_{t_i}^{t_f} \vec{\boldsymbol{\tau}}_S \, dt = \vec{\mathbf{L}}_{S,f} - \vec{\mathbf{L}}_{S,i}$$

When torque about *S* is zero, angular momentum is constant

$$\vec{\mathbf{L}}_{S,f} = \vec{\mathbf{L}}_{S,i}$$

Angular Momentum



8.01
W10D2

Methodology for Torque Law (Fixed Axis Rotation)

$$\vec{\tau}_S = I_S \vec{\alpha}$$

1. Identify all the different objects that are either rotating, translating or both.

2. For translational motion: choose coordinate system with unit vectors, draw force diagram and apply Newton's Second Law.

3. For rotational motion create a torque diagram which consists of the following:

a. Define a set of three unit vectors that satisfy the right hand rule. Two vectors are in the plane of rotation and the third is perpendicular to that plane. Example: $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}} = \hat{\mathbf{n}}_{RHR})$ or $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{k}} = \hat{\mathbf{n}}_{RHR})$. Your choice of $\hat{\mathbf{n}}_{RHR}$, corresponds to the direction of increasing angle *θ*. Then the angular acceleration in the **torque law** is $\vec{\alpha} = (d^2\theta / dt^2)\hat{\mathbf{k}} = \alpha_z \hat{\mathbf{k}}$

b. For each force $\vec{\mathbf{F}}$, draw an arrow where the force is acting.

c. Choose the point *S* about which you plan to calculate torque $\vec{\tau}_S = \vec{\mathbf{r}}_{S,F} \times \vec{\mathbf{F}}$

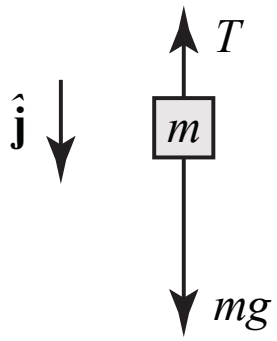
d. On your diagram draw the vector $\vec{\mathbf{r}}_{S,F}$ from the point *S* to where the force $\vec{\mathbf{F}}$ is acting.

e. Calculate the vector cross product for torque, $\vec{\tau}_S = \vec{\mathbf{r}}_{S,F} \times \vec{\mathbf{F}}$ be careful with signs.

f. Now apply $\vec{\tau}_S = I_S \vec{\alpha}$

Moment of Inertia of Rotor

Falling weight: Force and rotational equations while weight is descending:



$$mg - T = ma$$

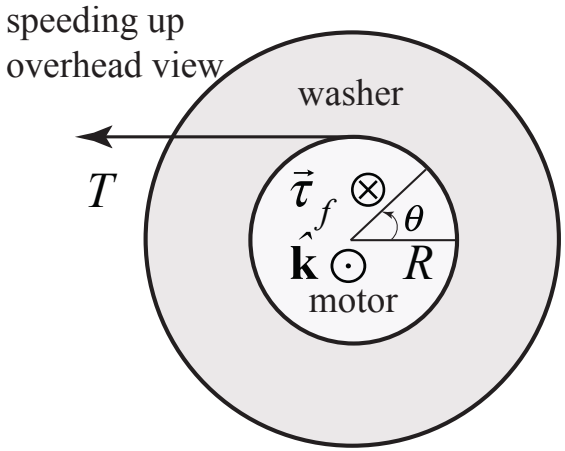
$$RT - \tau_f = I_r \alpha_1$$

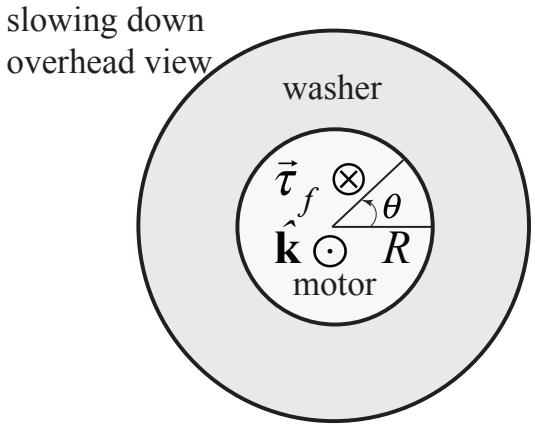
Constraint: $a = R\alpha_1$

Detached weight: $-\tau_f = I_r \alpha_2$

Solve for moment of inertia of the rotor:

$$Rm(g - R\alpha_1) + I_r \alpha_2 = I_r \alpha_1 \Rightarrow I_r = \frac{Rm(g - R\alpha_1)}{(\alpha_1 - \alpha_2)}$$





Constancy of Angular Momentum

Angular momentum about fixed axis $\vec{\mathbf{L}}_S^{\text{sys}} = I_S \vec{\omega}$

Assumption: all internal torques cancel in pairs

Rotational dynamics $\vec{\boldsymbol{\tau}}_S^{\text{ ext}} = \dfrac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt}$

No external torques $\vec{\mathbf{0}} = \vec{\boldsymbol{\tau}}_S^{\text{ ext}} = \dfrac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt}$

Change in Angular momentum is zero

$$\Delta\vec{\mathbf{L}}_S^{\text{sys}} \equiv \vec{\mathbf{L}}_{S,f}^{\text{sys}} - \vec{\mathbf{L}}_{S,i}^{\text{sys}} = \vec{\mathbf{0}}$$

Angular Momentum is constant $\vec{\mathbf{L}}_{S,f}^{\text{sys}} = \vec{\mathbf{L}}_{S,i}^{\text{sys}}$

Expt. Angular Impulse and Change in Angular Momentum

Angular impulse: $\vec{\mathbf{J}} = (\vec{\boldsymbol{\tau}}_S^{\,\text{ext}})_{\text{ave}} \Delta t_{\text{int}} = \Delta \vec{\mathbf{L}}_S^{\text{sys}}$

Change in angular momentum:

$$\Delta \vec{\mathbf{L}}_S^{\text{sys}} = \vec{\mathbf{L}}_{S,f}^{\text{sys}} - \vec{\mathbf{L}}_{S,i}^{\text{sys}} = (I_r + I_w)\vec{\omega}_f - I_r \vec{\omega}_i$$

Rotational dynamics:

$$\int_{t_i}^{t_f} \vec{\boldsymbol{\tau}}_S^{\,\text{ext}}\, dt = \vec{\mathbf{L}}_{S,f}^{\text{sys}} - \vec{\mathbf{L}}_{S,i}^{\text{sys}}$$

$$(\vec{\boldsymbol{\tau}}_S^{\,\text{ext}})_{\text{ave}} \Delta t_{\text{int}} = (I_r + I_w)\vec{\omega}_f - I_r \vec{\omega}_i$$

Ang. Mom.: Slow Angular Collision

At t_1 = 1.90 s, the magnitude of angular momentum of the system is

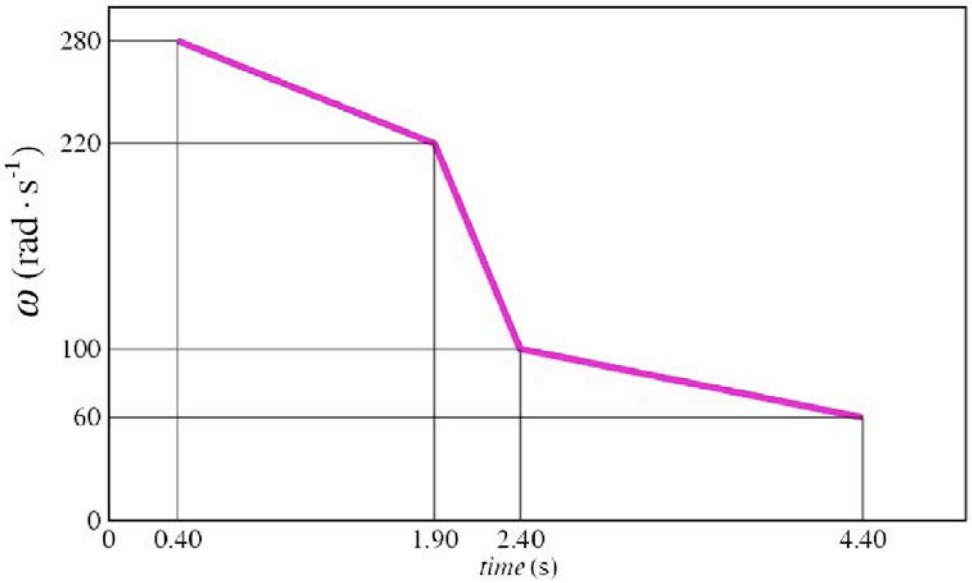
$$L(t_1) = I\omega_1 = I(220\ \text{rad}\cdot\text{s})$$

At t_2 = 2.40 s, the magnitude of angula momentum of the system is

$$L(t_2) = 2I\omega_2$$

$$= 2I(100\ \text{rad}\cdot\text{s}) = I(200\ \text{rad}\cdot\text{s})$$

So the magnitude of the angular momentum has decreased due to the frictional torque acting on the system during the interval [t_1, t_2].



$$-\tau_f \Delta t = L(t_2) - L(t_1) \Rightarrow \tau_f = -\frac{L(t_2) - L(t_1)}{\Delta t}$$

$$\tau_f = 1.7 \times 10^{-3}\ \text{N}\cdot\text{m}$$

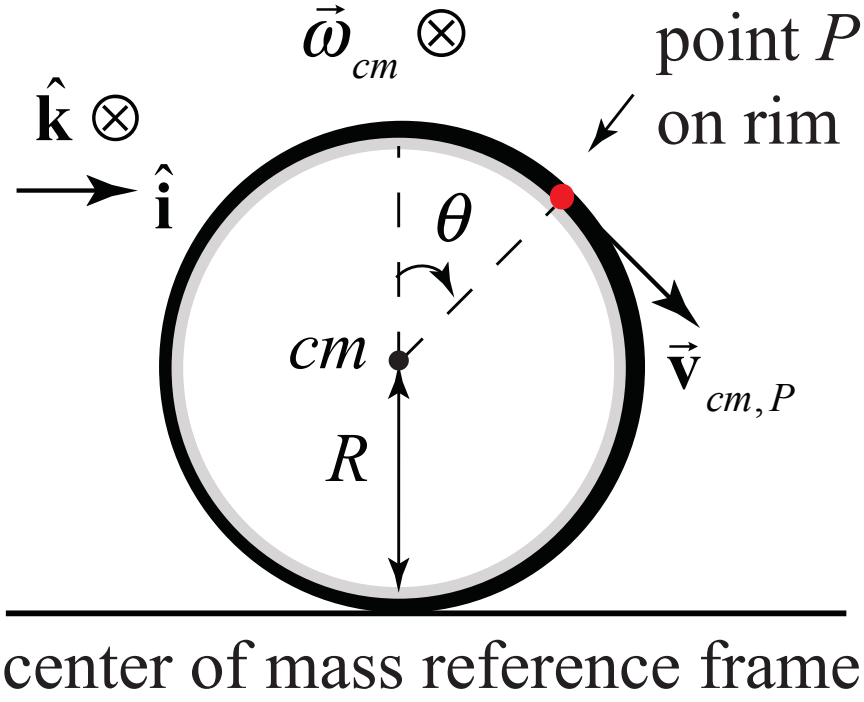
Rigid Body: Rotational and Translational



8.01
W11D1

Rolling Bicycle Wheel: Center of Mass Reference Frame

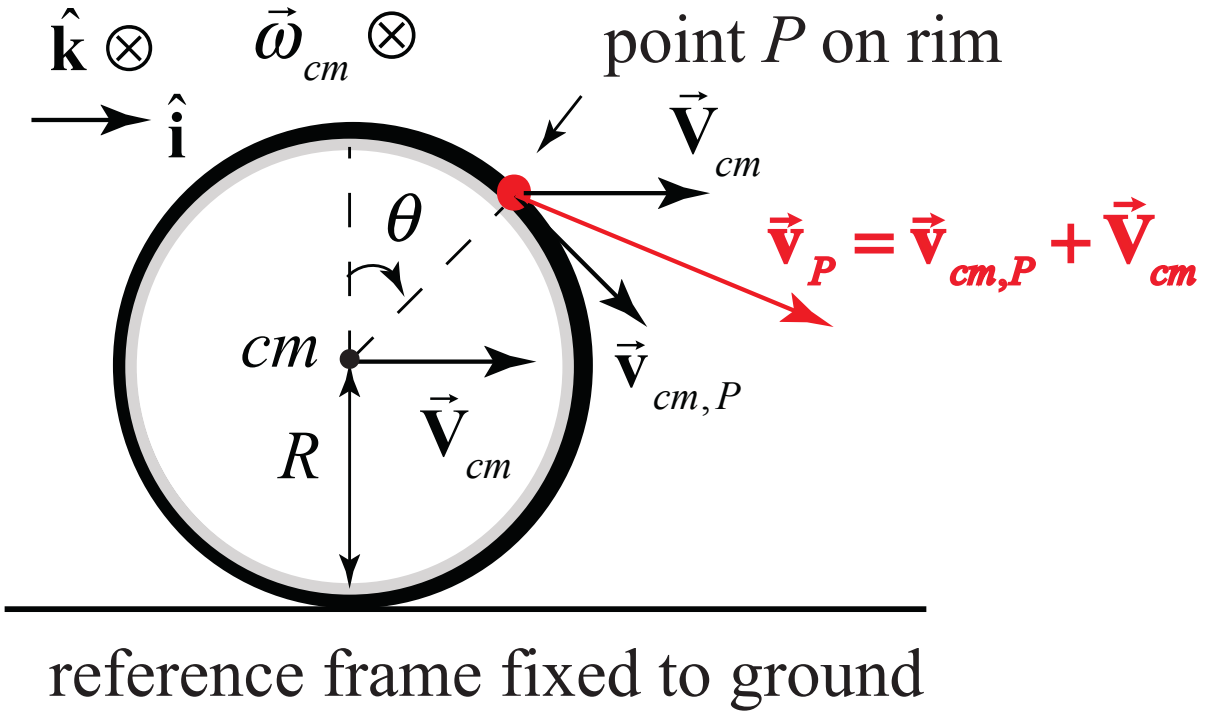
Motion of point *P* on rim of a rolling bicycle wheel



Rolling Bicycle Wheel: Ground Reference Frame

Motion of point *P* on rim of a rolling bicycle wheel

$$\vec{\mathbf{v}}_P = \vec{\mathbf{v}}_{cm,P} + \vec{\mathbf{V}}_{cm}$$



Rolling Bicycle Wheel: Constraint Relations

Rolling without slipping:

$$\Delta s = \Delta X_{\text{cm}}$$

$$R\omega_{\text{cm}} = V_{\text{cm}}$$

Rolling and skidding:

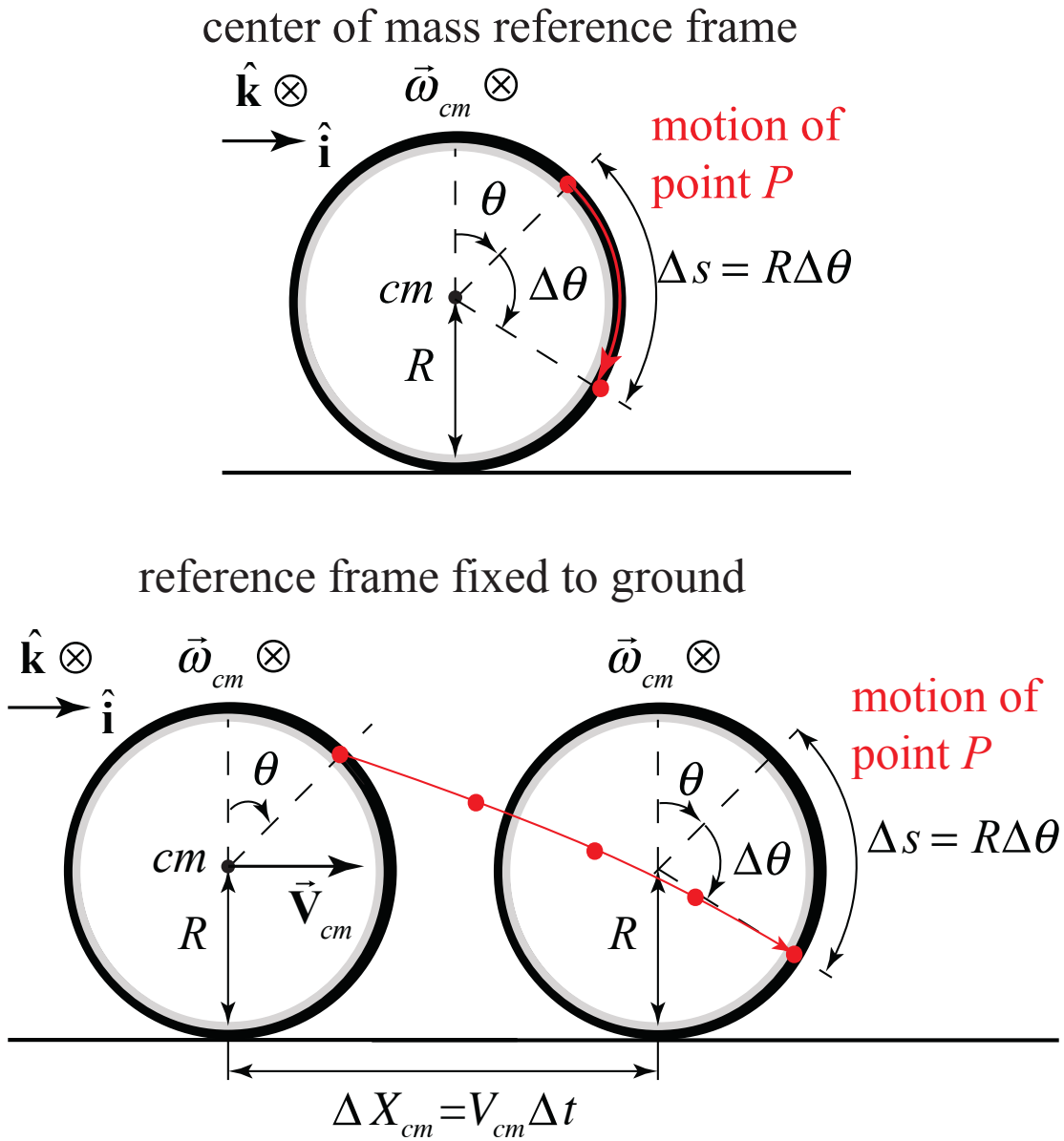
$$\Delta s < \Delta X_{\text{cm}}$$

$$R\omega_{\text{cm}} < V_{\text{cm}}$$

Rolling and slipping:

$$\Delta s > \Delta X_{\text{cm}}$$

$$R\omega_{\text{cm}} > V_{\text{cm}}$$



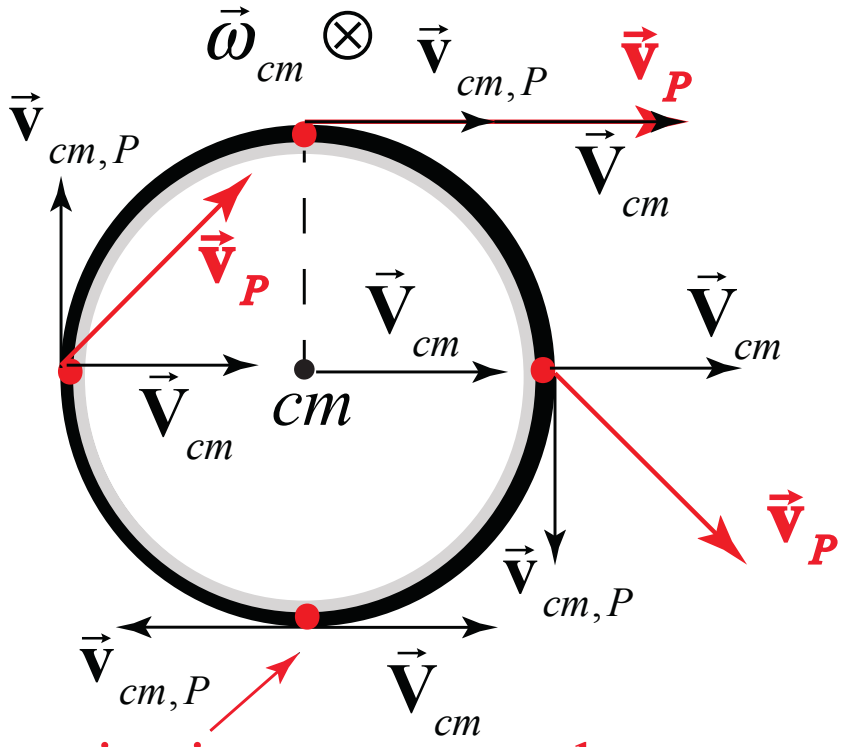
Rolling Without Slipping: Ground Reference Frame

The velocity of a point on the rim in the reference frame fixed to the ground is given by

$$\vec{\mathbf{v}}_P = \vec{\mathbf{v}}_{cm,P} + \vec{\mathbf{V}}_{cm}$$

Contact point with the ground is instantaneously at rest.

reference frame fixed to ground



contact point instantaneously at rest

Force and Torque Laws

The momentum of a system remains constant unless the system is acted on by an **external** force in which case the acceleration of center of mass satisfies

$$\vec{\mathbf{F}}_{\text{ext}} = \frac{d\vec{\mathbf{p}}_{\text{sys}}}{dt} = m_{\text{sys}} \frac{d\vec{\mathbf{V}}_{cm}}{dt} = m_{\text{sys}} \vec{\mathbf{A}}_{cm}$$

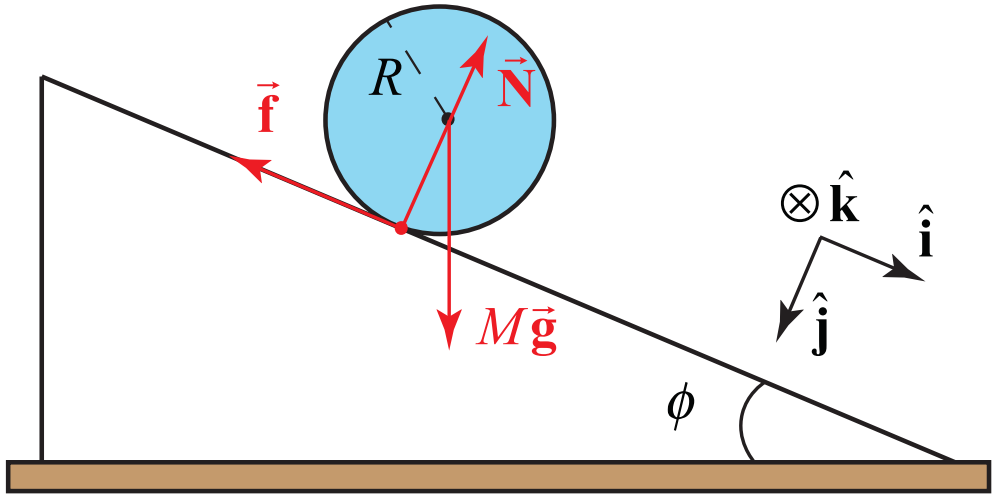
The total external torque produces an angular acceleration about the center-of-mass

$$\vec{\boldsymbol{\tau}}_{\text{cm}}^{\text{ext}} = I_{\text{cm}} \vec{\boldsymbol{\alpha}}_{\text{cm}} = \frac{d\vec{\mathbf{L}}_{\text{cm}}}{dt}$$

Cylinder Rolling w/o Slipping Down Inclined Plane - Torque Method Sol.

Torque equation:

$$f R\hat{\mathbf{k}} = I_{cm}\alpha_z\hat{\mathbf{k}} \Rightarrow f = I_{cm}\alpha_z / R$$



Newton's Second Law:

$$Mg\sin\phi - f = Ma_x$$

$$N - Mg\cos\phi = 0.$$

Constraint: $\alpha_z = a_x / R$

Acceleration:

$$Mg\sin\phi - (I_{cm} / R^2)a_x = Ma_x \Rightarrow$$

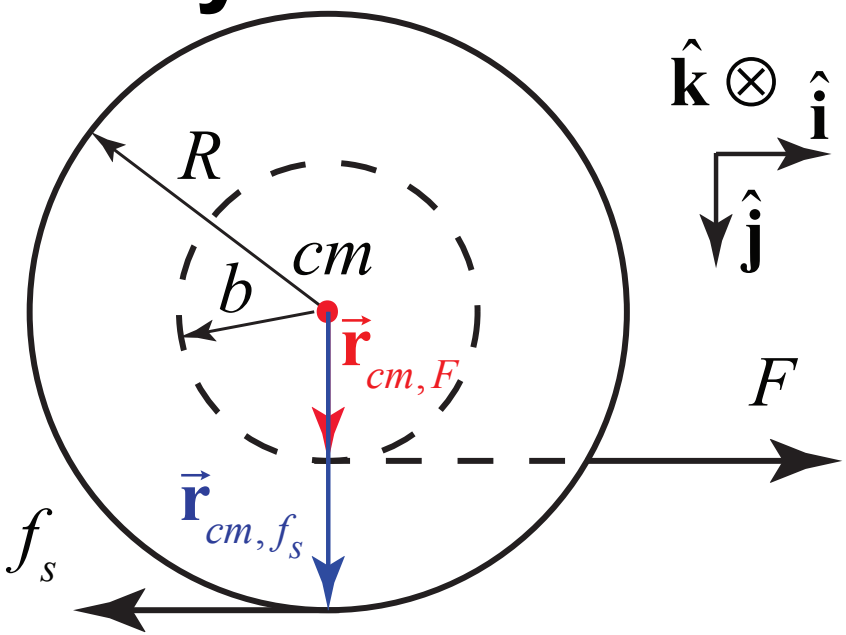
$$a_x = \frac{Mg\sin\phi}{M + (I_{cm} / R^2)}$$

Pulling a Yo-yo

Newton's Second Law:

$$F - f_s = ma_x \Rightarrow$$

$$a_x = \frac{F - f_s}{m}$$



Constraint:

$$\alpha_z = a_x / R = (F - f_s) / mR$$

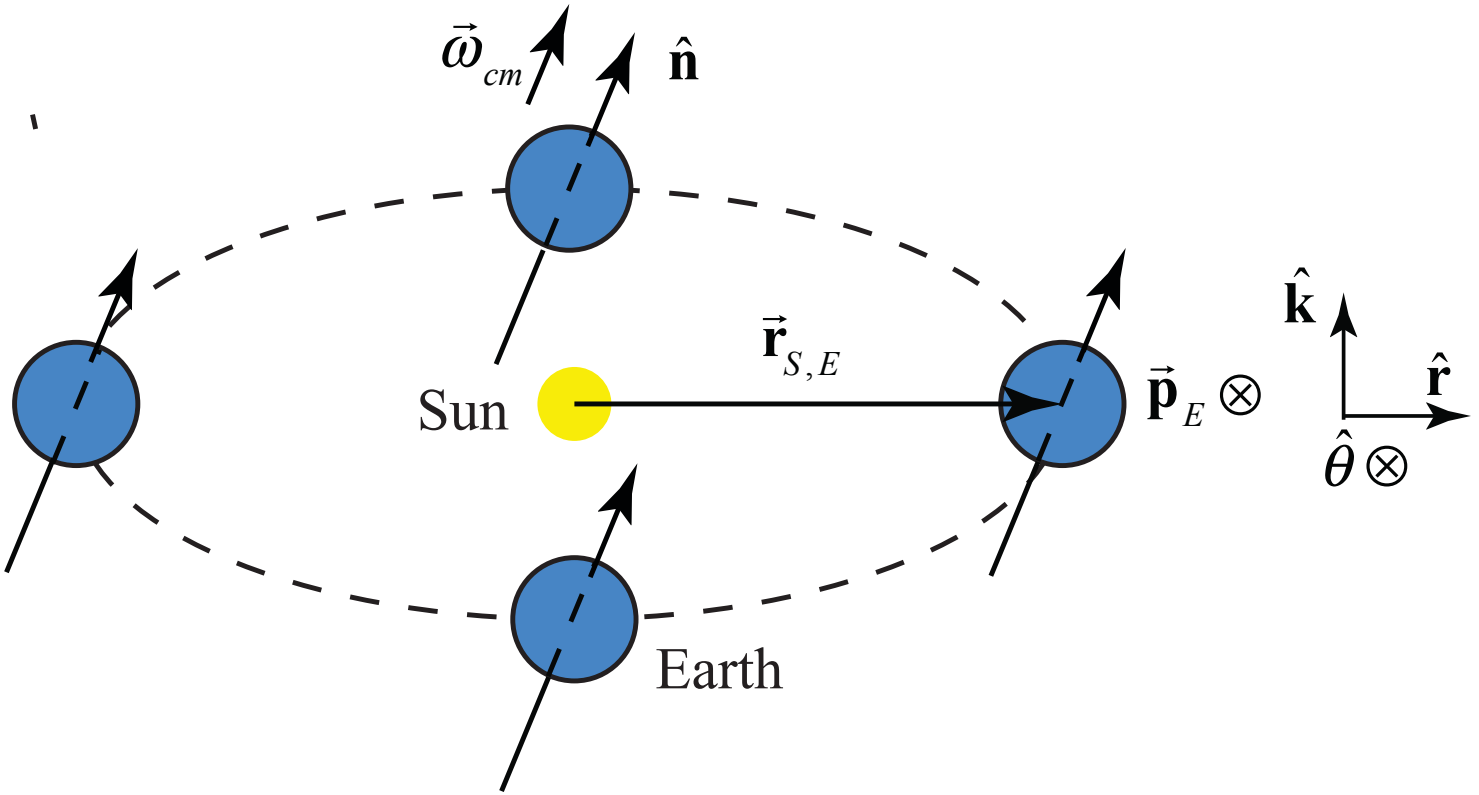
Torque equation:

$$Rf_s - bF = I_{cm}\alpha_z = (1/2)mR^2\alpha_z = (1/2)R(F - f_s) \Rightarrow$$

$$(3/2)f_s R = (b + (1/2)R)F \Rightarrow F = \frac{3f_s R}{2b + R}$$

Maximum Value of F: $f_s = \mu_s mg \Rightarrow F_{\max} = \dfrac{3\mu_s mgR}{2b + R}$

Rigid Body Motion: Energy and Angular Momentum



8.01
W11D2

Kinetic Energy of Fixed Axis Rotation and Translation

Angular momentum about center of mass

$$\vec{\mathbf{L}}_{cm,f} = I_{cm}\omega_f \hat{\mathbf{k}}$$

Kinetic energy of rotation about center-of-mass

$$K_{\text{rot}} = \frac{1}{2} I_{\text{cm}} \omega_{\text{cm}}^2 = \frac{L_{cm}^2}{2I_{\text{cm}}}$$

Translational kinetic energy

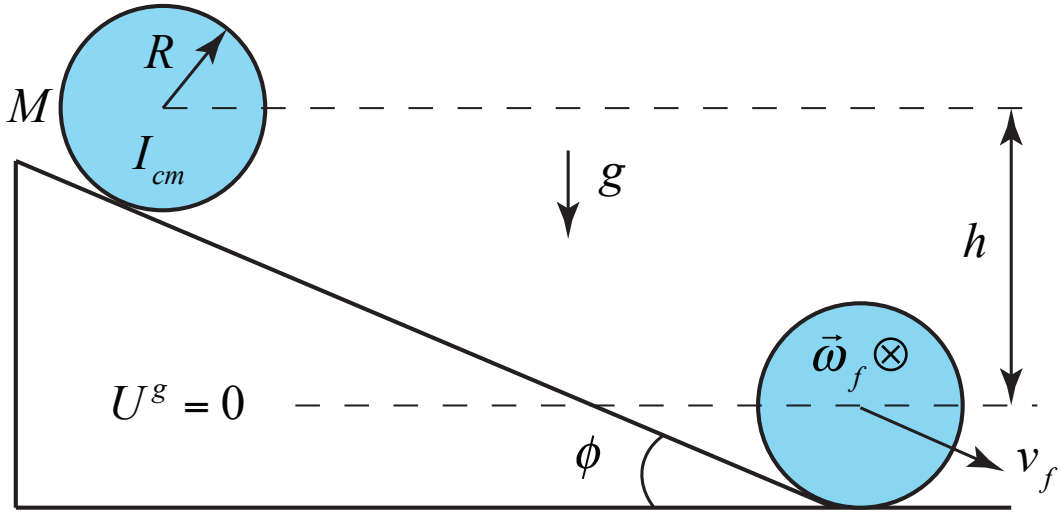
$$K_{\text{trans}} = \frac{1}{2} m v_{\text{cm}}^2 = \frac{p_{cm}^2}{2m}$$

Kinetic energy is sum

$$K = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2} m v_{\text{cm}}^2 + \frac{1}{2} I_{\text{cm}} \omega_{cm}^2 = \frac{p_{cm}^2}{2m} + \frac{L_{cm}^2}{2I_{\text{cm}}}$$

Cylinder Rolling Without Slipping Down an Inclined Plane − Energy

Initial energy: $E_i = Mgh$



Final energy;

$$E_f = (1/2)Mv_f^2 + (1/2)I_{cm}\omega_f^2$$

Rolling without slipping:

$$\omega_f = v_f / R \Rightarrow E_f = (1/2)(M + (I_{cm} / R^2))v_f^2$$

Conservation of energy: $E_i = Mgh = (1/2)(M + (I_{cm} / R^2))v_f^2$

Speed at bottom:

$$v_f = \sqrt{\frac{2hMg}{M + (I_{cm} / R^2)}}$$

Angular Momentum for 2-Dim. Rotation and Translation

The angular momentum for a translating and rotating object is given by

$$\vec{\mathbf{L}}_S = \vec{\mathbf{R}}_{S,cm} \times \vec{\mathbf{p}}^{\text{sys}} + \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{cm,i} \times m_i \vec{\mathbf{v}}_{cm,i}$$

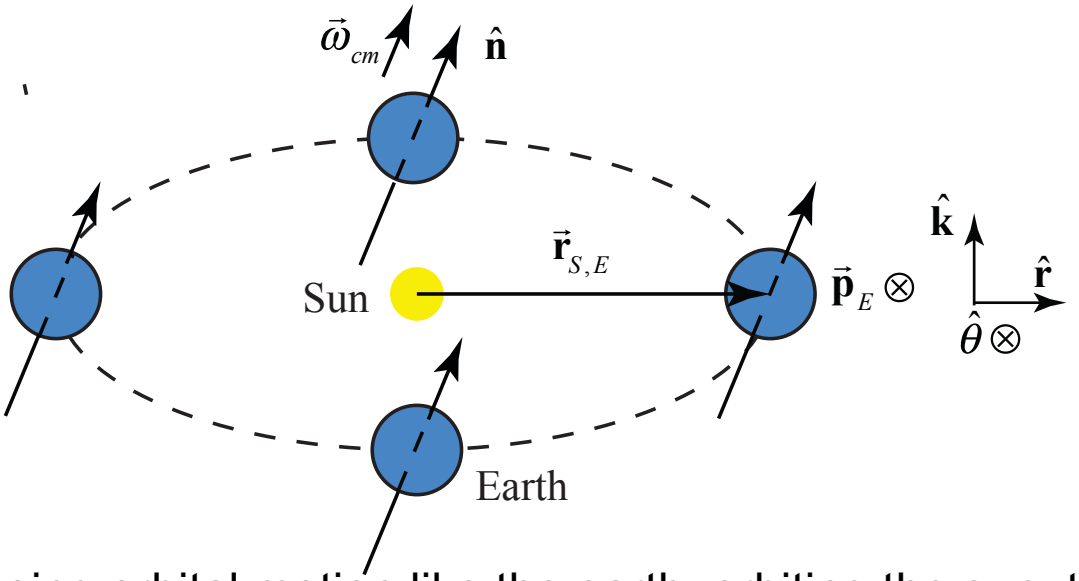
Angular momentum arising from translational of center-of-mass

$$\vec{\mathbf{L}}_{S,cm}^{trans} = \vec{\mathbf{R}}_{S,cm} \times \vec{\mathbf{p}}^{\text{sys}}$$

The second term is the angular momentum arising from rotation about center-of mass,

$$\vec{\mathbf{L}}_{cm}^{rot} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{cm,i} \times m_i \vec{\mathbf{v}}_{cm,i} = I_{cm} \vec{\boldsymbol{\omega}}_{cm}$$

Earth's Angular Momentum



For a body undergoing orbital motion like the earth orbiting the sun, the two terms can be thought of as an orbital angular momentum about the center-of-mass of the earth-sun system, denoted by *S*,

$$\vec{\mathbf{L}}_{S,\mathrm{cm}} = \vec{\mathbf{R}}_{S,\mathrm{cm}} \times \vec{\mathbf{p}}^{\,\mathrm{sys}} = r_{s,e} m_e v_{cm} \hat{\mathbf{k}}$$

Spin angular momentum about center-of-mass of earth

$$\vec{\mathbf{L}}_{cm}^{spin} = I_{\mathrm{cm}} \vec{\boldsymbol{\omega}}_{\mathrm{spin}} = \frac{2}{5} m_e R_e^{\,2} \omega_{\mathrm{spin}} \hat{\mathbf{n}}$$

Total angular momentum about *S*

$$\vec{\mathbf{L}}_S^{\mathrm{total}} = r_{s,e} m_e v_{cm} \hat{\mathbf{k}} + \frac{2}{5} m_e R_e^2 \omega_{\mathrm{spin}} \hat{\mathbf{n}}$$

Rotational and Translational Motion

Quantity	Fixed Axis Rotation	Translation
Momentum (mag.)		$p = mV_{\text{cm}} = \sqrt{2mK_{tran}}$
Ang. Momentum (mag.)	$L_{\text{cm}} = I_{\text{cm}}\boldsymbol{\omega} = \sqrt{2I_{cm}K_{rot}}$	
Force		$\vec{\mathbf{F}}_{\text{ext}} = d\vec{\mathbf{p}}_{\text{sys}} / dt = m_{\text{total}}\vec{\mathbf{A}}_{\text{cm}}$
Torque	$\vec{\boldsymbol{\tau}}_{\text{cm}} = d\vec{\mathbf{L}}_{\text{cm}} / dt = I_{cm}\vec{\boldsymbol{\alpha}}$	
Kinetic Energy	$K_{\text{rot}} = (1/2)I_{\text{cm}}\boldsymbol{\omega}^2$ $K_{\text{rot}} = L_{\text{cm}}^2 / 2I_{\text{cm}}$	$K_{\text{trans}} = (1/2)mV_{\text{cm}}^2$ $K_{\text{trans}} = p^2 / 2m$
Work	$W = \int_{\theta_i}^{\theta_f} \tau_S \, d\theta$	$W = \int_i^f \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$
Power	$P_{rot} = \tau_S \, \omega$	$P = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}$

Rolling Cylinder

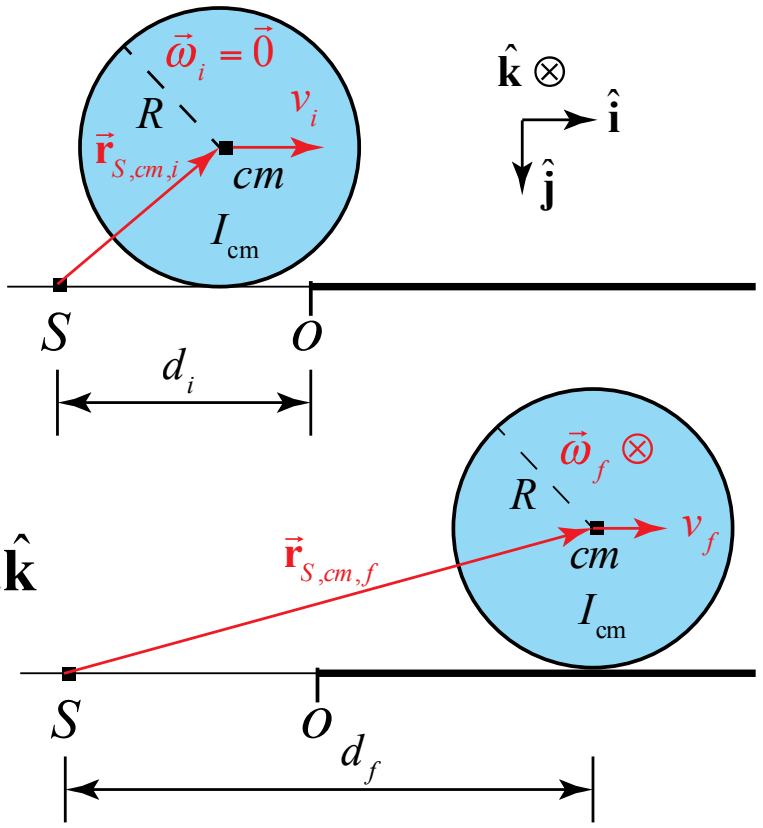
Initial angular momentum about S;

$$\vec{\mathbf{L}}_{S,i} = \vec{\mathbf{r}}_{S,\mathrm{cm,i}} \times m\vec{\mathbf{v}}_i = (d_i\hat{\mathbf{i}} - R\hat{\mathbf{j}}) \times mv_i\hat{\mathbf{i}} = Rmv_i\hat{\mathbf{k}}$$

The final angular momentum about S;

$$\vec{\mathbf{L}}_{S,f} = \vec{\mathbf{r}}_{S,\mathrm{cm,f}} \times m\vec{\mathbf{v}}_f + I_{cm}\vec{\omega}_f$$

$$= (d_f\hat{\mathbf{i}} - R\hat{\mathbf{j}}) \times mv_f\hat{\mathbf{i}} + I_{cm}\omega_f\hat{\mathbf{k}} = Rmv_f\hat{\mathbf{k}} + I_{cm}\omega_f\hat{\mathbf{k}}$$



Rolling without slipping condition

$$\omega_f = v_f / R$$

Angular momentum is constant so

$$mRv_i\hat{\mathbf{k}} = (mRv_f + (1/2)MR^2v_f / R)\hat{\mathbf{k}} = (3/2)MRv_f$$

Conclusion: $v_f = (2/3)v_i$

Kepler's Laws and Newtonian Gravitation



8.01
W12D1

http://wwwmpa.mpa-garching.mpg.de/gadget/collision.mpg

Announcements

Week 12:
LS 1 and LS 2 due Sunday at 11 pm

Week 12: Problem Set 11 due Wednesday at 11 pm

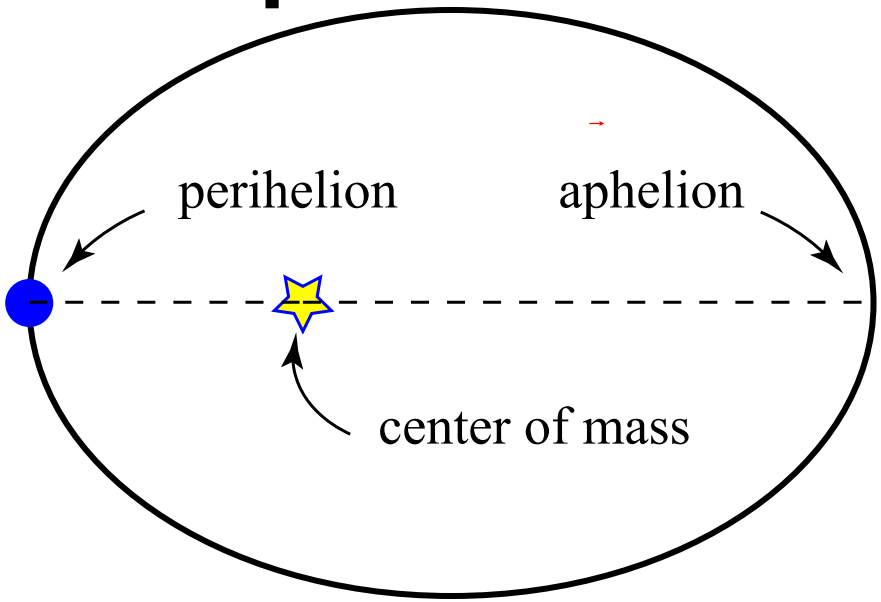
Wednesday Nov 22 Open Class

Thurs-Fri No Class Holiday

Week 13:
LS 1 and LS 2 due Sunday at 11 pm
LS 3 and LS 4 due Tuesday at 11 pm

Week 13: Problem Set 12 due Friday Dec 1 at 11 pm

Kepler's Laws

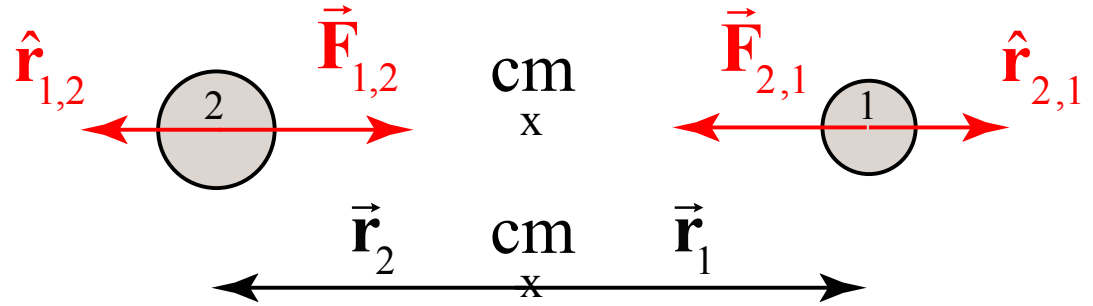


1. The orbit of every planet is an ellipse with the Sun at one of the two foci.

2. A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

3. The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

Kepler Problem



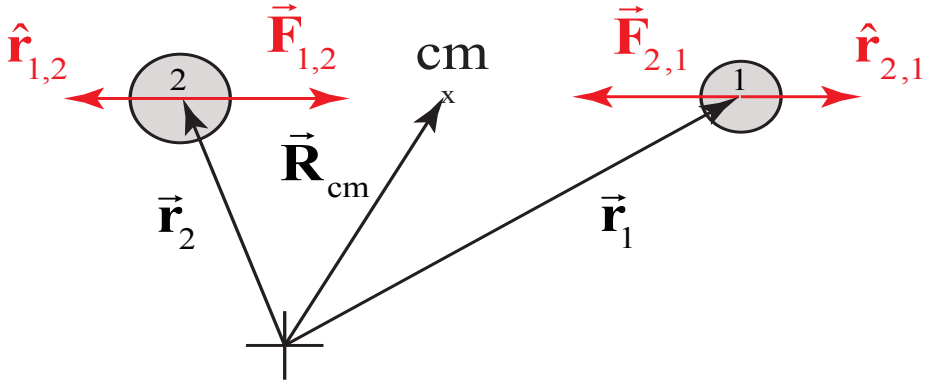
$$\vec{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1$$

Goal: Can we describe the motion of two celestial bodies under the influence of gravitation?

$$\vec{\mathbf{F}}_{1,2}(r) = -G\frac{m_1 m_2}{r_{1,2}^2}\hat{\mathbf{r}}_{1,2}$$

Approach: We will reduce the two body problem to an "effective" one body problem and then apply energy methods to obtain the equations describing the motion. The solution for the orbit equation is complicated. If you would like to see the details, we recommend Chapter 25 in the course textbook.

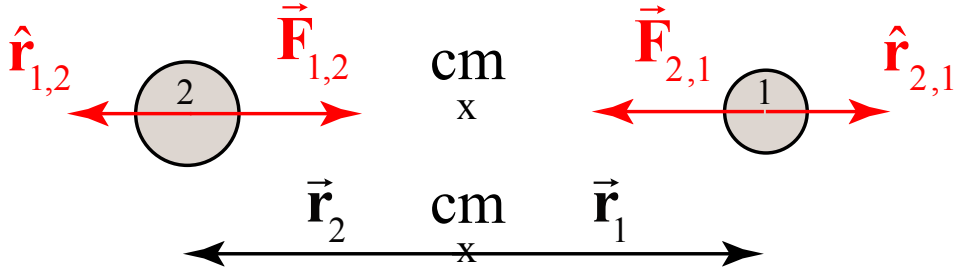
Center of Mass & Relative Distance



Recall the definition of the center of mass of two bodies:

$$(m_1 + m_2)\vec{\mathbf{R}}_{cm} = m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2$$

We will work in the center of mass reference frame. This is because we will be more interested in the ***relative*** motion for this problem.



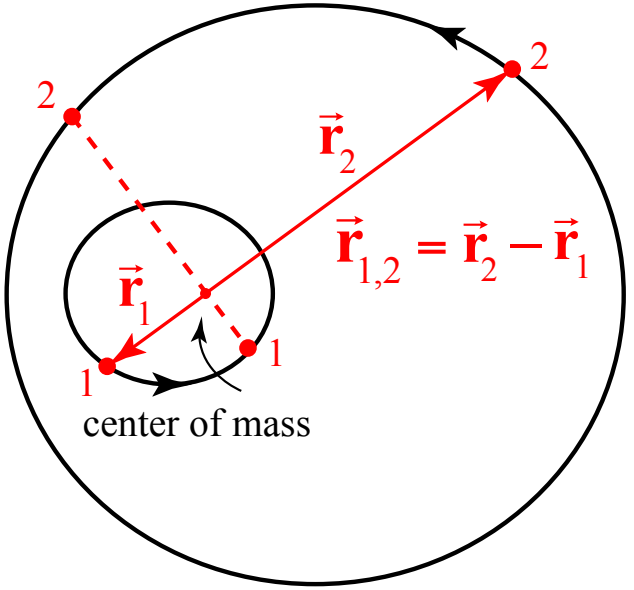
$$\vec{\mathbf{R}}_{cm} = \vec{\mathbf{0}} \Rightarrow \vec{\mathbf{r}}_1 = -(m_2 / m_1)\vec{\mathbf{r}}_2$$

Reduced Mass

Define $\vec{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1$ .Use the condition that to write $\vec{\mathbf{R}}_{cm} = \vec{\mathbf{0}} \Rightarrow \vec{\mathbf{r}}_1 = -(m_2 / m_1)\vec{\mathbf{r}}_2$

$$\vec{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1 = -\left(\frac{m_1}{m_2} + 1\right)\vec{\mathbf{r}}_1 = -\left(\frac{m_1 + m_2}{m_2}\right)\vec{\mathbf{r}}_1 \Rightarrow$$

$$\vec{\mathbf{r}}_1 = -\left(\frac{m_2}{m_2 + m_1}\right)\vec{\mathbf{r}}_{1,2} = -\left(\frac{m_1 m_2}{m_2 + m_1}\right)\frac{\vec{\mathbf{r}}_{1,2}}{m_1} = -\frac{\mu}{m_1}\vec{\mathbf{r}}_{1,2}$$



where the reduced mass of the system μ is defined by

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \Rightarrow \mu = \frac{m_1 m_2}{m_1 + m_2}$$

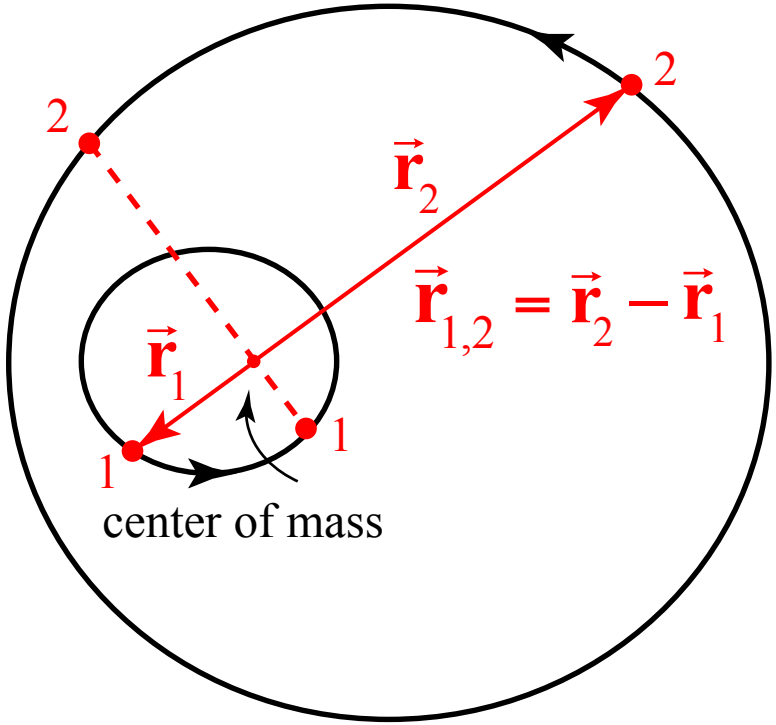
Similarly

$$\vec{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1 = \left(1 + \frac{m_2}{m_1}\right)\vec{\mathbf{r}}_2 \Rightarrow \vec{\mathbf{r}}_2 = \left(\frac{m_1}{m_2 + m_1}\right)\vec{\mathbf{r}}_{1,2} = \frac{\mu}{m_2}\vec{\mathbf{r}}_{1,2}$$

Relative Distance and Reduced Mass

The two position vectors are then

$$\vec{\mathbf{r}}_1 = -\frac{\mu}{m_1}\vec{\mathbf{r}}_{1,2}; \quad \vec{\mathbf{r}}_2 = \frac{\mu}{m_2}\vec{\mathbf{r}}_{1,2}$$

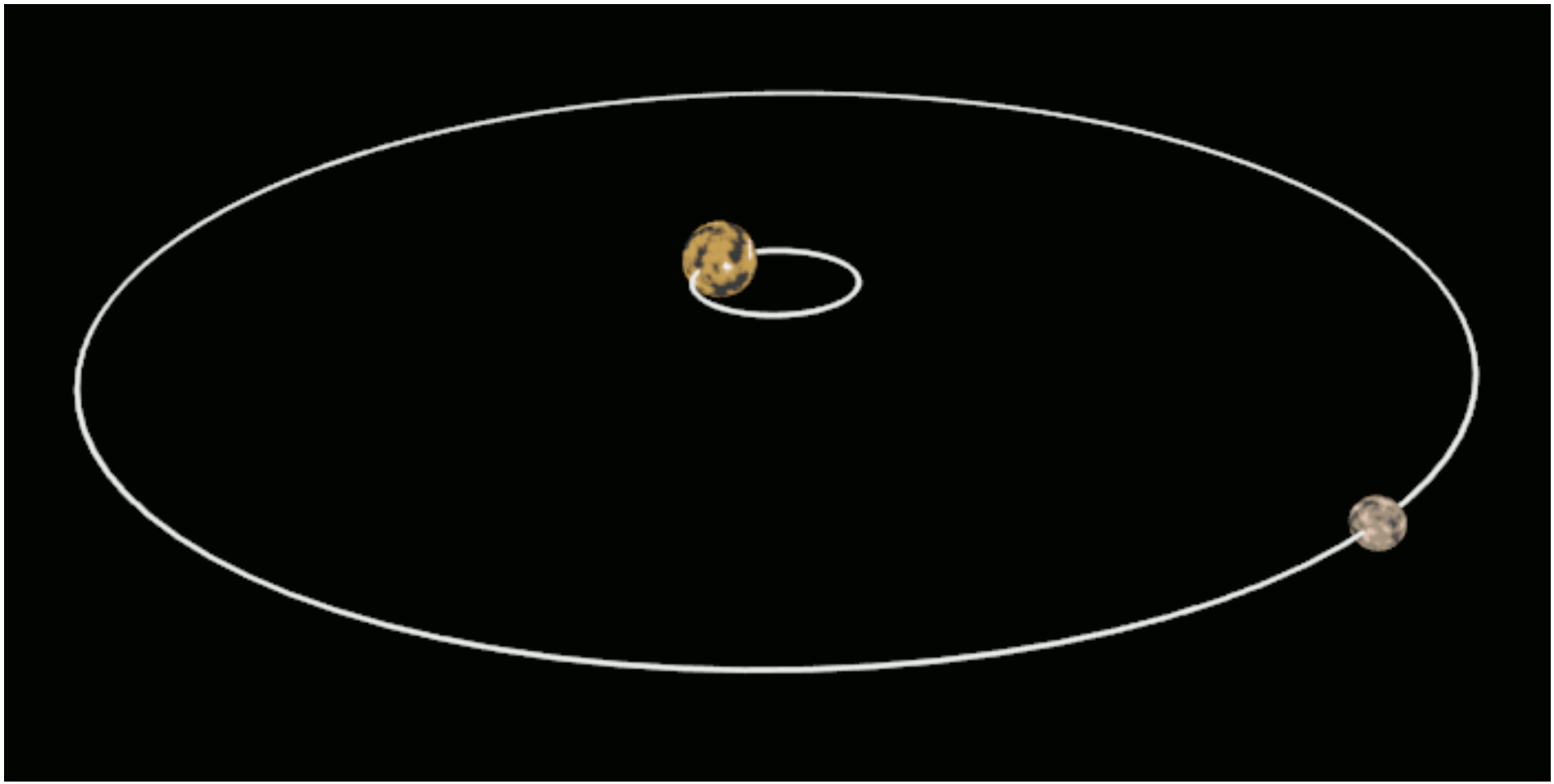


In the case where one mass (say m_2) dominates, the center-of-mass gets *close* to the center of m_2. If $m_2 >> m_1$ then

$$\mu \sim m_1; \quad \vec{\mathbf{r}}_1 \sim \vec{\mathbf{r}}_{1,2}; \quad \vec{\mathbf{r}}_2 \sim \vec{\mathbf{0}}$$

Pluto- Charon Orbit

$$m_2 > m_1$$

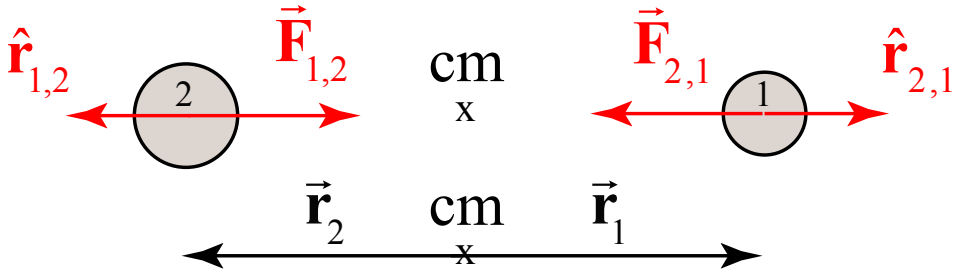


https://i.stack.imgur.com/YM3TW.gif

Reduction of Two Body Problem

Newton's Second Law

$$\vec{\mathbf{F}}_{2,1} = m_1 \frac{d^2\vec{\mathbf{r}}_1}{dt^2}; \quad \vec{\mathbf{F}}_{1,2} = m_2 \frac{d^2\vec{\mathbf{r}}_2}{dt^2}$$



Divide by mass $\frac{\vec{\mathbf{F}}_{2,1}}{m_1} = \frac{d^2\vec{\mathbf{r}}_1}{dt^2}; \quad \frac{\vec{\mathbf{F}}_{1,2}}{m_2} = \frac{d^2\vec{\mathbf{r}}_2}{dt^2}$

$$\vec{\mathbf{F}}_{1,2}(r) = -G\frac{m_1 m_2}{r_{1,2}^2}\hat{\mathbf{r}}_{1,2} = F_{1,2}\hat{\mathbf{r}}_{1,2}$$

Subtract: $\left(\frac{\vec{\mathbf{F}}_{1,2}}{m_2} - \frac{\vec{\mathbf{F}}_{2,1}}{m_1}\right) = \frac{d^2(\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1)}{dt^2} = \frac{d^2\vec{\mathbf{r}}_{1,2}}{dt^2}$

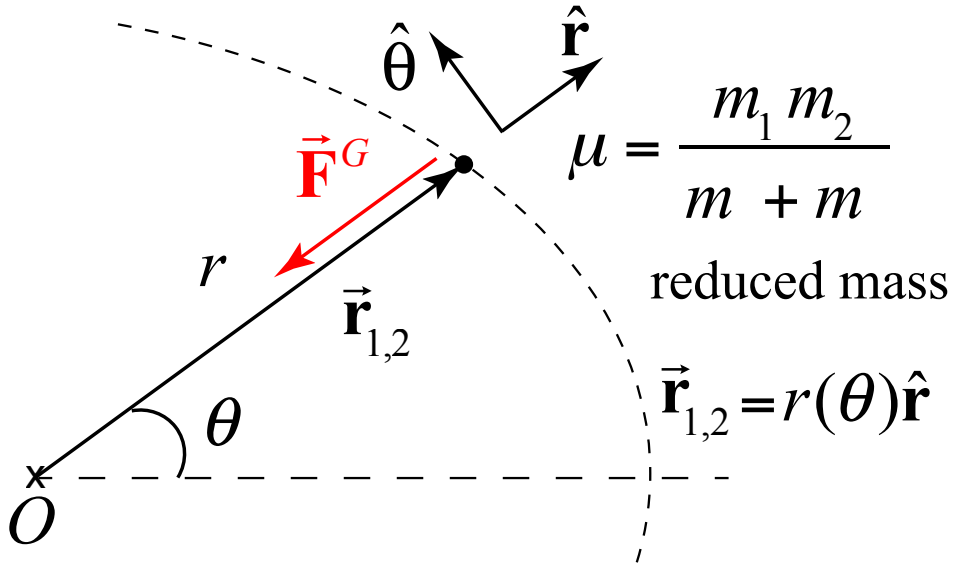
$$\vec{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1$$

Use Newton's Third Law $\vec{\mathbf{F}}_{2,1} = -\vec{\mathbf{F}}_{1,2}$

Reduced mass: $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \Rightarrow \mu = \frac{m_1 m_2}{m_1 + m_2}$

Summary: $\frac{\vec{\mathbf{F}}_{1,2}}{\mu} = \frac{d^2\vec{\mathbf{r}}_{1,2}}{dt^2} \Rightarrow -G\frac{m_1 m_2}{r_{1,2}^2}\hat{\mathbf{r}}_{1,2} = \mu\frac{d^2\vec{\mathbf{r}}_{1,2}}{dt^2}$

"Effective" One-body Problem



The two-body system behaves as if it were a single body with a "reduced" mass μ. Choose polar coordinates then

$$\vec{\mathbf{r}}_{1,2} = r(\theta)\hat{\mathbf{r}}$$

Newton's Second Law (Orbit Equation):

$$-G\frac{m_1 m_2}{r_{1,2}^2}\hat{\mathbf{r}} = \mu\frac{d^2}{dt^2}(r(\theta)\hat{\mathbf{r}})$$

Solving the orbit equation means finding the equation for the distance $r(\theta)$ from the origin (as a function of the angle θ).

Angular Momentum about Center of Mass

Suppose that two bodies, possessing mass m_1 and m_2, are orbiting about their center of mass. Their position and velocity vectors are described by $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$, and $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$, respectively.

The angular momentum of the two bodies about the center-of-mass can be re-expressed in terms of relative positions and velocities and reduced mass, i.e in terms of and

$$\vec{\mathbf{v}}_{1,2} = (\vec{\mathbf{v}}_2 - \vec{\mathbf{v}}_1)$$

Use the relations: $\vec{\mathbf{r}}_1 = -(\mu / m_1)\vec{\mathbf{r}}_{1,2}; \quad \vec{\mathbf{r}}_2 = (\mu / m_2)\vec{\mathbf{r}}_{1,2}$ The angular momentum about the center of mass is then

$$\vec{\mathbf{L}}_{cm} = \vec{\mathbf{L}}_{cm,1} + \vec{\mathbf{L}}_{cm,2} = \vec{\mathbf{r}}_1 \times m_1\vec{\mathbf{v}}_1 + \vec{\mathbf{r}}_2 \times m_2\vec{\mathbf{v}}_2$$

$$\vec{\mathbf{L}}_{cm} = \vec{\mathbf{r}}_{1,2} \times \mu(\vec{\mathbf{v}}_2 - \vec{\mathbf{v}}_1) = \vec{\mathbf{r}}_{1,2} \times \mu\vec{\mathbf{v}}_{1,2}$$

Solving for Angular Momentum

Now let's solve for the angular momentum. Start with the separation between the two masses. Choose polar coordinates then

$$\mathbf{r}_{1,2} = r(\theta)\,\hat{\mathbf{r}}$$

$\hat{\mathbf{r}}_{1,2}$

The time derivative now acts on the distance and the unit vector

$$\vec{\mathbf{v}}_{1,2} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}$$

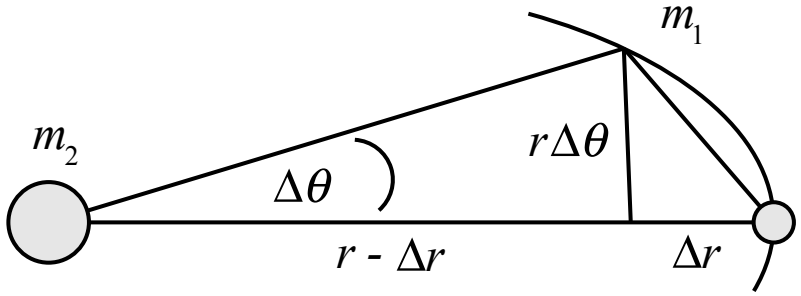
Now let's solve for the angular momentum:

$$\vec{\mathbf{L}}_{cm} = \mu\vec{\mathbf{r}}_{1,2} \times \vec{\mathbf{v}}_{1,2} = \mu r\hat{\mathbf{r}} \times \left(\frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} \right) = \mu r^2 \frac{d\theta}{dt}\hat{\mathbf{k}} = L_z\hat{\mathbf{k}}$$

where

$$L_z = \mu r^2 \frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{L_z}{\mu r^2}$$

Kepler Equal Area Law



$$(r-\Delta r)\tan(\Delta\theta) \simeq (r-\Delta r)\Delta\theta \approx r\Delta\theta;$$

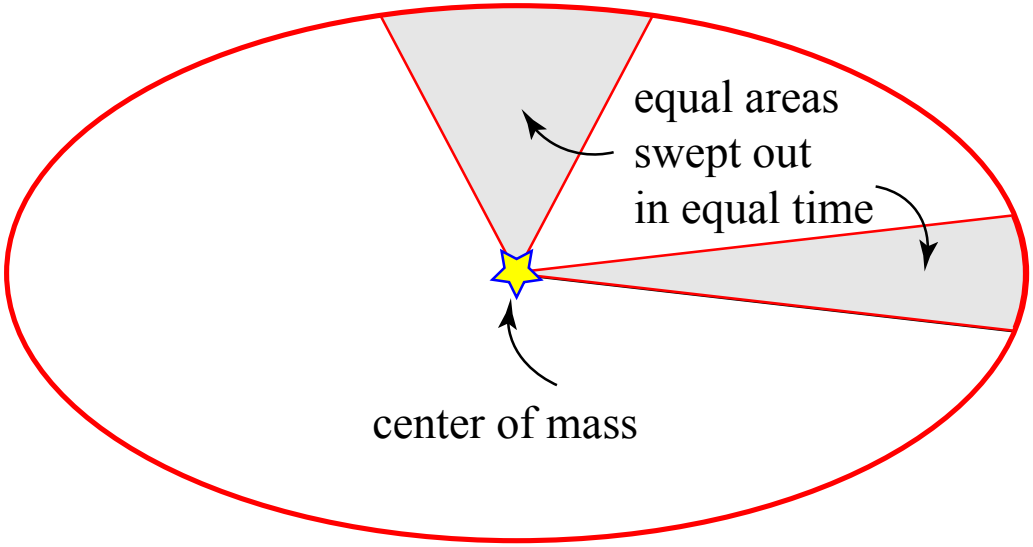
$$\Delta A \approx \frac{1}{2}(r\Delta\theta)\Delta r + \frac{1}{2}(r\Delta\theta)(r-\Delta r)$$

$$\Delta A \approx \frac{1}{2}(r^2\Delta\theta)$$

Rate of change of area:

$$\frac{dA}{dt} = \lim_{\Delta t\to 0}\frac{\Delta A}{\Delta t} = \lim_{\Delta t\to 0}\frac{1}{2}\left(r^2\frac{\Delta\theta}{\Delta t}\right)$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{L_z}{2\mu} = constant$$



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Kinetic Energy of Orbiting System

We can define the kinetic energy of the two bodies in terms of relative velocities

$$\vec{\mathbf{v}}_{1,2} = \vec{\mathbf{v}}_2 - \vec{\mathbf{v}}_1$$

In polar coordinates

$$\vec{\mathbf{v}}_{1,2} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}$$

The kinetic energy is: $K = \frac{1}{2}\mu\left|\vec{\mathbf{v}}_{1,2}\right|^2$

$$K = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\mu\left(r\frac{d\theta}{dt}\right)^2$$

Energy

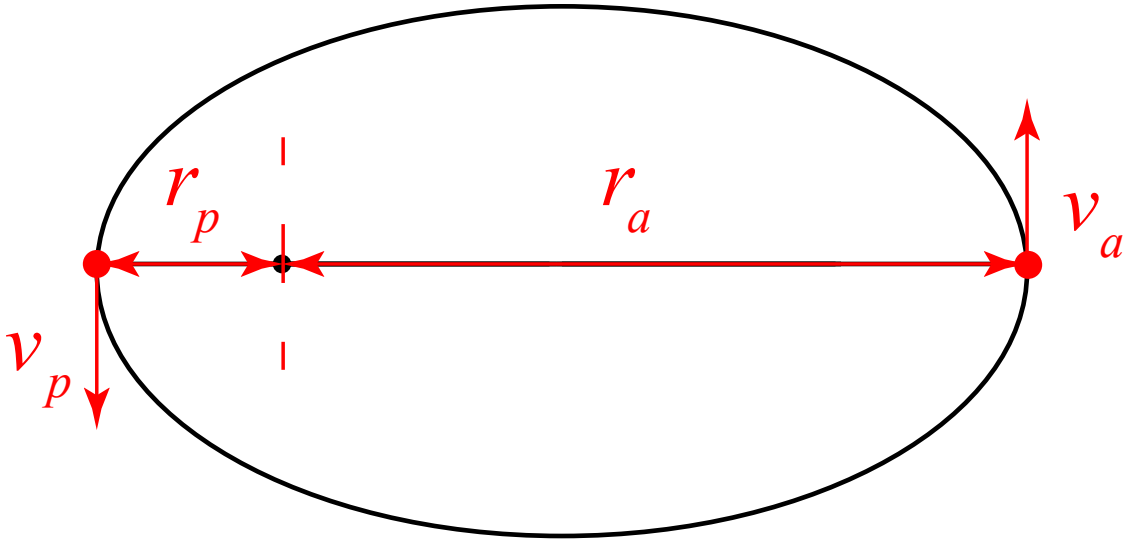
Kinetic Energy $$K = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\mu\left(r\frac{d\theta}{dt}\right)^2 = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{L_z^2}{2\mu r^2}$$

Potential Energy $$U(r) = -\frac{Gm_1m_2}{r}$$

Mechanical Energy $$E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{L_z^2}{2\mu r^2} - \frac{Gm_1m_2}{r}$$

The energy depends only on the distance between the two bodies and it's derivative.

Satellite in Elliptic Orbit



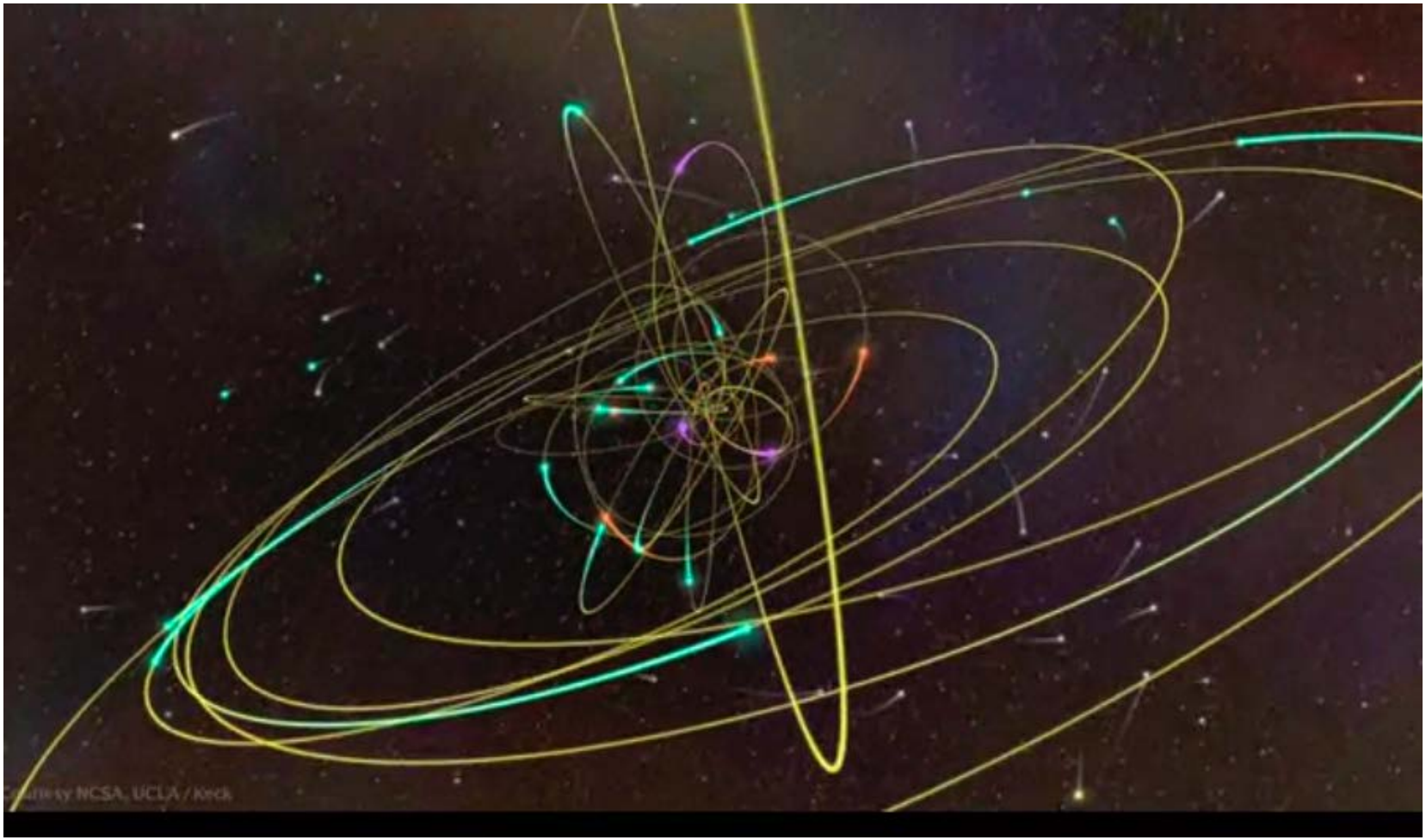
Energy:

$$\frac{1}{2}m_s v_a^2 - G\frac{mm_s}{r_a} = \frac{1}{2}m_s v_p^2 - G\frac{mm_s}{r_p}$$

Angular Momentum:

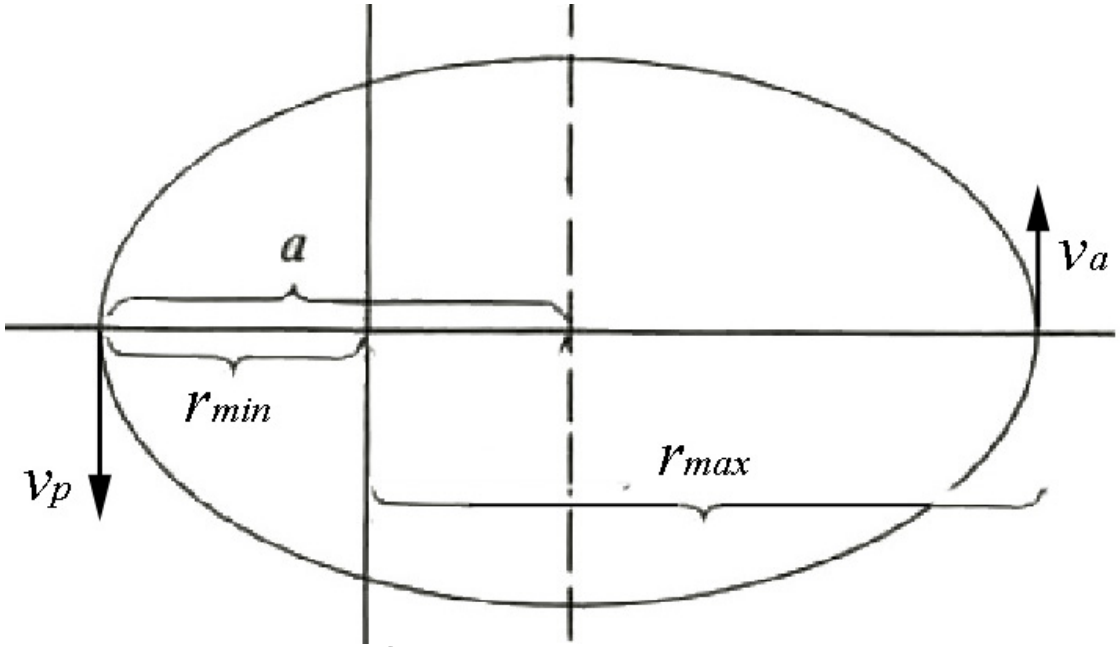
$$m_s r_a v_a = m_s r_p v_p$$

Orbits of Stars around Black Hole at Galactic Center



https://www.youtube.com/watch?v=SGJY0aeWBNc&t=5s

Kepler's Period Law



Period squared is proportional to cube of the major axis but depends on both masses.

For a circular orbit: Newton's Second Law becomes

$$G\frac{m_S m_P}{r^2} = m_P r\left(\frac{2\pi}{T}\right)^2 \Rightarrow T^2 = \frac{4\pi^2}{Gm_S}r^3$$

For an elliptic orbit:

$$T^2 = \frac{4\pi^2 a^3}{G\left(m_S + m_P\right)}$$

where a is the semi-major axis of the ellipse.

Reduction to One-dimension Motion

Energy:
$$E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{L_z^2}{2\mu r^2} - \frac{Gm_1m_2}{r}$$

Effective Potential Energy:
$$U_{eff} = \frac{L_z^2}{2\mu r^2} - \frac{Gm_1m_2}{r}$$

The new term represents an "effective" centrifugal force in one-dimension (variable r) along with the usual gravitational force

$$F_{eff} = -\frac{dU_{eff}}{dr} = \frac{L_z^2}{\mu r^3} - \frac{Gm_1m_2}{r^2}$$

Energy:
$$E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + U_{eff}$$

The energy describes one dimension motion (variable r) under the interaction of repulsive and attractive forces.

Effective Force

The effective potential energy is:

$$U_{eff} = \frac{L_z^2}{2\mu r^2} - \frac{Gm_1 m_2}{r}$$

The associated effective force is

$$F_{eff} = -\frac{dU_{eff}}{dr} = \frac{L_z^2}{\mu r^3} - \frac{Gm_1 m_2}{r^2}$$

The first term represents an “effective” centrifugal force in one-dimension (variable r) and the second term is the centripetal gravitational force

Energy Diagram: Circular Orbit

Case 4: $E = E_{min}$

$U_{eff}(r)$

Circular orbit

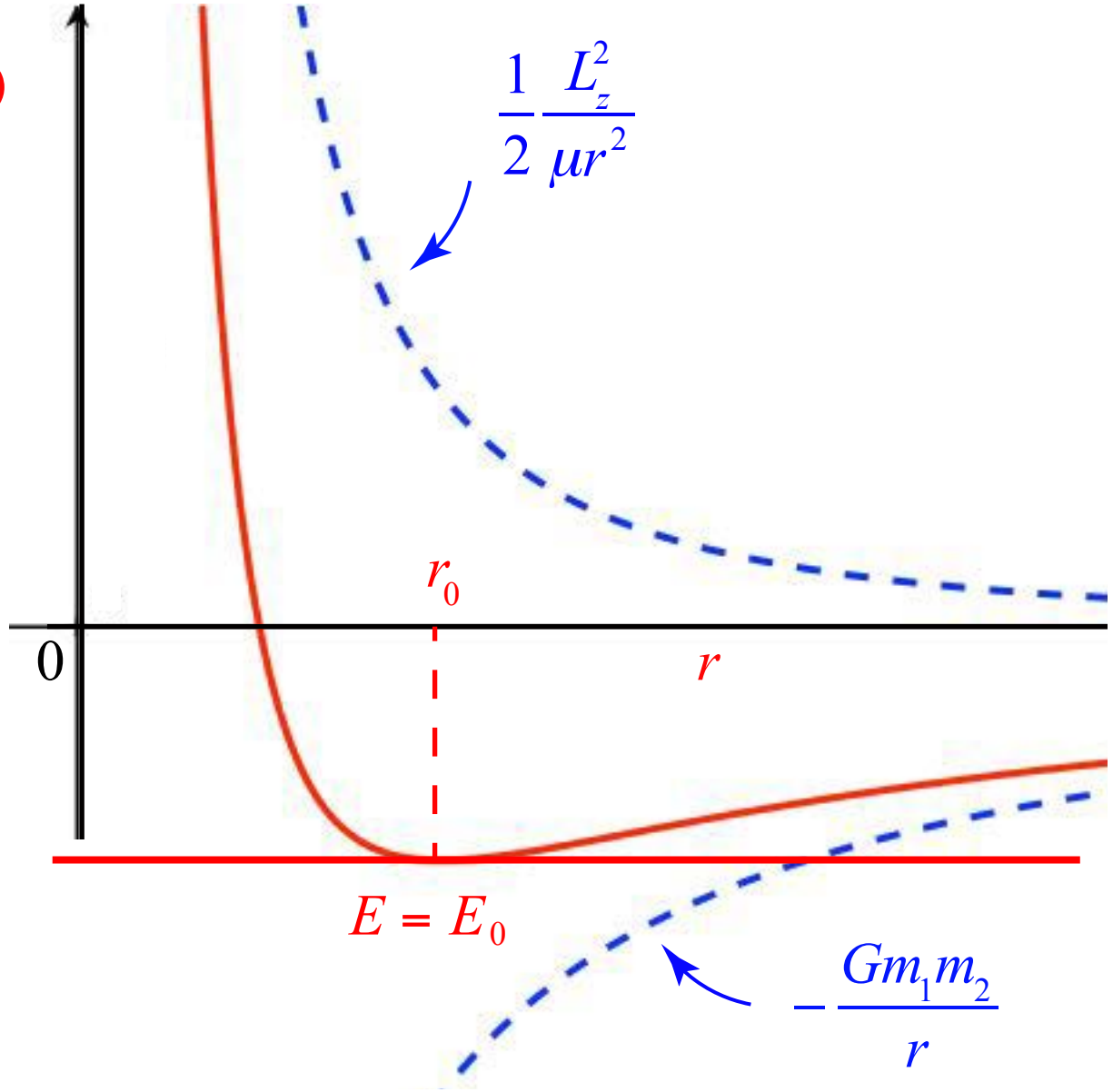
$$U_{eff}(r) = \frac{L_z^2}{2\mu r^2} - \frac{Gm_1m_2}{r}$$

Extrema occur when

$$\frac{dU_{eff}(r)}{dt} = 0$$

so one can solve for r_{min} and

$$E_{min} = U_{eff}(r_{min})$$



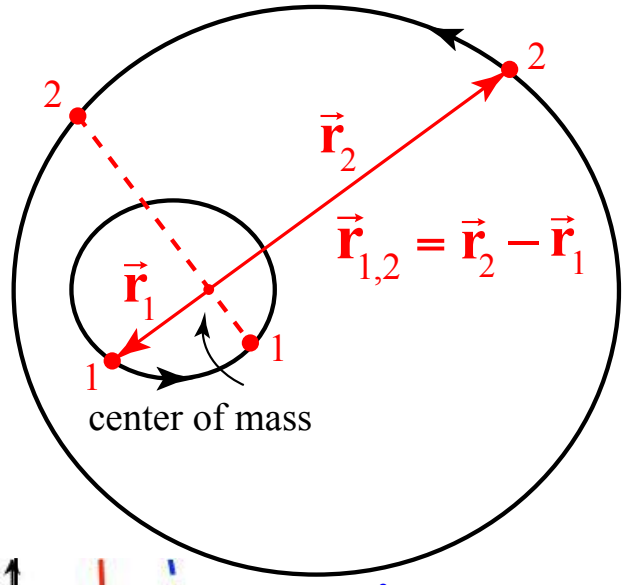
Radius of Circular Orbit

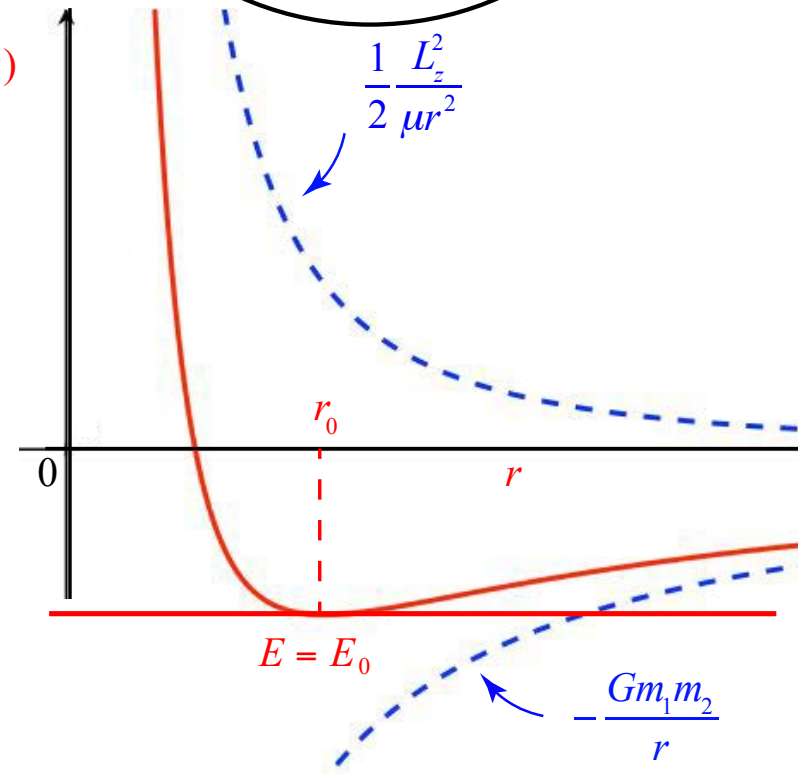
The effective potential is given by

$$U_{eff}(r) = \frac{L_z^2}{2\mu r^2} - \frac{Gm_1m_2}{r}$$

The value of *r* at the minimum of the effective potential is the radius of the circular orbit. Thus

$$0 = \frac{dU_{eff}(r)}{dr} = -\frac{L_z^2}{\mu r^3} + \frac{Gm_1m_2}{r^2} \Rightarrow r = \frac{L_z^2}{\mu G m_1 m_2}$$





Newton's Second Law:

$$-Gm_1m_2 / r^2 = \mu v^2 / r \Rightarrow (1/2)\mu v^2 = -Gm_1m_2 / r$$

The energy is then

$$E = (1/2)\mu v^2 - Gm_1m_2 / r = -Gm_1m_2 / 2r$$

Thus $$Gm_1m_2 = -2rE$$

$$r = \frac{L_z^2}{\mu G m_1 m_2} = \frac{L_z^2}{\mu(-2Er)} \Rightarrow r = \frac{L_z}{\sqrt{-2E\mu}}$$

Energy Diagram: Elliptic Orbit

Case 3: $E < 0$

$U_{eff}(r)$

Elliptical orbit

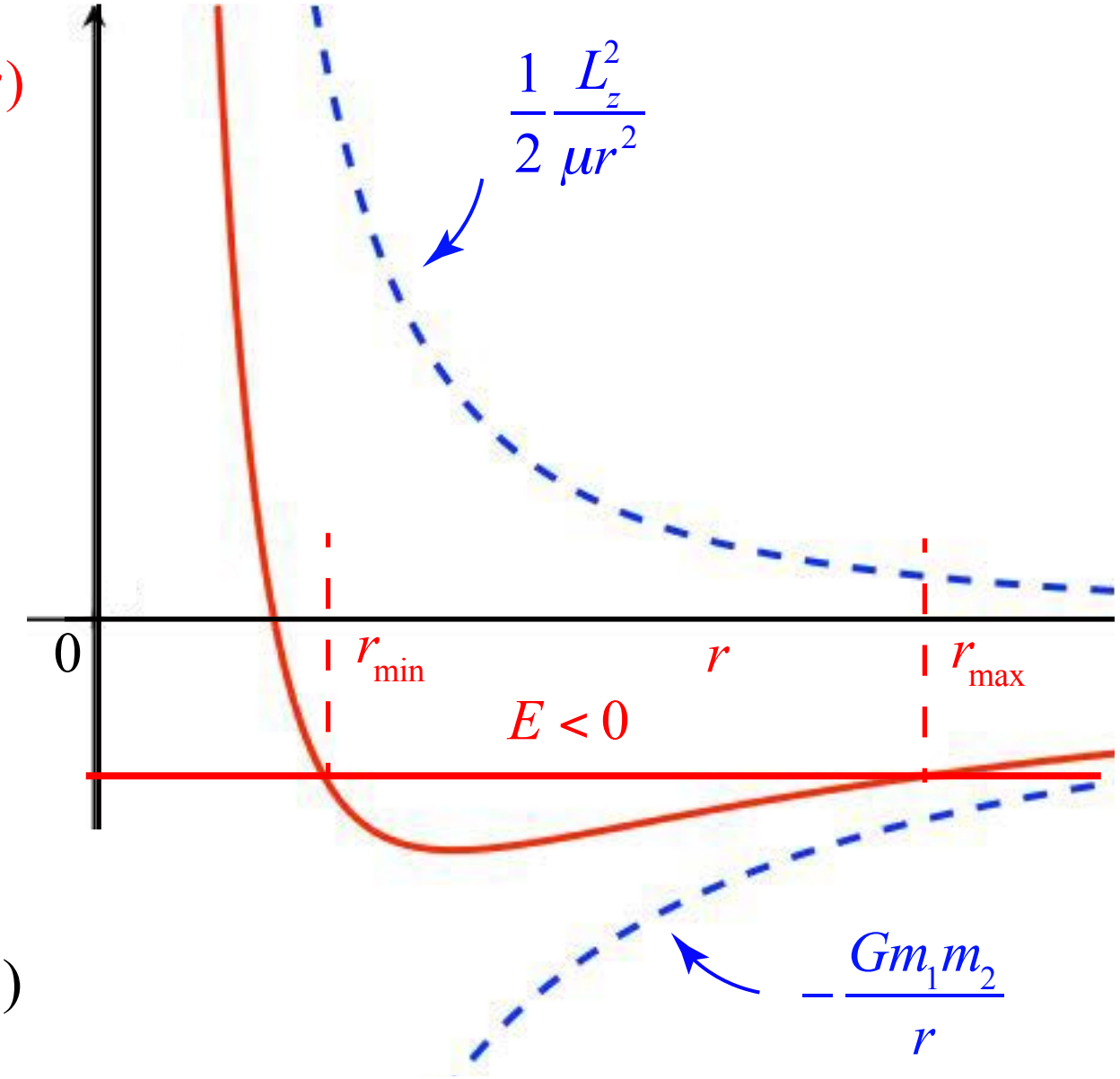
$$U_{eff}(r) = \frac{L_z^2}{2\mu r^2} - \frac{Gm_1m_2}{r}$$

Nearest and furthest approach occur when

$$E = U_{eff}(r)$$

so one can solve for r_{min} and r_{max}

$$E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + U_{eff}(r)$$



Energy Diagram: Parabolic Orbit

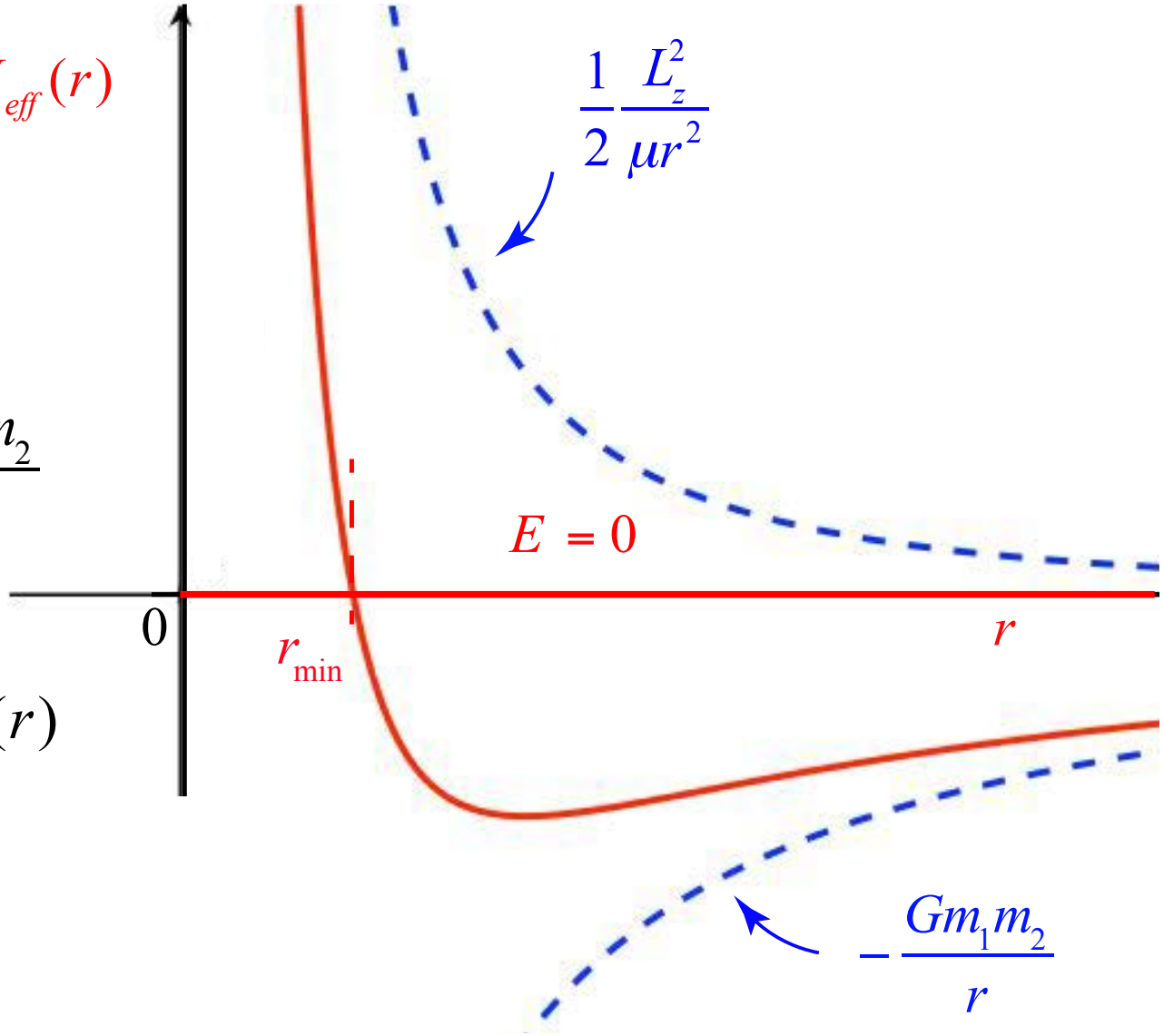
Case 2: $E = 0$

$U_{eff}(r)$

Parabolic orbit

$$U_{eff}(r) = \frac{L_z^2}{2\mu r^2} - \frac{Gm_1m_2}{r}$$

$$E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + U_{eff}(r)$$



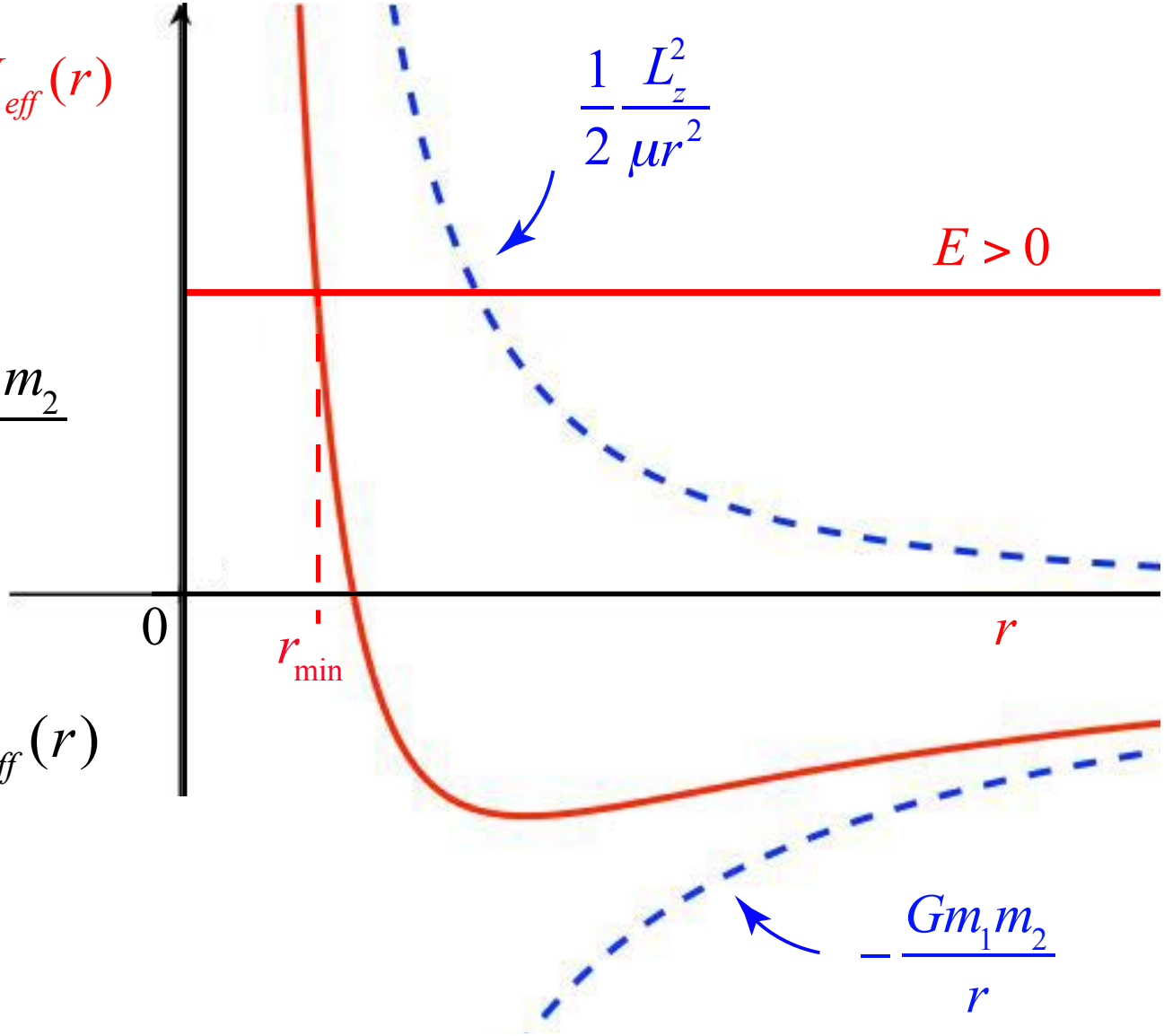
Energy Diagram: Hyperbolic Orbit

Case 1: $E > 0$

Hyperbolic orbit

$$U_{eff}(r) = \frac{L_z^2}{2\mu r^2} - \frac{Gm_1m_2}{r}$$

$$E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + U_{eff}(r)$$



Kinetic Energy of Orbiting System

We can define the kinetic energy of the two bodies in terms of relative velocities as well.

$$\vec{\mathbf{v}}_{1,2} = \vec{\mathbf{v}}_2 - \vec{\mathbf{v}}_1$$

where

$$\vec{\mathbf{v}}_1 = \frac{d\vec{\mathbf{r}}_1}{dt} = -\left(\frac{m_2}{m_2 + m_1}\right)\vec{\mathbf{v}}_{1,2} \qquad \vec{\mathbf{v}}_2 = \frac{d\vec{\mathbf{r}}_2}{dt} = \left(\frac{m_2}{m_2 + m_1}\right)\vec{\mathbf{v}}_{1,2}$$

The kinetic energy is:

$$K = \frac{1}{2}m_1\left|\vec{\mathbf{v}}_1\right|^2 + \frac{1}{2}m_2\left|\vec{\mathbf{v}}_2\right|^2 = \frac{1}{2}\left(\frac{m_1 m_2{}^2}{(m_2 + m_1)^2} + \frac{m_2 m_1{}^2}{(m_2 + m_1)^2}\right)\left|\vec{\mathbf{v}}_{1,2}\right|^2$$

$$= \frac{1}{2}\frac{(m_1 m_2)(m_2 + m_1)}{(m_2 + m_1)^2}\left|\vec{\mathbf{v}}_{1,2}\right|^2 = \frac{1}{2}\left(\frac{m_1 m_2}{m_2 + m_1}\right)\left|\vec{\mathbf{v}}_{1,2}\right|^2 = K = \frac{1}{2}\mu\left|\vec{\mathbf{v}}_{1,2}\right|^2$$

Kepler's First Law: Orbit Equation

To get the analytic behavior of the orbit, one can use the energy equation

$$\frac{dr}{dt} = \frac{2}{\mu}\left(E - \frac{L^2}{2\mu r^2} + \frac{Gm_1 m_2}{r} \right)^{1/2}$$

and integrate:

$$\int_{t_0}^{t} dt = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - \frac{L^2}{2\mu r^2} + \frac{Gm_1 m_2}{r} \right)}}$$

The solution can be expressed in terms of a function of angle (which depends on time).

$$r = \frac{r_0}{1 - \varepsilon \cos\theta}$$

where r_0 is radius of circular orbit with energy E_0, and ε is the eccentricity given by

$$\varepsilon = \left(1 + \frac{2EL^2}{\mu\left(Gm_1 m_2\right)^2} \right)^{1/2} = \left(1 - \frac{E}{E_0} \right)^{1/2} \quad ; \quad r_0 = -\frac{Gm_1 m_2}{2E_0}$$

Newton's Second Law in Non-Inertial Reference Frames



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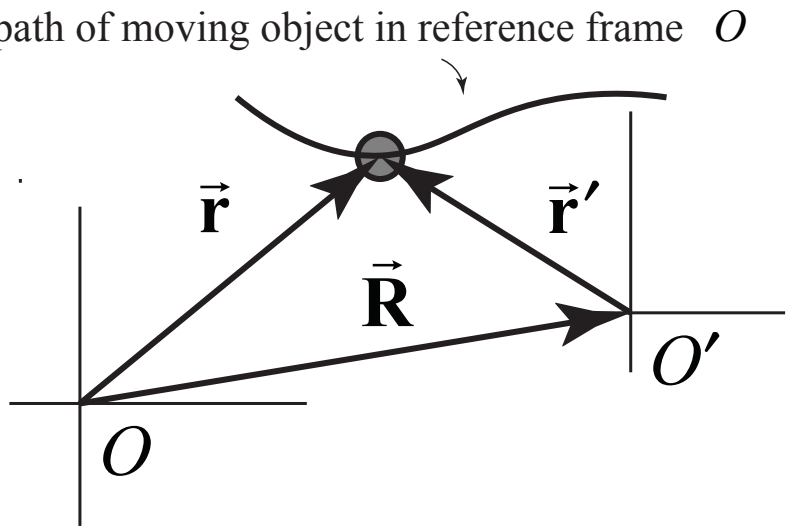
Acceleration of an Object in Different Reference Frames

Suppose an object is moving; observers in different reference frames may measure different accelerations

$$\vec{A} = d\vec{V} / dt$$



$$\vec{\mathbf{r}}' = \vec{\mathbf{r}} - \vec{\mathbf{R}}$$

$$\vec{\mathbf{v}}' = \vec{\mathbf{v}} - \vec{\mathbf{V}}$$

$$\vec{\mathbf{a}}' = \vec{\mathbf{a}} - \vec{\mathbf{A}}$$

Acceleration of Frame O' with respect to Frame O:

$$\vec{a} = d\vec{v} / dt$$

$$\vec{\mathbf{a}}' = d\vec{\mathbf{v}}' / dt$$

Acceleration of the object in Frame O':

2nd Law in Non-Inertial Reference Frames

Consider an object that is accelerating in an inertial frame *O* . In that frame, Newton's Second Law is

$$\vec{\mathbf{F}}_{physical} = m\vec{\mathbf{a}}$$

where $\vec{\mathbf{F}}_{physical}$ are the physical forces acting on the object.

Now let *O*' be an **accelerated** reference frame with respect to Frame *O*. The mass times acceleration of the object in *O*' is given by

$$m\vec{\mathbf{a}}' = m\vec{\mathbf{a}} - m\vec{\mathbf{A}} = \vec{\mathbf{F}}_{physical} - m\vec{\mathbf{A}}$$

Define

$$\vec{\mathbf{F}}_{\text{fictitious}} = -m\vec{\mathbf{A}}$$

Then a modified Newton's Second Law in Frame *O*' becomes

$$\vec{\mathbf{F}}_{\text{physical}} + \vec{\mathbf{F}}_{\text{fictitious}} = m\vec{\mathbf{a}}'$$

Effective Gravity

The fictitious force in the non-inertial frame is given by

$$\vec{\mathbf{F}}_{fic} = -m\vec{\mathbf{A}} = -mA_x\hat{\mathbf{i}}$$

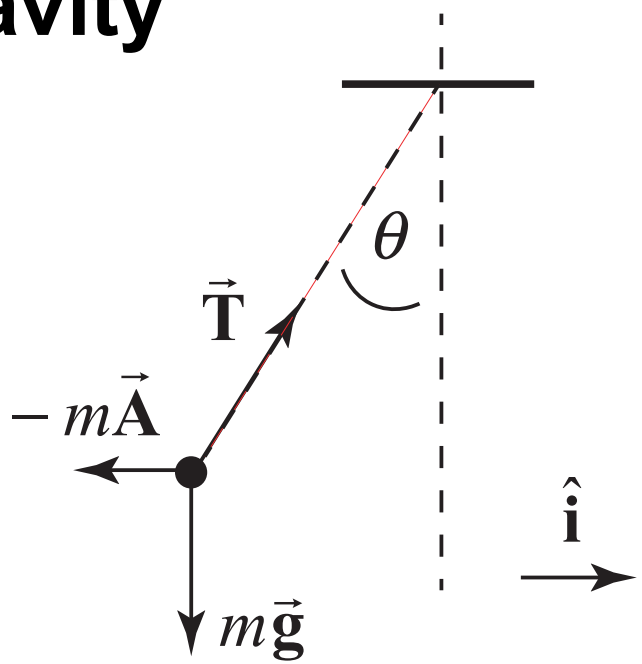
Because the bob is at rest in the non-inertial frame, the sum of the physical and fictitious force is zero

$$\vec{\mathbf{T}} + m\vec{\mathbf{g}} - m\vec{\mathbf{A}} = \vec{\mathbf{0}}$$

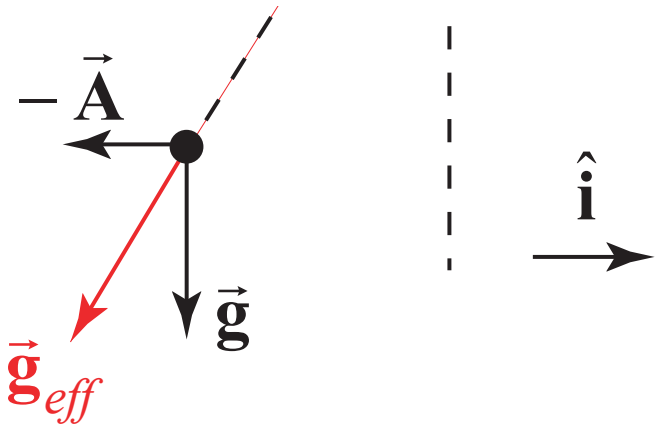
$$\Rightarrow \vec{\mathbf{T}} = -m\vec{\mathbf{g}} + m\vec{\mathbf{A}}$$

In the non-inertial frame the effective gravitational acceleration is

$$\vec{\mathbf{g}}_{eff} = \vec{\mathbf{g}} + (\vec{\mathbf{F}}_{fic} / m) = \vec{\mathbf{g}} - \vec{\mathbf{A}}$$



non-inertial frame accelerating $\vec{\mathbf{A}} = A_x\hat{\mathbf{i}}$ to the right with respect to inertial frame



Accelerating Truck

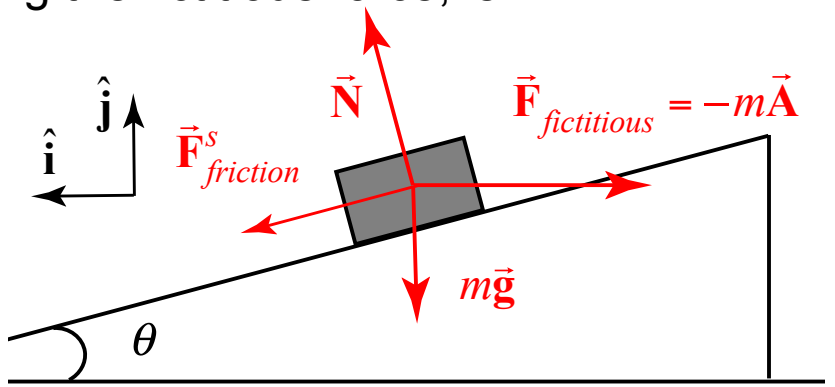
a) In the reference frame accelerating with the truck, the block is at rest, hence

$$\vec{\mathbf{a}}' = \vec{\mathbf{0}}$$

b) The vector expression for the fictitious force acting on the block is

$$\vec{\mathbf{F}}_{fictitious} = -m\vec{\mathbf{A}} = -mA\hat{\mathbf{i}}$$

c) The free body force diagram, including the fictitious force, is



d) Static friction is at it's maximal

$$f^{s}_{friction} = \mu N$$

Newton's Second Law:

Horizontal direction:

$$\mu N\cos\theta - mA + N\sin\theta = 0 \Rightarrow A = \frac{N(\mu\cos\theta + \sin\theta)}{m}$$

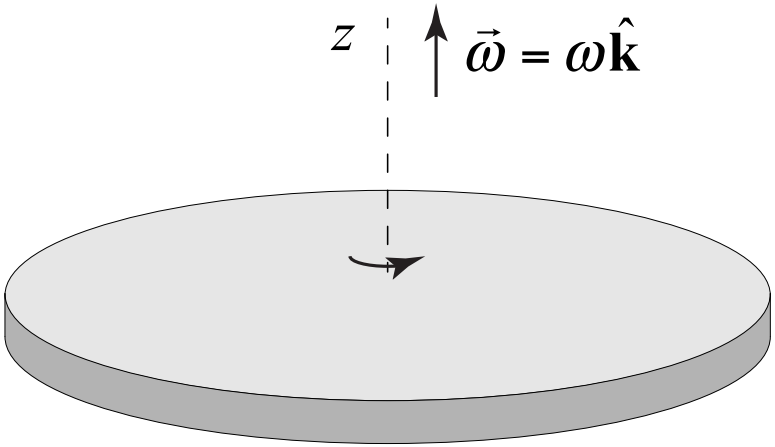
Vertical direction: $$-\mu N\sin\theta - mg + N\cos\theta = 0 \Rightarrow N = \frac{mg}{\cos\theta - \mu\sin\theta}$$

The magnitude of the acceleration is then

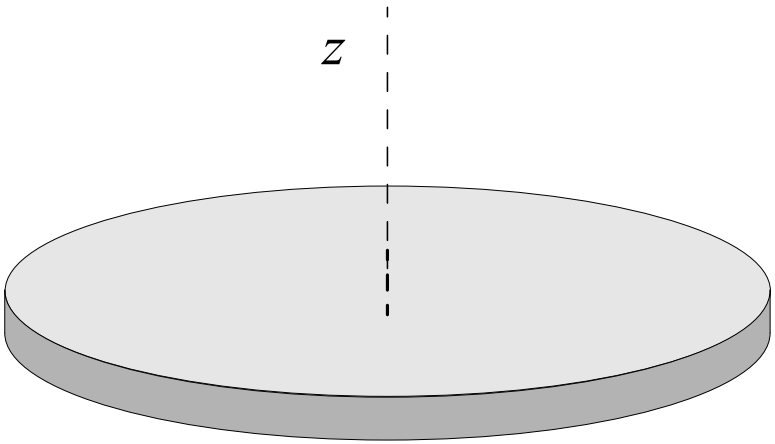
$$A = \frac{\mu\cos\theta + \sin\theta}{\cos\theta - \mu\sin\theta}g$$

Non-Inertial Rotating Reference Frame: Rotating Platform

A reference frame O' fixed to a rotating platform.



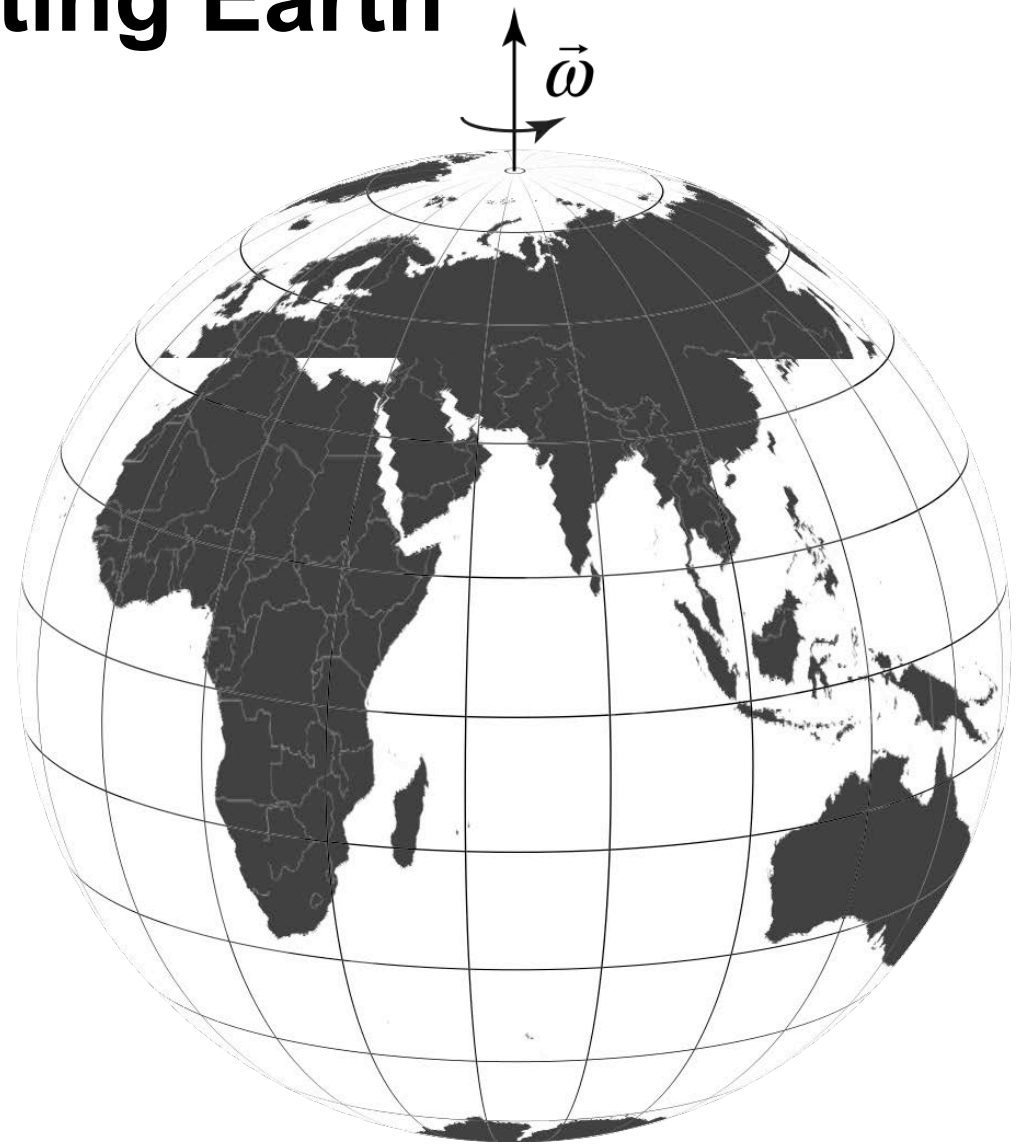
inertial frame O, platform is rotating with angular velocity $\vec{\omega}$



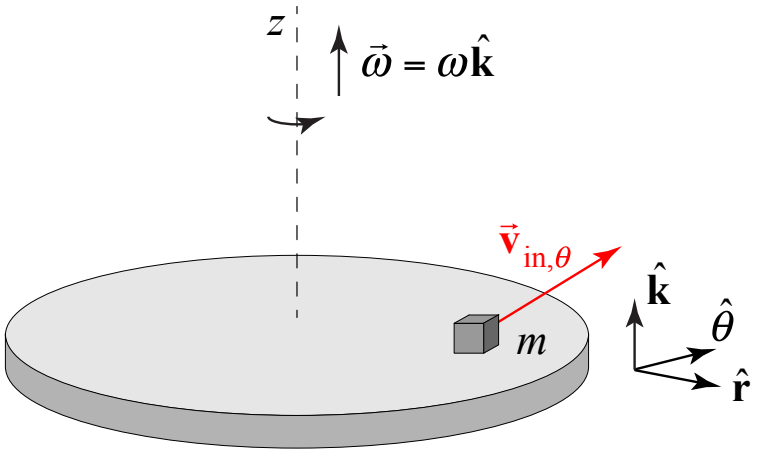
frame O' is rotating with angular velocity $\vec{\omega}$ with respect to inertial frame O

Non-Inertial Rotating Reference Frame: Rotating Earth

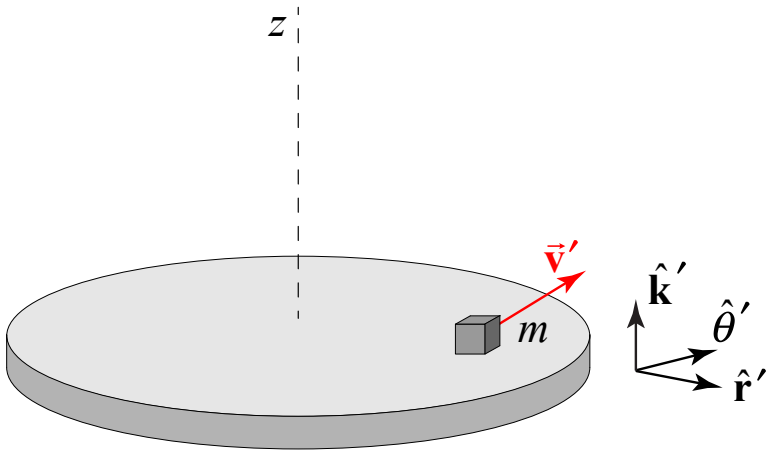
Reference fixed to the earth which is rotating about its axis, the earth is a non-inertial rotating reference frame.



Velocity in Inertial Frame



inertial frame O

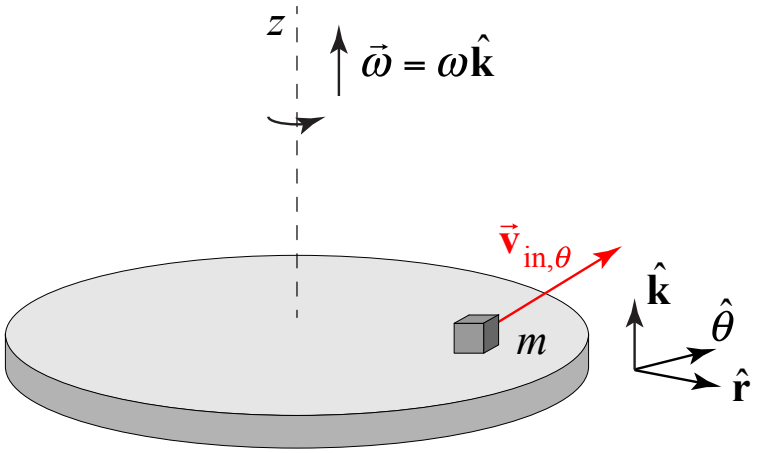


frame O' is rotating
with angular velocity $\vec{\omega}\odot$
with respect to inertial frame O

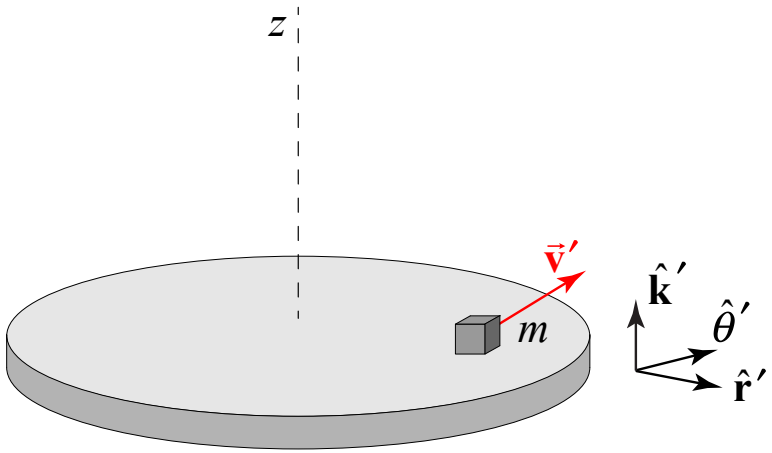
Consider a platform that is rotating about the positive z-axis with angular velocity $\vec{\omega} = \omega\hat{\mathbf{k}}$ in the inertial reference frame O. A point on the platform a distance r from the center is moving with velocity

$$\vec{\mathbf{V}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}} = \omega\hat{\mathbf{k}} \times r\hat{\mathbf{r}} = r\omega\hat{\boldsymbol{\theta}}$$

Velocity Transformation Law



inertial frame O



frame O' is rotating
with angular velocity $\vec{\omega}\odot$
with respect to inertial frame O

Consider a platform that is rotating about the positive z-axis with angular velocity $\vec{\omega} = \omega\hat{\mathbf{k}}$ in the inertial reference frame O. Let O' denote a reference frame that is rotating with the platform. A point P on the platform a distance r from the center is moving with velocity $\vec{\mathbf{V}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}} = \omega\hat{\mathbf{k}} \times r\hat{\mathbf{r}} = r\omega\hat{\boldsymbol{\theta}}$

Consider an object that is moving in a circle of radius r on the rotating platform with a constant velocity $\vec{\mathbf{v}}' = v'\hat{\boldsymbol{\theta}}'$ in the inertial frame O'. The velocity of the object in the inertial reference frame O is then

$$\vec{\mathbf{v}} = \vec{\mathbf{v}}' + \vec{\mathbf{V}} = \vec{\mathbf{v}}' + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}} = (v' + r\omega)\hat{\boldsymbol{\theta}}$$

Time Derivatives in Inertial and Non-inertial Reference Frames

The time derivatives of any vector quantity $\vec{\mathbf{C}}$ in a reference frame *O*' rotating with constant angular velocity $\vec{\boldsymbol{\omega}}$ with respect to an inertial frame *O*, (the origins of the two frames coincide) are related by

$$\left(\frac{d\vec{\mathbf{C}}}{dt}\right)_{inertial} = \left(\frac{d\vec{\mathbf{C}}}{dt}\right)_{rot} + (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{C}})$$

Example: When $\vec{\mathbf{C}} = \vec{\mathbf{r}}$

$$\left(\frac{d\vec{\mathbf{r}}}{dt}\right)_{inertial} = \left(\frac{d\vec{\mathbf{r}}}{dt}\right)_{rot} + (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}})$$

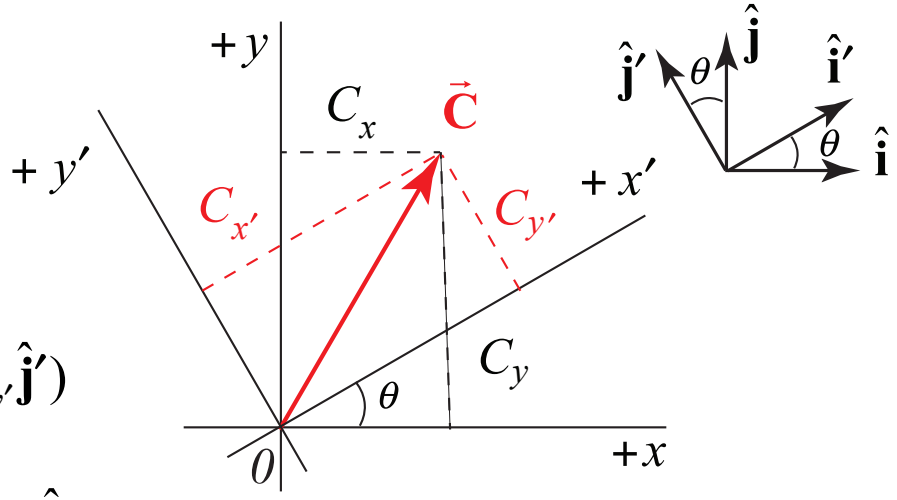
Hence

$$\vec{\mathbf{v}} = \vec{\mathbf{v}}' + (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}})$$

$$\vec{\mathbf{v}}' = \vec{\mathbf{v}} - (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}})$$

Time Derivatives in Inertial and Non-inertial Reference Frames

$$\vec{\mathbf{C}}(t) = C_x\hat{\mathbf{i}} + C_y\hat{\mathbf{j}} = C_{x'}\hat{\mathbf{i}}' + C_{y'}\hat{\mathbf{j}}'$$



$$\left(\frac{d\vec{\mathbf{C}}(t)}{dt}\right)_{in} = \frac{d}{dt}(C_x\hat{\mathbf{i}} + C_y\hat{\mathbf{j}}) = \frac{d}{dt}(C_{x'}\hat{\mathbf{i}}' + C_{y'}\hat{\mathbf{j}}')$$

$$= \frac{dC_{x'}}{dt}\hat{\mathbf{i}}' + \frac{dC_{y'}}{dt}\hat{\mathbf{j}}' + C_{x'}\frac{d\hat{\mathbf{i}}'}{dt} + C_{y'}\frac{d\hat{\mathbf{j}}'}{dt}$$

$$= \left(\frac{d\vec{\mathbf{C}}(t)}{dt}\right)_{rot} + \vec{\boldsymbol{\omega}} \times (C_{x'}\hat{\mathbf{i}}' + C_{y'}\hat{\mathbf{j}}')$$

$$= \left(\frac{d\vec{\mathbf{C}}(t)}{dt}\right)_{rot} + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{C}}$$

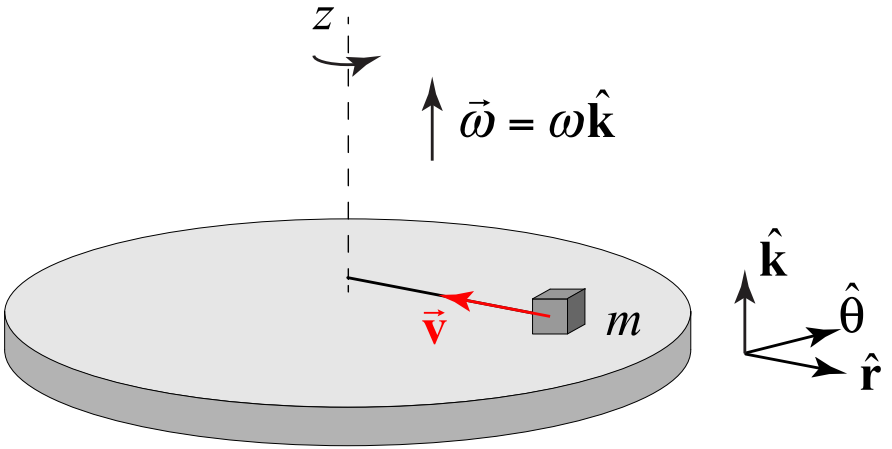
$$\hat{\mathbf{i}}' = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$$

$$\hat{\mathbf{j}}' = -\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}$$

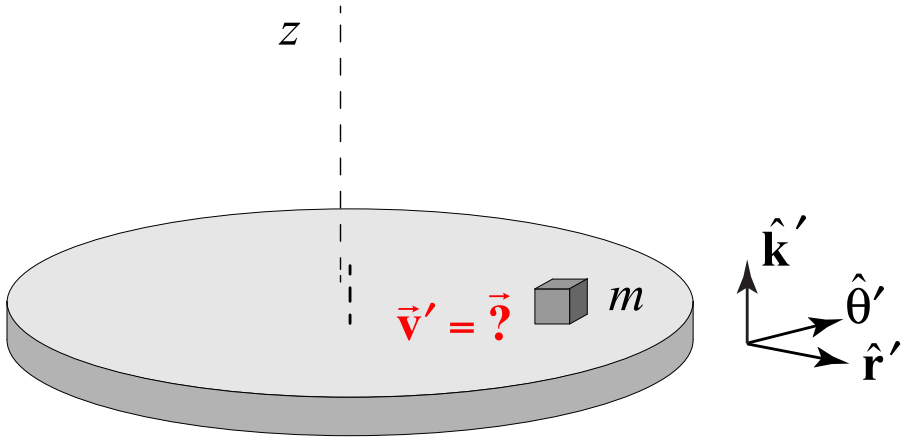
$$\frac{d\hat{\mathbf{i}}'}{dt} = \frac{d\theta}{dt}\hat{\mathbf{j}}' = \frac{d\theta}{dt}\hat{\mathbf{k}} \times \hat{\mathbf{i}}' = \vec{\boldsymbol{\omega}} \times \hat{\mathbf{i}}'$$

$$\frac{d\hat{\mathbf{j}}'}{dt} = -\frac{d\theta}{dt}\hat{\mathbf{i}}' = \frac{d\theta}{dt}\hat{\mathbf{k}} \times \hat{\mathbf{j}}' = \vec{\boldsymbol{\omega}} \times \hat{\mathbf{j}}'$$

Velocity in Inertial and Rotating Frames



inertial frame O



frame O' is rotating
with angular velocity $\vec{\omega}$
with respect to inertial frame O

At the instance shown the unit vectors $\hat{\mathbf{r}} = \hat{\mathbf{r}}'$ and $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}'$. The velocity in O is then

$$\vec{\mathbf{v}} = -v\hat{\mathbf{r}}$$

The velocity of the object in O' is then

$$\vec{\mathbf{v}}' = \vec{\mathbf{v}} - \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}} = -v\hat{\mathbf{r}} - \omega\hat{\mathbf{k}} \times r\hat{\mathbf{r}}$$

$$= -v\hat{\mathbf{r}} - r\omega\,\hat{\boldsymbol{\theta}}$$

Fictitious Forces in a Rotating Reference Frame

Assume $\vec{\omega}$ is constant. The derivative of the velocity in the inertial frame is equal to

$$\vec{\mathbf{a}} = \left(\frac{d\vec{\mathbf{v}}}{dt}\right)_{in} = \left(\frac{d\vec{\mathbf{v}}}{dt}\right)_{non-in} + (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}_{in})$$

which becomes $$\vec{\mathbf{a}} = \left(\frac{d(\vec{\mathbf{v}}' + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}})}{dt}\right)_{non-in} + \vec{\boldsymbol{\omega}} \times (\vec{\mathbf{v}}' + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}})$$

Expanding the first term:

$$\vec{\mathbf{a}} = \left(\frac{d\vec{\mathbf{v}}'}{dt}\right)_{non-in} + \vec{\boldsymbol{\omega}} \times \left(\frac{d\vec{\mathbf{r}}}{dt}\right)_{non-in} + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}' + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}})$$

Thus

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}' + 2\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}' + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}})$$

$$\vec{\mathbf{a}}' = \vec{\mathbf{a}} - 2\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}' - \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}})$$

Newton's Second Law in Rotating Frame

In the non-inertial frame, Newton's Second Law is

$$\vec{\mathbf{F}}_{\text{physical}} + \vec{\mathbf{F}}_{\text{fictitious}} = m\vec{\mathbf{a}}'$$

Using the transformation law for acceleration

$$\vec{\mathbf{a}}' = \vec{\mathbf{a}} - 2\vec{\omega} \times \vec{\mathbf{v}}' - \vec{\omega} \times (\vec{\omega} \times \vec{\mathbf{r}})$$

The Second Law becomes

$$\vec{\mathbf{F}}_{\text{physical}} + \vec{\mathbf{F}}_{\text{fictitious}} = m\vec{\mathbf{a}} - m2\vec{\omega} \times \vec{\mathbf{v}}' - m\vec{\omega} \times (\vec{\omega} \times \vec{\mathbf{r}})\vec{\mathbf{a}}'$$

Because $\vec{\mathbf{F}}_{\text{physical}} = m\vec{\mathbf{a}}$. The fictitious forces are

$$\vec{\mathbf{F}}_{\text{fictitious}} = -2m\vec{\omega} \times \vec{\mathbf{v}}' - m\vec{\omega} \times (\vec{\omega} \times \vec{\mathbf{r}})$$

(1) the centrifugal force is $\vec{\mathbf{F}}_{\text{centrifugal}} = -m(\vec{\omega} \times (\vec{\omega} \times \vec{\mathbf{r}}))$

(2) the Coriolis force is $\vec{\mathbf{F}}_{\text{coriolis}} = -2m\vec{\omega} \times \vec{\mathbf{v}}'$

Newton's Second Law in a Rotating Reference Frame

Therefore Newton's Second Law

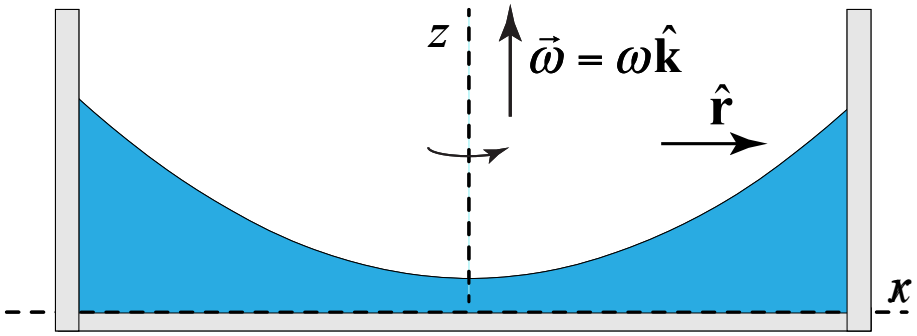
$$\vec{\mathbf{F}}_{phyiscal} + \vec{\mathbf{F}}_{fictitious} = m\vec{\mathbf{a}}'$$

in the rotating frame takes on the form

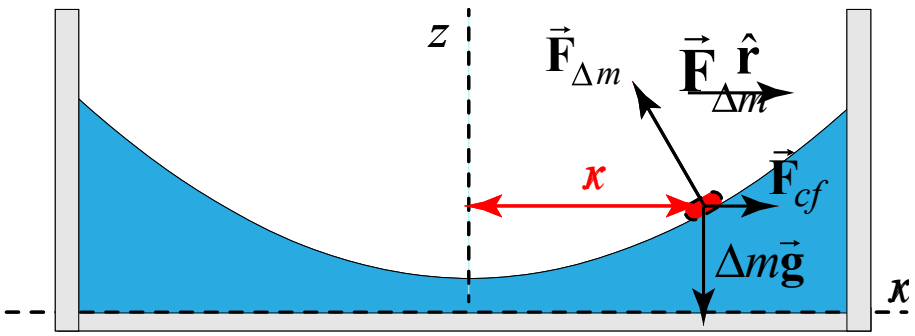
$$\vec{\mathbf{F}}_{\text{physical}} - m(\vec{\omega} \times (\vec{\omega} \times \vec{\mathbf{r}})) - 2m\vec{\omega} \times \vec{\mathbf{v}}' = m\vec{\mathbf{a}}'$$

where $\vec{\mathbf{v}}'$ and $\vec{\mathbf{a}}'$ are the velocity and accelerations of the object in the rotating frame.

Effective Gravity



inertial frame O



rotating non-inertial frame O'

Answer 3: In the non-inertial frame, the vector sum of the centrifugal force divided mass Δm and the gravitational acceleration called the effective gravitational acceleration is

$$\vec{\mathbf{g}}_{eff} = \vec{\mathbf{g}} + (\vec{\mathbf{F}}_{cf} / m) = \vec{\mathbf{g}} - \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}})$$

$$\vec{\mathbf{g}}_{eff}(x) = -g\hat{\mathbf{k}} + r\omega^2\hat{\mathbf{r}}$$

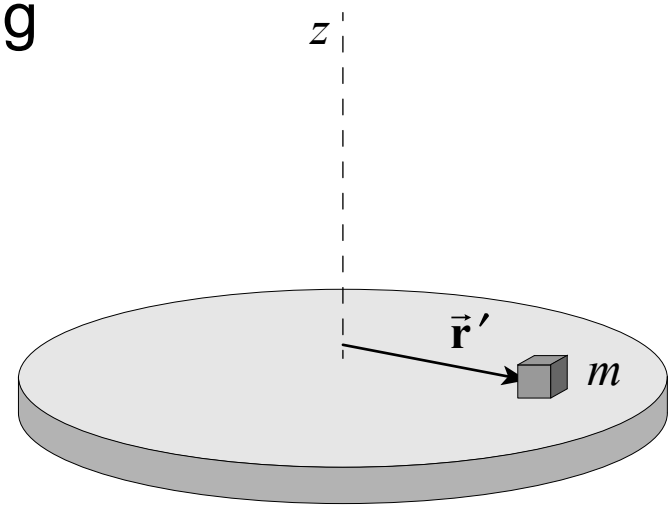
Newton's Second Law in Rotating Reference Frame

In the reference frame *O*' fixed to a rotating platform the object is at rest, $\vec{\mathbf{a}}' = \vec{\mathbf{0}}$ hence the fictitious force is

$$\vec{\mathbf{F}}_{fictitious} = \vec{\mathbf{F}}_{centrifugal} = -m(\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}))$$

$$= -m(\omega\, \hat{\mathbf{k}} \times (\omega\, \hat{\mathbf{k}} \times r\, \hat{\mathbf{r}}))$$

$$= -m(\omega\, \hat{\mathbf{k}} \times r\omega\, \hat{\theta}) = mr\omega^2 \hat{\mathbf{r}}$$



non-inertial frame O' rotating with angular velocity $\vec{\omega}$ with respect to O

The physical force is static friction

$$\vec{\mathbf{F}}_{phyiscal} = \vec{\mathbf{f}}_s = -f_s\, \hat{\mathbf{r}}$$

Newton's Second Law in the rotating frame is then

$$\vec{\mathbf{f}}_s + \vec{\mathbf{F}}_{fictitious} = \vec{\mathbf{0}} \Rightarrow \vec{\mathbf{f}}_s + mr\omega^2 \hat{\mathbf{r}} = \vec{\mathbf{0}}$$

The friction force is then $\vec{\mathbf{f}}_s = f_s\, \hat{\mathbf{r}} = -mr\omega^2 \hat{\mathbf{r}}$

Coriolis Force

An object in the rotating frame has velocity

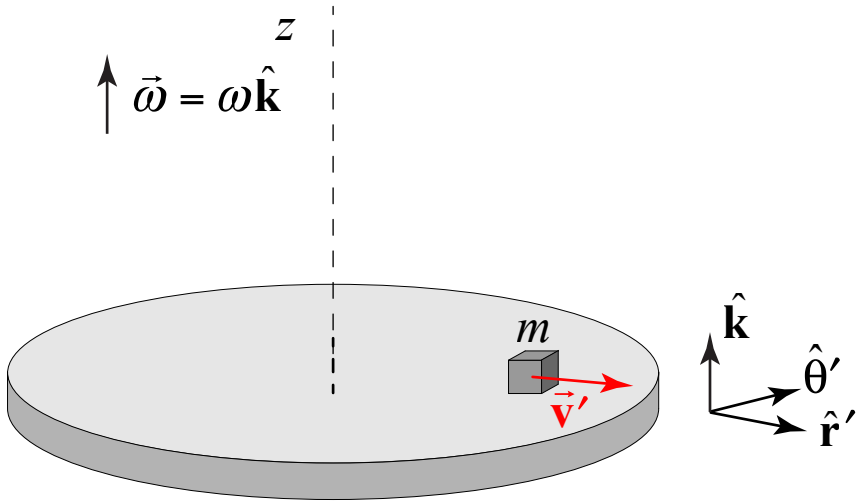
$$\vec{\mathbf{v}}' = v'\hat{\mathbf{r}}'$$

The Coriolis force acting on the object is

$$\vec{\mathbf{F}}_{\text{cor}} = -2m\vec{\omega} \times \vec{\mathbf{v}}' = -2m\omega\, \hat{\mathbf{k}} \times (v'\hat{\mathbf{r}}')$$

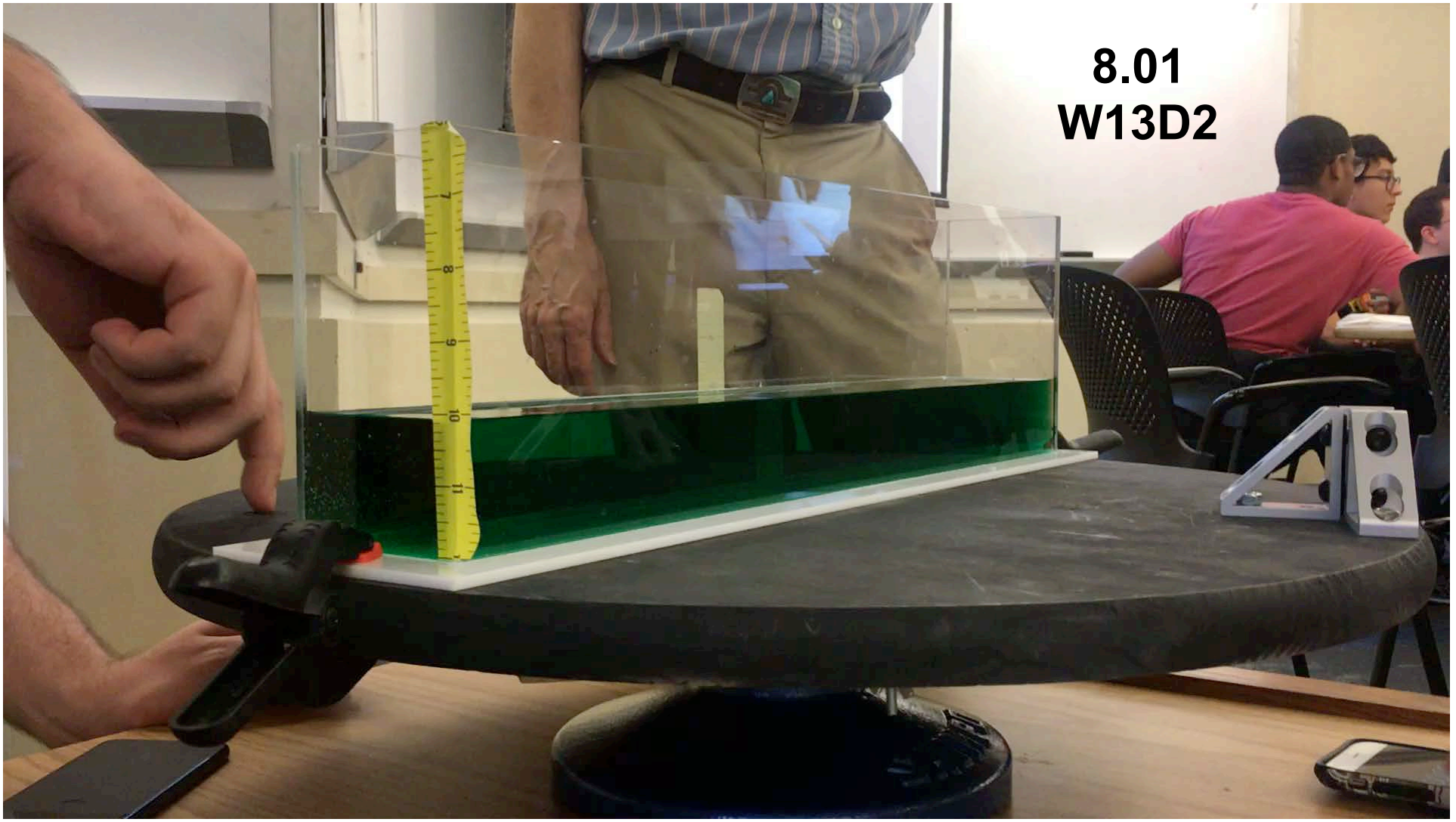
$$= -2m\omega v'\hat{\boldsymbol{\theta}}'$$

The force is in the negative $\hat{\boldsymbol{\theta}}'$ -direction, opposite the counterclockwise rotation of the platform (as seen from above).

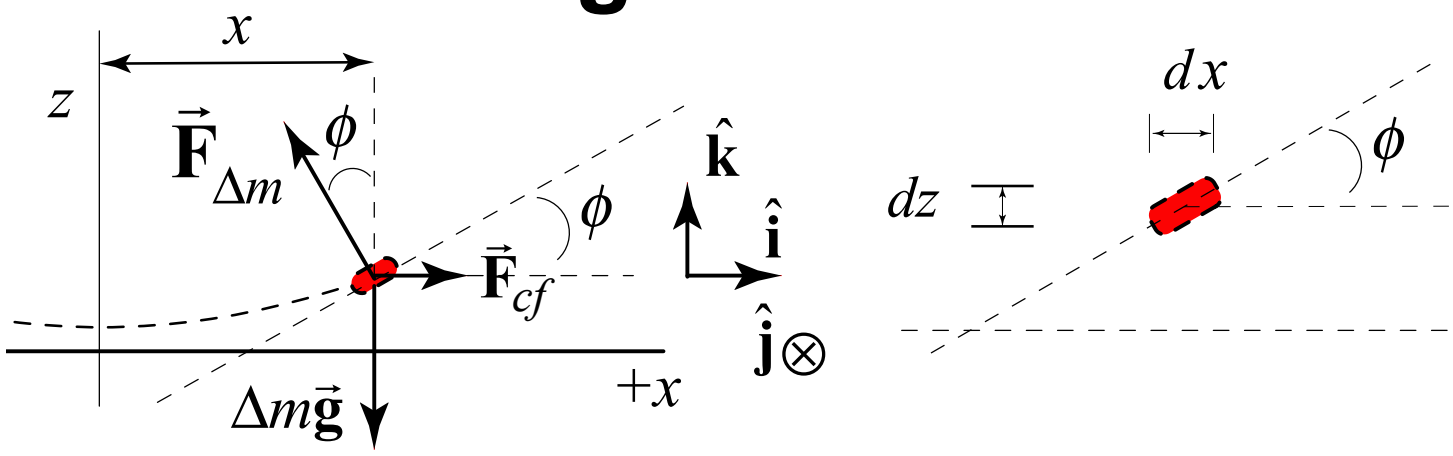


frame O' is rotating with angular velocity $\vec{\omega}$ with respect to inertial frame O

Rotating Water Tank Experiment



Rotating Water Tank



The fluid is in static equilibrium in the rotating frame. Newton's Second Law in rotating frame becomes

$$-F_{\Delta m}\sin\phi + \Delta m r\omega^2 = 0 \text{ and } F_{\Delta m}\cos\phi - \Delta m g = 0 \Rightarrow \tan\phi = \frac{r\omega^2}{g}$$

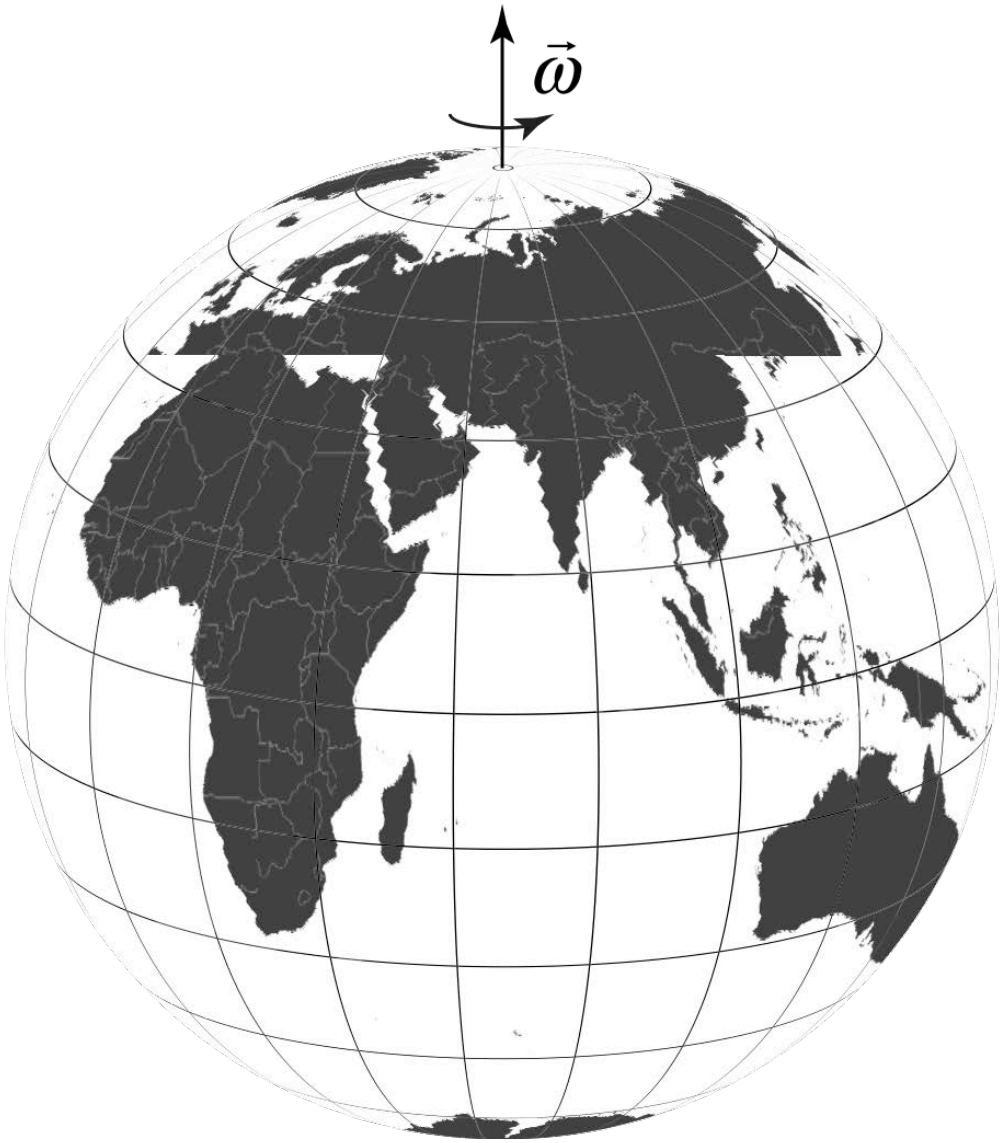
Slope satisfies the condition: $\tan\phi = dz/dr = r\omega^2/g$

Integrate to get equation for surface

$$\int_{z=h_0}^{h(L/2))} dz = \frac{\omega^2}{g}\int_{r=0}^{L/2} r dr$$

$$h(L/2) - h_0 = \frac{1}{8}\frac{\omega^2 L^2}{g} \Rightarrow g = \frac{\omega^2 L^2}{8(h(L/2) - h_0)}$$

Rotating Spherical Reference Frame



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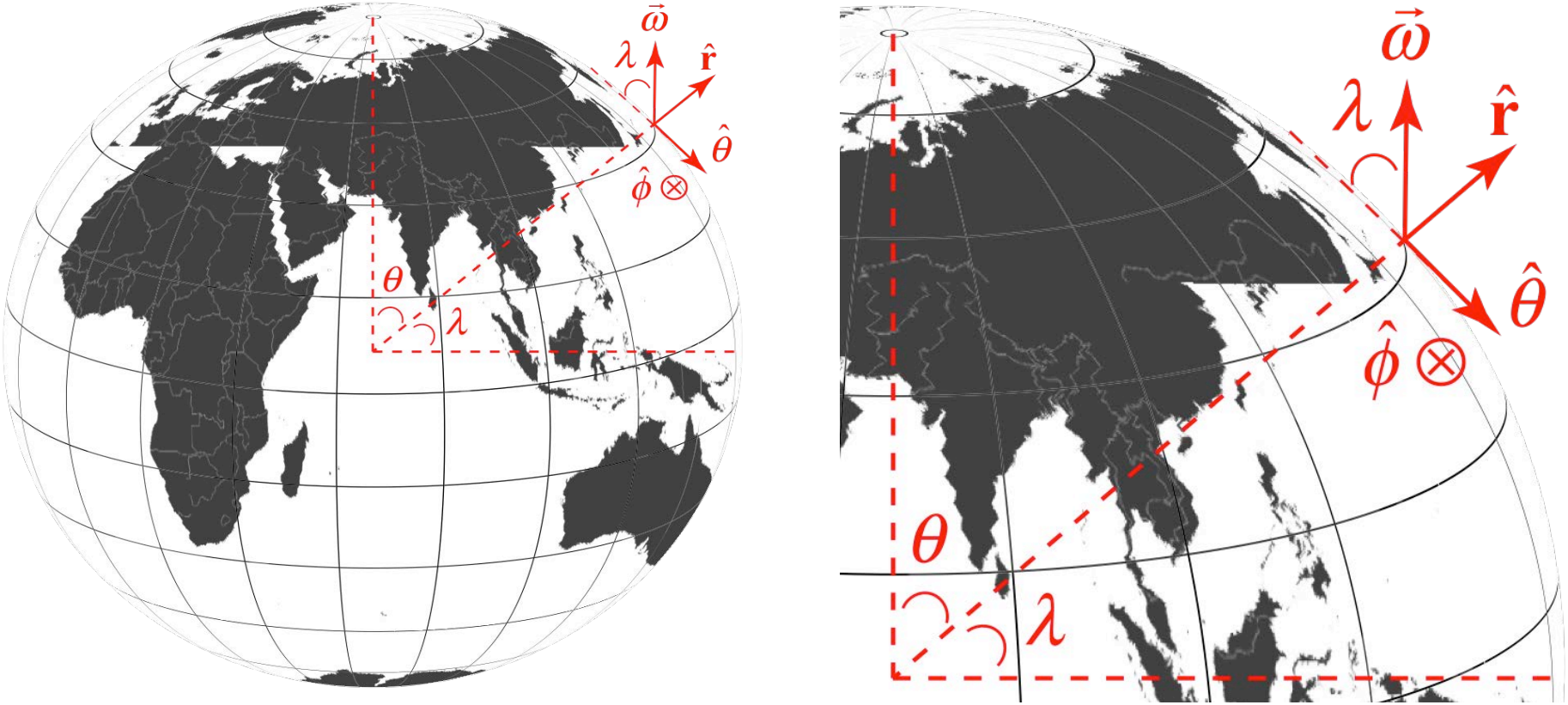
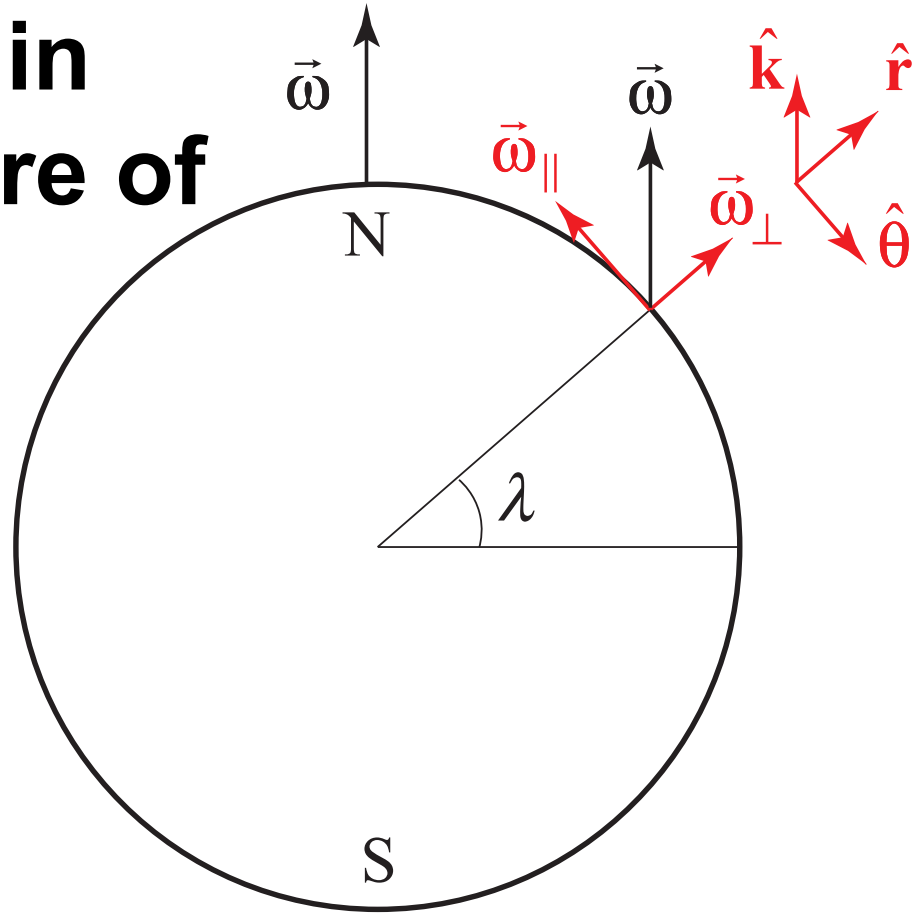
Spherical Coordinates



Each point on the Earth has coordinates (r,θ,ϕ) with unit vectors $(\hat{\mathbf{r}},\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}})$ such that $\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}=\hat{\boldsymbol{\phi}}$.

Note we are dropping primes on unit vectors and coordinates in the rotating frame.

Angular Velocity in Northern Hemisphere of Earth



At north latitude λ, the angular velocity vector can be written as the vector sum

$$\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}} = \omega\sin\lambda\,\hat{\mathbf{r}} - \omega\cos\lambda\hat{\boldsymbol{\theta}}$$

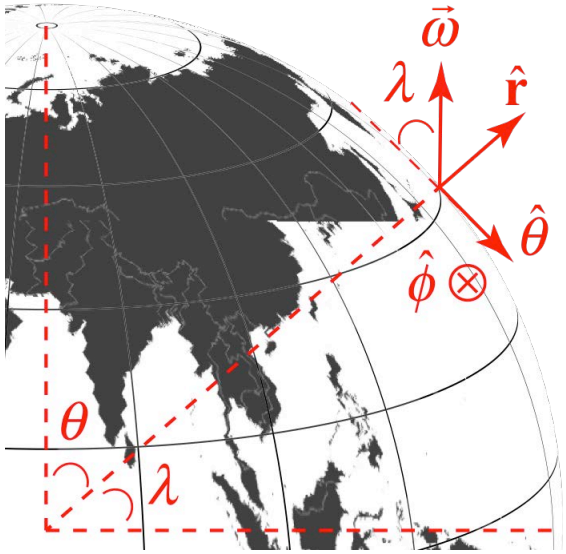
The angular velocity has a component perpendicular to the surface of the Earth:

$$\vec{\omega}_{\perp} = \omega\sin\lambda\,\hat{\mathbf{r}}$$

And a component parallel to the surface of the Earth

$$\vec{\boldsymbol{\omega}}_{\parallel} = -\omega\cos\lambda\hat{\boldsymbol{\theta}}$$

Particle Moving Tangent to Surface of the Earth: Magnitude of Components of Coriolis Force in Northern Hemisphere



$$\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}} = \omega\sin\lambda\,\hat{\mathbf{r}} - \omega\cos\lambda\hat{\boldsymbol{\theta}}$$

$$\vec{\mathbf{F}}_{\text{coriolis}} = -2m\vec{\boldsymbol{\omega}}\times\vec{\mathbf{v}}' = -2m(\vec{\boldsymbol{\omega}}_{\perp} + \vec{\boldsymbol{\omega}}_{\parallel})\times\vec{\mathbf{v}}'$$

$$= -2m\vec{\boldsymbol{\omega}}_{\perp}\times\vec{\mathbf{v}}' - 2m\vec{\boldsymbol{\omega}}_{\parallel}\times\vec{\mathbf{v}}'$$

The magnitude of the component tangent to the surface of the Earth is

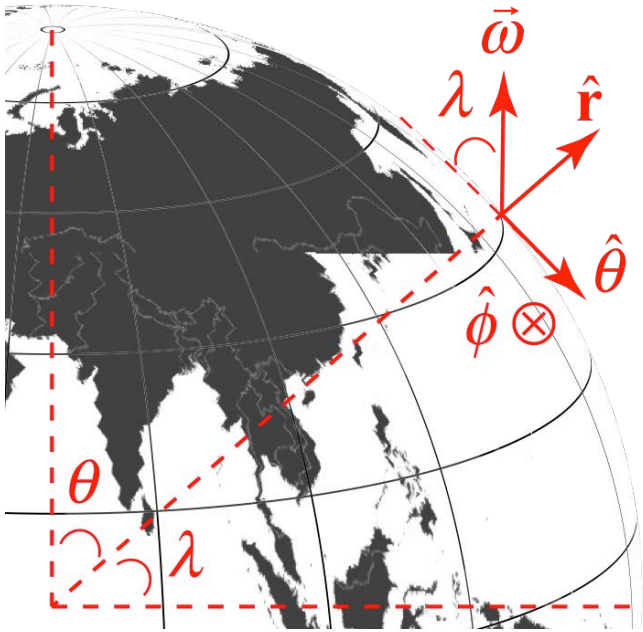
$$\left|-2m\vec{\boldsymbol{\omega}}_{\perp}\times\vec{\mathbf{v}}'\right| = 2m\omega\sin\lambda v' = 2m\omega_{\perp}v'$$

The component perpendicular to the surface of the Earth is

$$-2m\vec{\boldsymbol{\omega}}_{\parallel}\times\vec{\mathbf{v}}' = -2m\omega\cos(\lambda)\hat{\boldsymbol{\theta}}\times(v'_{\theta}\hat{\boldsymbol{\theta}} + v'_{\phi}\hat{\phi}) = -2m\omega_{\parallel}v'_{\phi}\hat{\mathbf{r}}$$

Direction of Coriolis Force

A particle is located at north latitude λ and traveling due west with speed v'? What is the direction of the component of the Coriolis force tangent to the surface of the earth?



The Coriolis force is given by

$$\vec{\mathbf{F}}_{\text{cor}} = -2m\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}' = -2m(\omega \sin(\lambda)\hat{\mathbf{r}} - \omega \cos(\lambda)\hat{\boldsymbol{\theta}}) \times (-v'\hat{\boldsymbol{\phi}})$$

$$= -2m\omega v' \sin(\lambda)\,\hat{\boldsymbol{\theta}} - 2m\omega v' \cos(\lambda)\hat{\mathbf{r}} = -2m\omega_{\perp} v'\,\hat{\boldsymbol{\theta}} - 2m\omega_{\parallel} v'\hat{\mathbf{r}}$$

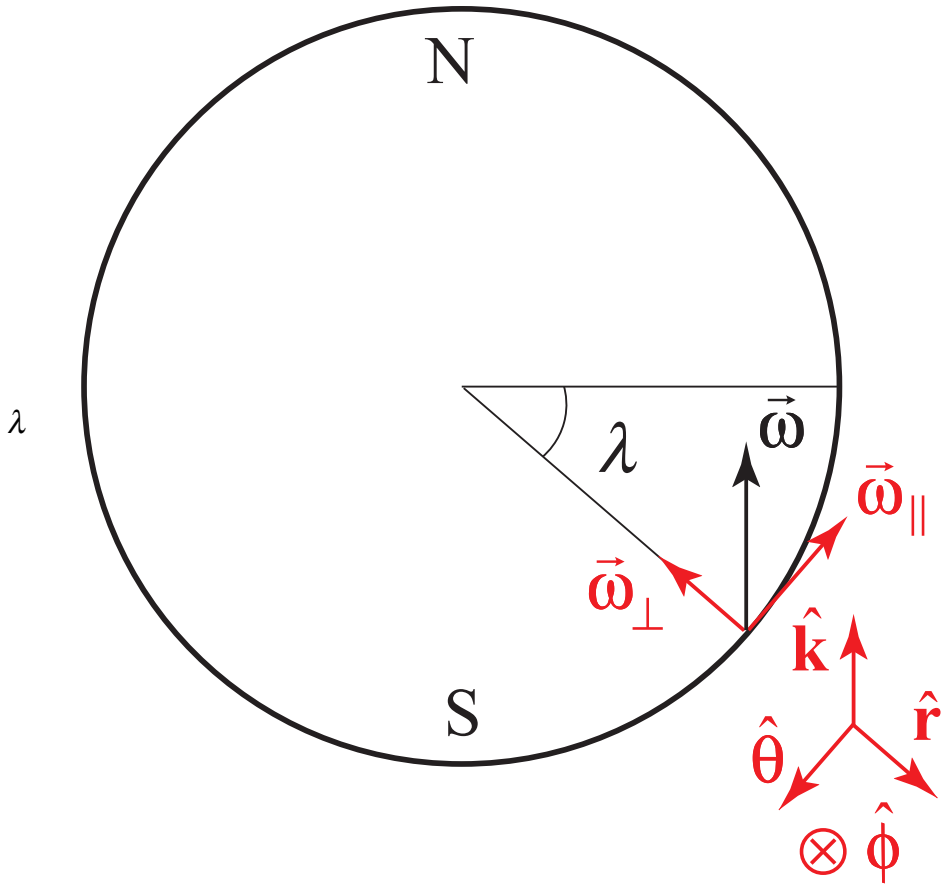
The tangential component is $\vec{\mathbf{F}}_{\text{cor},\theta} = -2m\omega v' \sin(\lambda)\,\hat{\boldsymbol{\theta}} = -2m\omega_{\perp} v'\,\hat{\boldsymbol{\theta}}$ which is pointing due north which is in the negative $\hat{\boldsymbol{\theta}}$ direction.

Motion in the Southern Hemisphere

The angular velocity is

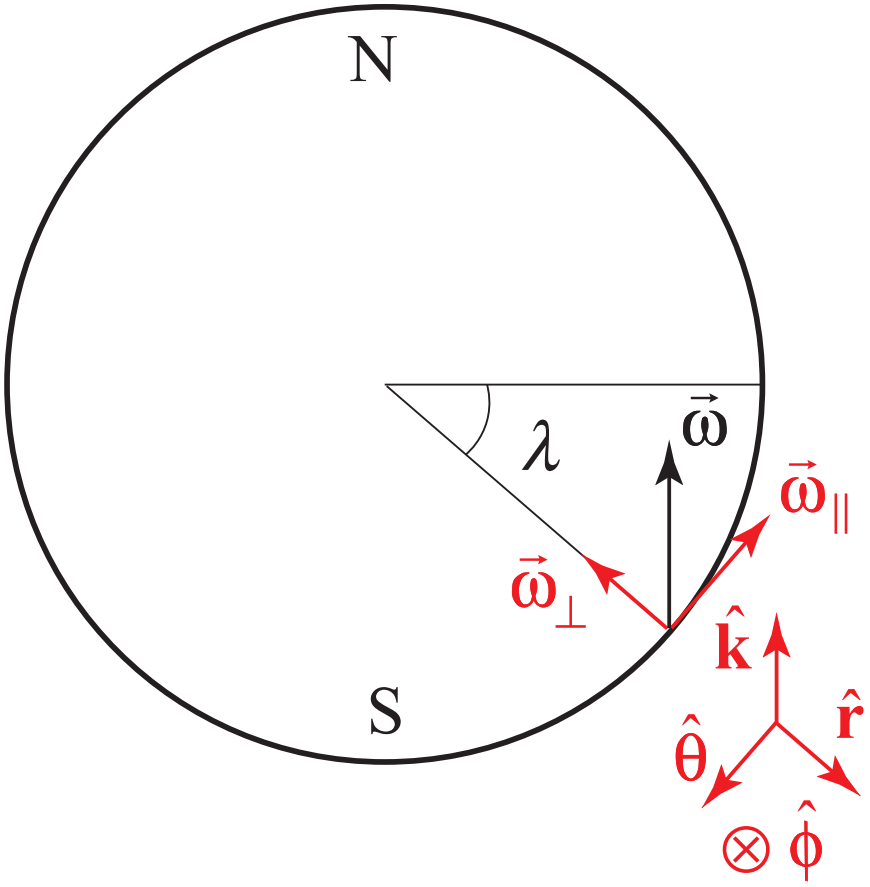
$$\vec{\omega} = \omega\hat{\mathbf{k}} = -\omega\sin\lambda\,\hat{\mathbf{r}} - \omega\cos\lambda\hat{\boldsymbol{\theta}}$$

Key fact: The radial vector component points inward.



Gp. Prob. Motion in the Southern Hemisphere

Consider a particle located along south latitude λ > 0 is traveling with southward with velocity $\vec{\mathbf{v}}' = v'\hat{\boldsymbol{\theta}}$



The angular velocity of Earth at that latitude is

$$\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}} = -\omega\sin\lambda\,\hat{\mathbf{r}} - \omega\cos\lambda\hat{\boldsymbol{\theta}}$$

Find a vector expression for the Coriolis force acting on the particle.

$$\vec{\mathbf{F}}_{\text{coriolis}} = -2m\vec{\boldsymbol{\omega}}\times\vec{\mathbf{v}}'$$

Express your answers in terms of $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \omega$ and λ, as needed.

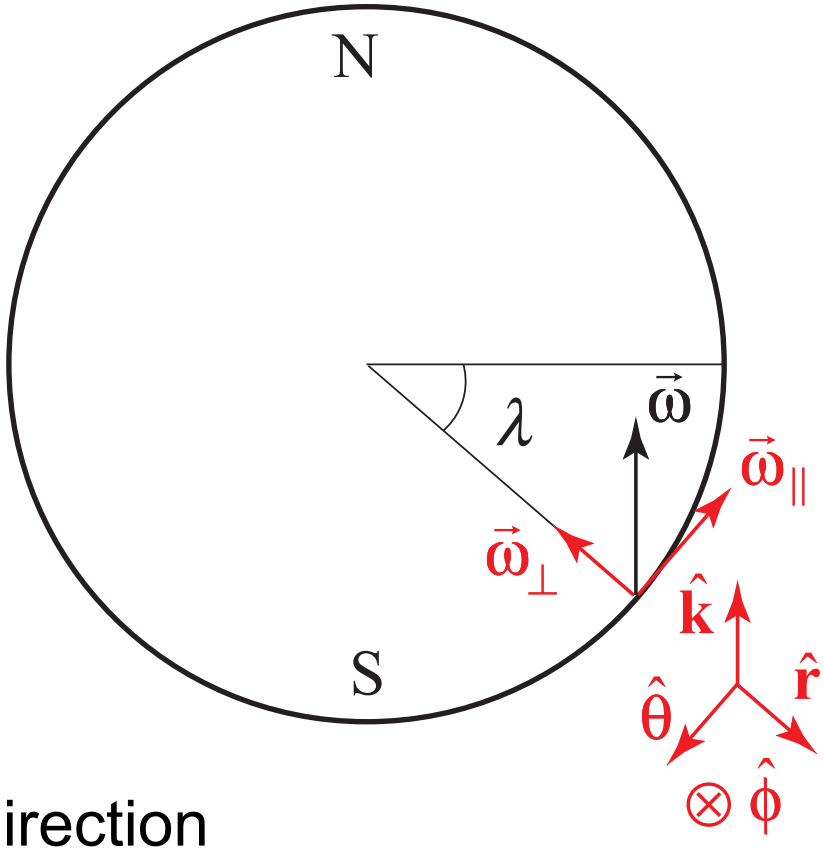
Motion in the Southern Hemisphere

The angular velocity is

$$\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}} = -\omega\sin\lambda\,\hat{\mathbf{r}} - \omega\cos\lambda\hat{\boldsymbol{\theta}}$$

b) The Coriolis force is given by

$$\vec{\mathbf{F}}_{\text{coriolis}} = -2m\vec{\boldsymbol{\omega}}\times\vec{\mathbf{v}}'$$
$$= -2m(-\omega\sin\lambda\,\hat{\mathbf{r}} - \omega\cos\lambda\hat{\boldsymbol{\theta}})\times v'\hat{\boldsymbol{\theta}}$$
$$= +2m\omega\sin\lambda v'\hat{\boldsymbol{\phi}}$$



which is due east in the positive $\hat{\boldsymbol{\phi}}$-direction

Note that the particle curves to the left, opposite the case in the northern hemisphere

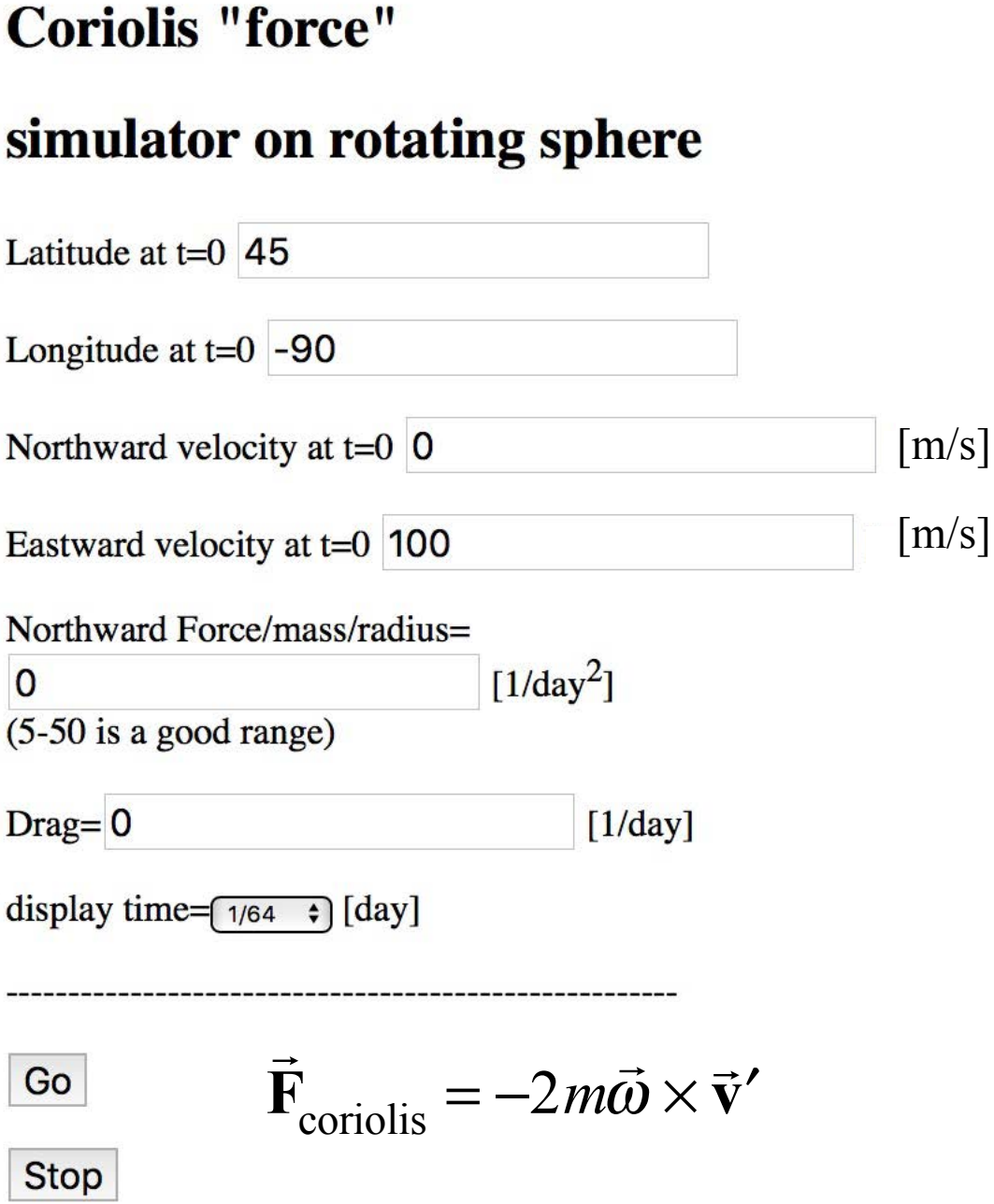
Coriolis with Pressure Force

Open up Coriolis on Sphere at
http://cmpo5.mit.edu/8.01/

a) What happens in the absence of a pressure force?

b) Try to find a northward pressure force such that the initial 100 m/s velocity makes the particle just circle the globe.

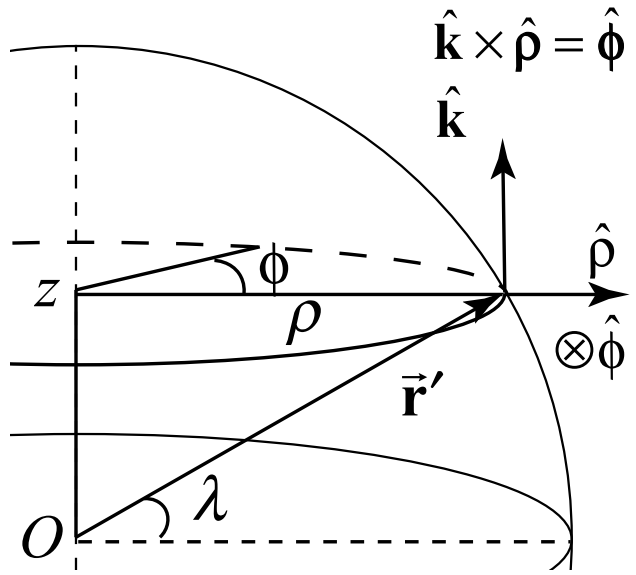
c) Consider how the Coriolis force acts on a particle moving east at one latitude and west at a lower latitude Use negative value) . If the speeds are the same, which will have the larger curvature?



Centrifugal Force on Earth in Cylindrical Coordinates

The angular velocity of the Earth is

$$\vec{\boldsymbol{\omega}} = \omega \hat{\mathbf{k}}$$



The centrifugal force is

$$\vec{\mathbf{F}}_{centrifugal} = -m(\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}'))$$

The vector $\vec{\mathbf{r}}' = R_E \cos\lambda \hat{\boldsymbol{\rho}} + R_E \sin\lambda \hat{\mathbf{k}}$

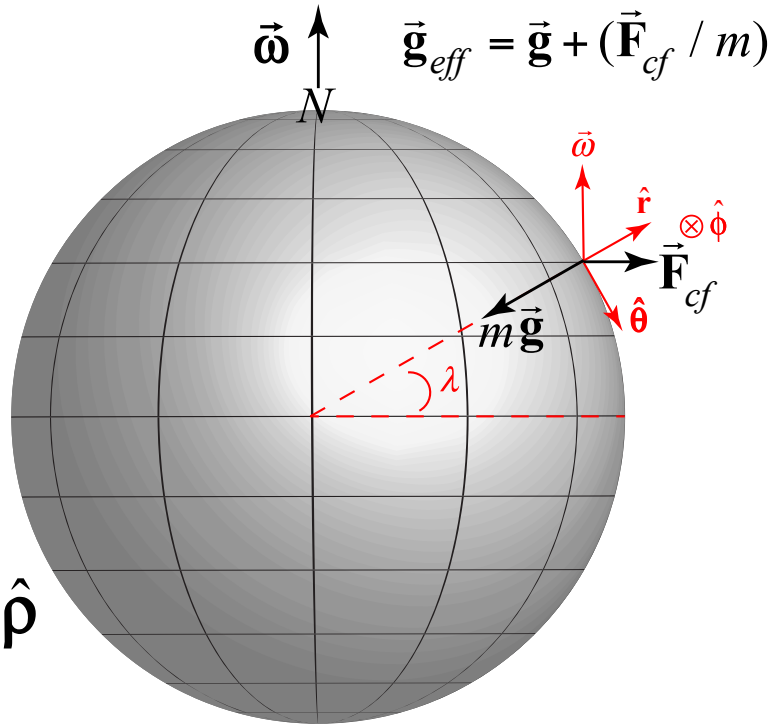
Thus $\vec{\mathbf{F}}_{centrifugal} = -m(\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}'))$

$$= -m(\omega\hat{\mathbf{k}} \times (\omega\hat{\mathbf{k}} \times (R_E \cos\lambda \hat{\boldsymbol{\rho}} + R_E \sin\lambda \hat{\mathbf{k}}))$$

$$= -m(\omega\hat{\mathbf{k}} \times R_E \cos\lambda\omega\hat{\boldsymbol{\phi}}) = mR_E \cos\lambda\omega^2\hat{\boldsymbol{\rho}}$$

In the rotating reference frame the centrifugal force points radially away from the axis of rotation.

Effective Gravity on Earth



In the non-inertial frame fixed to the Earth, effective gravitational acceleration is

$$\vec{\mathbf{g}}_{eff} = \vec{\mathbf{g}} + (\vec{\mathbf{F}}_{cf} / m)$$

$$\vec{\mathbf{g}}_{eff} = \vec{\mathbf{g}} - \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}') = -g\hat{\mathbf{r}} + r\cos\lambda\omega^2\hat{\boldsymbol{\rho}}$$

$$\vec{\mathbf{g}}_{eff} = -g\hat{\mathbf{r}} + r\cos\lambda\omega^2(\cos\lambda\hat{\mathbf{r}} + \sin\lambda\hat{\boldsymbol{\theta}})$$

$$= -(g - mr\cos^2\lambda\omega^2)\hat{\mathbf{r}} + mr\cos\lambda\sin\lambda\omega^2\hat{\boldsymbol{\theta}}$$

The magnitude is

$$g_{eff} = ((g - r\cos^2\lambda\omega^2)^2 + (r\cos\lambda\sin\lambda\omega^2)^2)^{1/2}$$

At the latitude of MIT $\lambda = 42.36^\circ\,\mathrm{N}$: $g = 9.82\ \mathrm{m\cdot s^{-2}}$

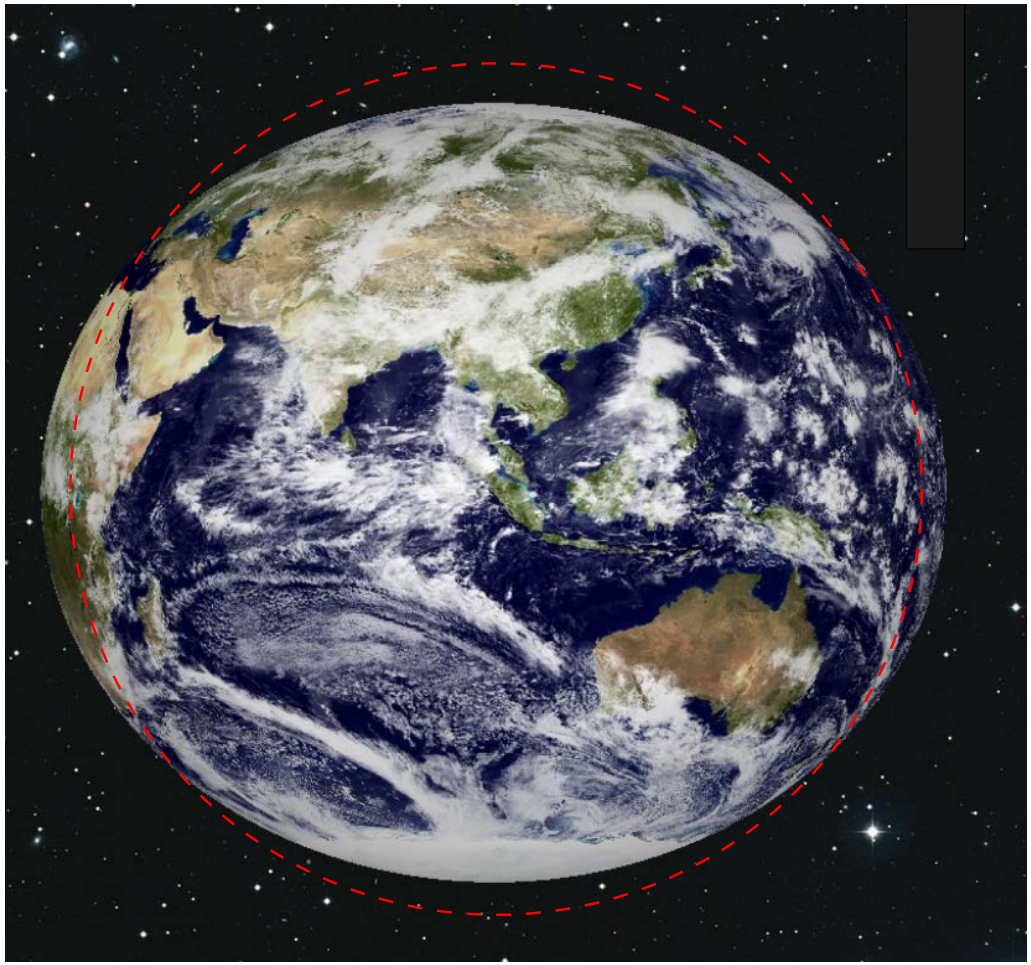
$g_{eff} = 9.801\ \mathrm{m\cdot s^{-2}}$

Rotational Oblateness

Open up

http://cmpo5.mit.edu/8.01/

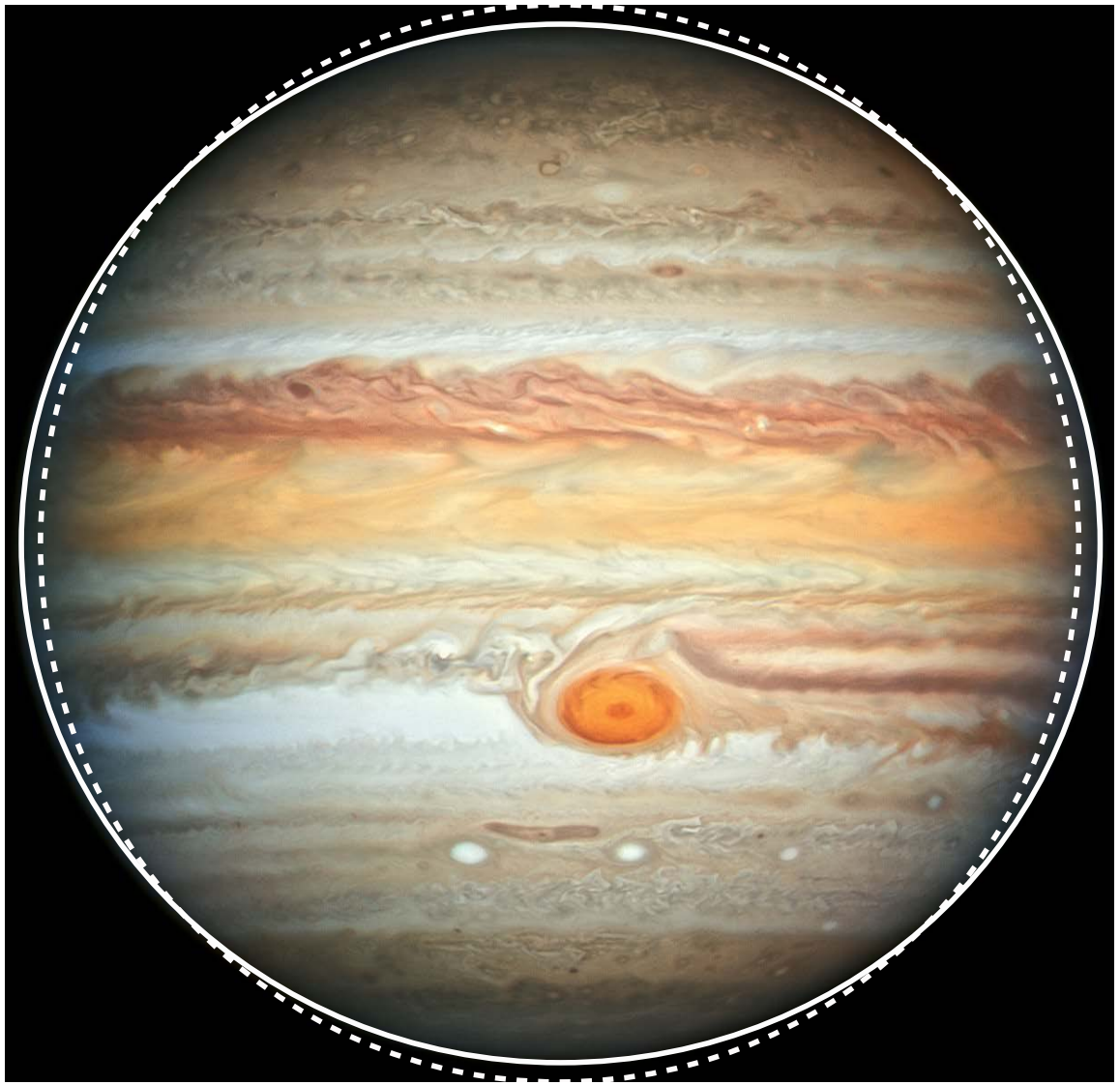
And click on Centrifugal force Rotational Oblateness. Try a few rotational periods, especially ten hours.



Rotational period four hours

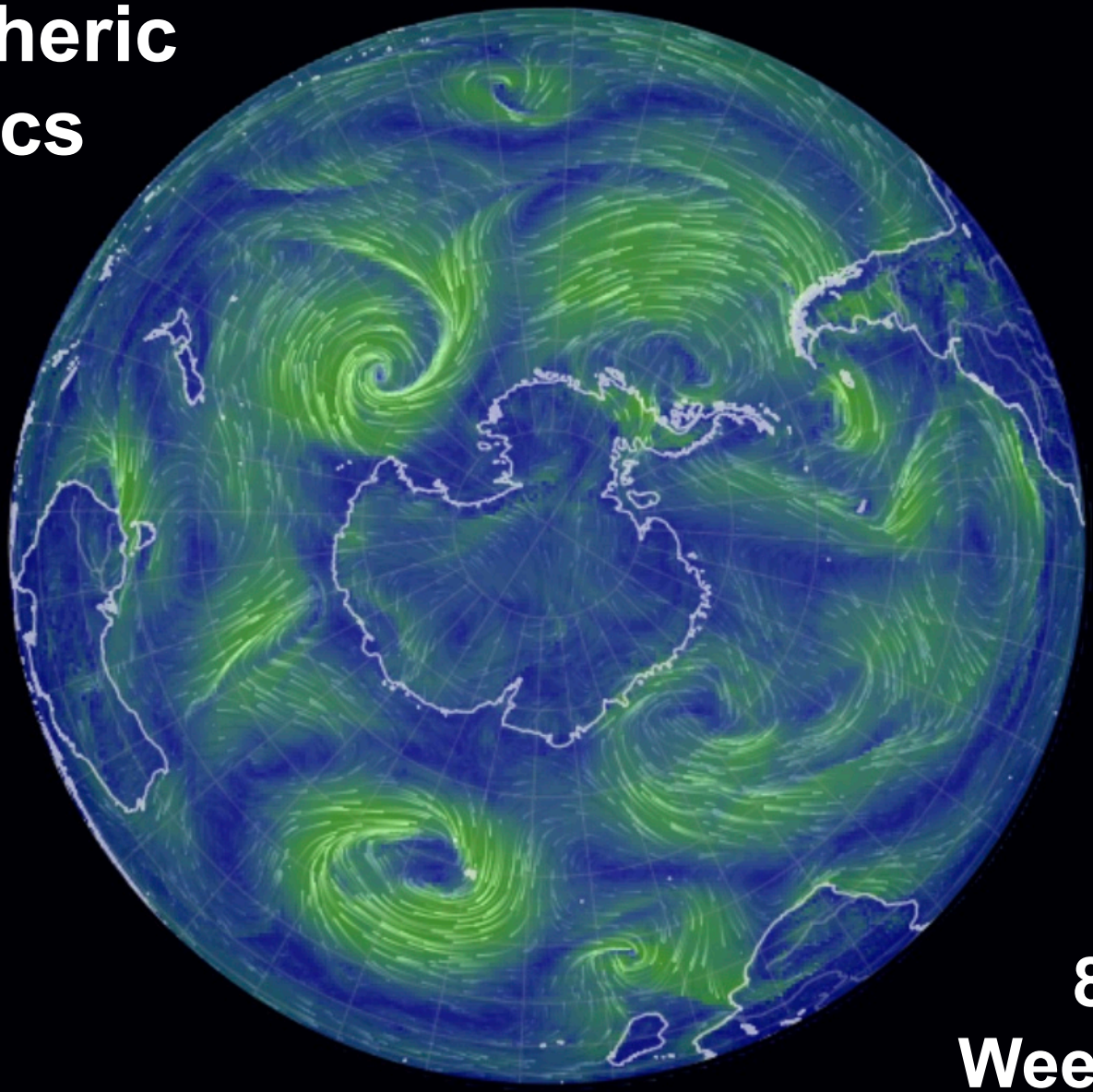
http://cmpo5.mit.edu/esg1/index.html?8.01/cor.html;r=425

Rotational period of Jupiter: 10 hours



http://cmpo5.mit.edu/esg1/index.html?8.01/cor.html;r=425

Atmospheric Physics



8.01
Week 14D2

earth ≡

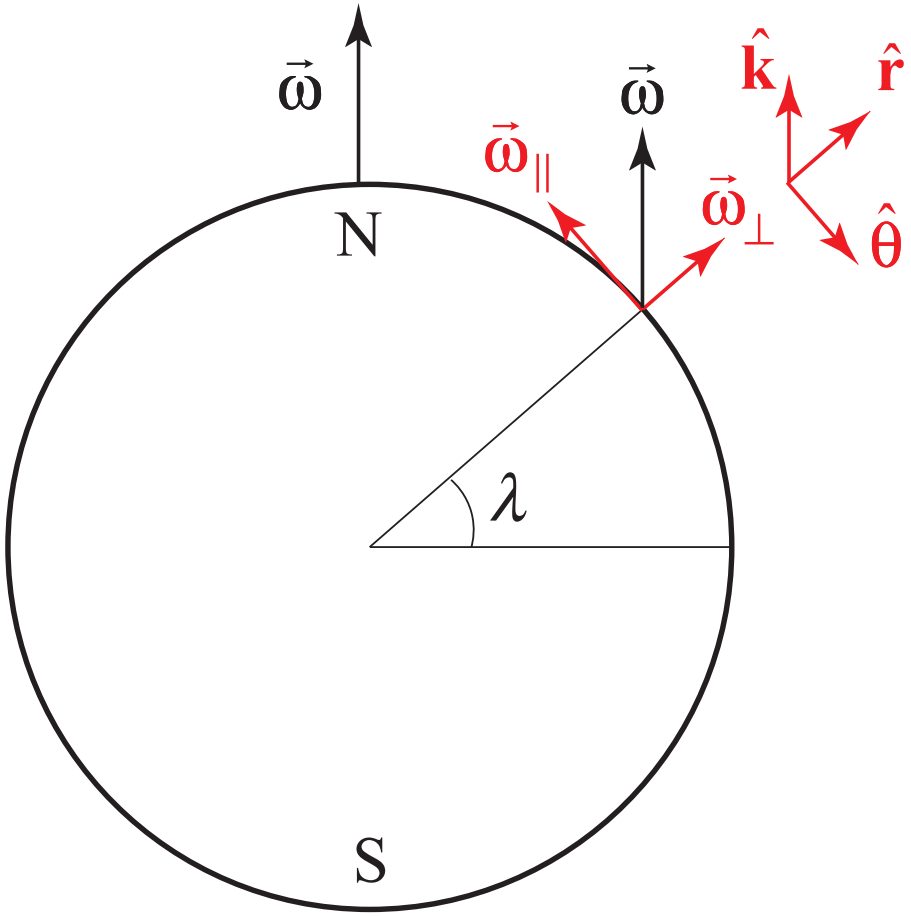
Recall: Motion on the Earth: Angular velocity decomposition

At the north latitude λ, the angular velocity vector can be written as the vector sum

$$\vec{\omega} = \vec{\omega}_{\perp} + \vec{\omega}_{\parallel}$$

$$\vec{\omega}_{\perp} = \omega \sin\lambda \hat{\mathbf{r}}$$

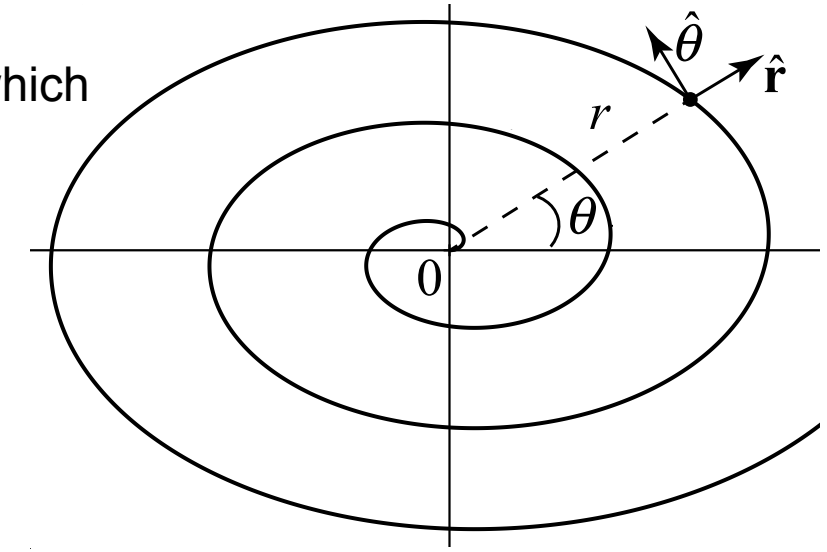
$$\vec{\omega}_{\parallel} = -\omega \sin\lambda \hat{\boldsymbol{\theta}}$$



Position, Velocity and Acceleration in Polar Coordinates

Consider an object moving in the plane in which

$$\frac{dr}{dt} \neq 0 \ \ and \ \frac{d\theta}{dt} \neq 0$$



Recall that

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}} \ \text{and} \ \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\frac{d\theta}{dt}\hat{\mathbf{r}}$$

Starting with the position vector

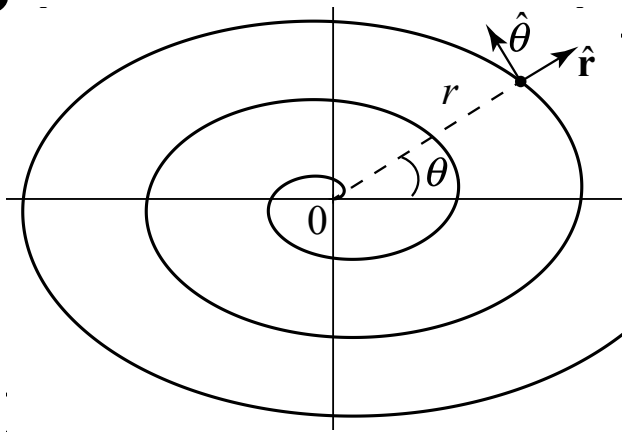
$$\vec{\mathbf{r}}(t) = r(t)\hat{\mathbf{r}}(t)$$

Use the product rule to derive vector expressions for

a) $\vec{\mathbf{v}}(t) = \dfrac{d\vec{\mathbf{r}}(t)}{dt}$

b) $\vec{\mathbf{a}}(t) = \dfrac{d\vec{\mathbf{v}}(t)}{dt} = \dfrac{d^2\vec{\mathbf{r}}(t)}{dt^2}$

Velocity and Acceleration: Polar Coordinates



a) The velocity is

$$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{r}}(t)}{dt} = \frac{d}{dt}(r(t)\hat{\mathbf{r}}(t))$$

$$= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}$$

b) The acceleration is then

$$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}(t)}{dt} = \frac{d}{dt}\left(\frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}\right)$$

$$= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\hat{\mathbf{r}}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{dt}$$

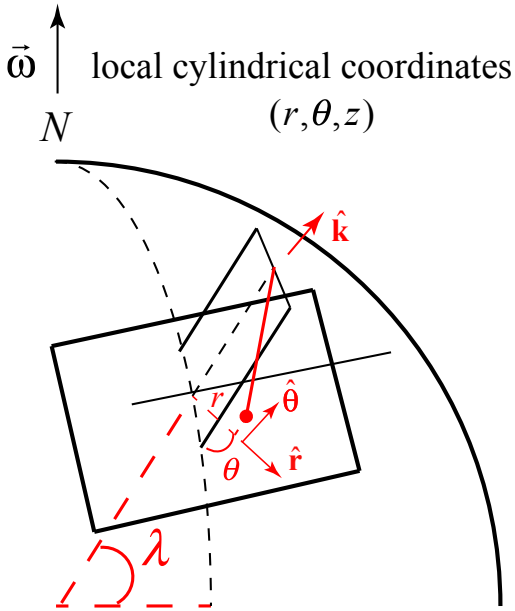
$$= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}} - r\left(\frac{d\theta}{dt}\right)^2\hat{\mathbf{r}}$$

$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\mathbf{r}} + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\hat{\boldsymbol{\theta}}$$

Foucault pendulum: kinematics

For small angles of oscillation, the pendulum undergoes simple harmonic motion with the position of the bob in the horizontal plane is given by

$$r(t) = A\sin(\omega_0 t)$$



The angular rotational velocity of Earth at latitude λ is given by

$$\vec{\boldsymbol{\omega}} = \vec{\boldsymbol{\omega}}_{||} + \vec{\boldsymbol{\omega}}_{\perp} = \vec{\boldsymbol{\omega}}_{||} + \omega\sin\lambda\hat{\mathbf{k}}$$

where $\vec{\boldsymbol{\omega}}_{||}$ is the component in the horizontal plane. The Coriolis force is

$$\vec{\mathbf{F}}_{cor} = -2m(\vec{\boldsymbol{\omega}}_{\perp} + \vec{\boldsymbol{\omega}}_{||}) \times \vec{\mathbf{v}}$$

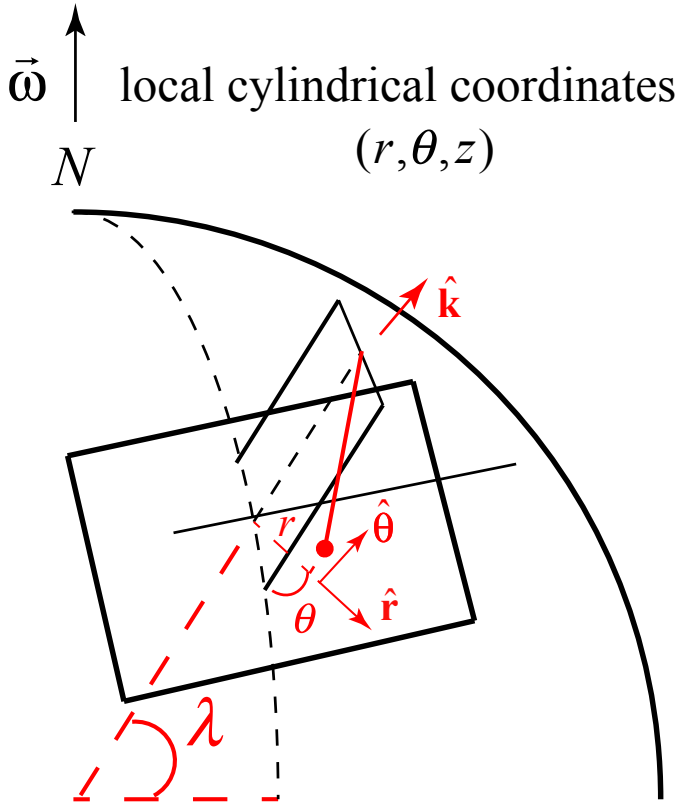
We are assuming that dz/dt is very small, so the component of the Coriolis force in the tangential plane is

$$\vec{\mathbf{F}}_{cor,||} = -2m(\vec{\boldsymbol{\omega}}_{\perp} \times \vec{\mathbf{v}})$$

Foucault pendulum: Coriolis force in horizontal plane

The velocity of the bob in the horizontal plane is

$$\vec{\mathbf{v}}_{hor}(t) = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}$$



Therefore Coriolis force is

$$-2m\vec{\omega}_{\perp} \times \vec{\mathbf{v}} = -2m\left[\omega \sin\lambda \hat{\mathbf{k}} \times \left(\frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}\right)\right]$$

$$= -2m\left(\omega \sin\lambda \frac{dr}{dt}\hat{\boldsymbol{\theta}} + \omega \sin\lambda r\frac{d\theta}{dt}\hat{\mathbf{r}}\right)$$

The component of the Coriolis force in the tangential direction is

$$F_{\theta}^{cor} = -2m\omega \sin\lambda \frac{dr}{dt}$$

Foucault pendulum: Newton's 2nd Law

The acceleration the tangential direction is $a_\theta = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$

Therefore Newton's Second Law is $-2m\omega\sin\lambda\frac{dr}{dt} = m\left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)$

One possible solution occurs when $\frac{d\theta}{dt} = \text{constant} \Rightarrow \frac{d^2\theta}{dt^2} = 0$

Hence $\frac{d\theta}{dt} = -\omega\sin\lambda$

The pendulum precesses in the negative $\hat{\boldsymbol{\theta}}$ -direction. With a period

$$T = 2\pi / \omega\sin\lambda = 24\text{ h}/\sin\lambda$$

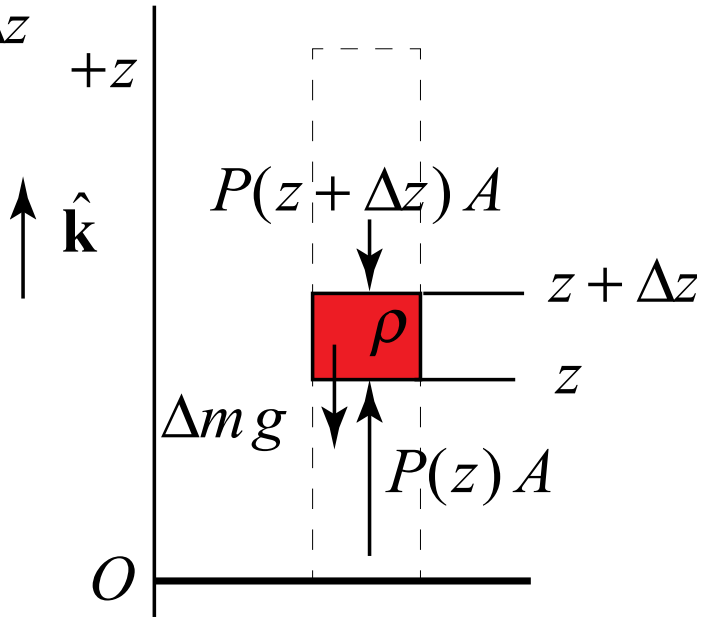
When $\lambda = 90^\circ, \text{then } T = 24\text{ h}$. Earth rotates underneath the pendulum. The plane of the pendulum's oscillation is not changing with respect to the local inertial space

General: Pressure Gradient Forces

Pressure Force: $A = \Delta x \Delta y \quad \Delta m = \rho \Delta V = \rho A \Delta z$

$$\vec{\mathbf{F}}(z+\Delta z) + \vec{\mathbf{F}}(z) = (-P(z+\Delta z) + P(z))A\hat{\mathbf{k}}$$

$$\vec{\mathbf{F}}(z+\Delta z) + \vec{\mathbf{F}}(z) = -\Delta P A \hat{\mathbf{k}}$$



Pressure Force per unit mass:

$$\frac{\vec{\mathbf{F}}(z+\Delta z) + \vec{\mathbf{F}}(z)}{\Delta m} = -\frac{\Delta P}{\Delta m} A\hat{\mathbf{k}}$$

Limit:

$$f_z = \lim_{\Delta m \to 0} \frac{F_z(z+\Delta z) + F_z(z)}{\Delta m} = -\frac{1}{\rho} \lim_{\Delta z \to 0} \frac{\Delta P}{\Delta z} = -\frac{1}{\rho}\frac{dP}{dz}$$

Generalization:

$$f_x(x,y,z) = -\frac{1}{\rho}\left(\frac{\partial P}{\partial x}\right)_{y,z} \qquad f_y(x,y,z) = -\frac{1}{\rho}\left(\frac{\partial P}{\partial y}\right)_{x,z} \qquad f_z(x,y,z) = -\frac{1}{\rho}\left(\frac{\partial P}{\partial z}\right)_{x,y}$$

Pressure and Height

In meteorology, pressure, which is easily measured, is used as a vertical coordinate rather than height.

$$dz = (dz/dx)dx$$

The difference in height between the points B and C is given by

$$\Delta P = P_C - P_B = \rho g dz = \rho g \frac{dz}{dx} dx$$

Because the fluid is in hydrostatic equilibrium, the positive pressure difference between the two points C and B is

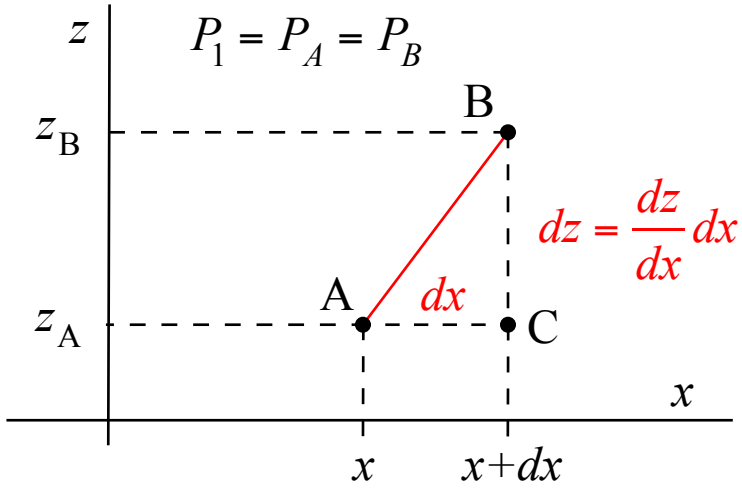
$$\Delta P = P_C - P_A = P_C - P_B = \frac{dP}{dx} dx$$

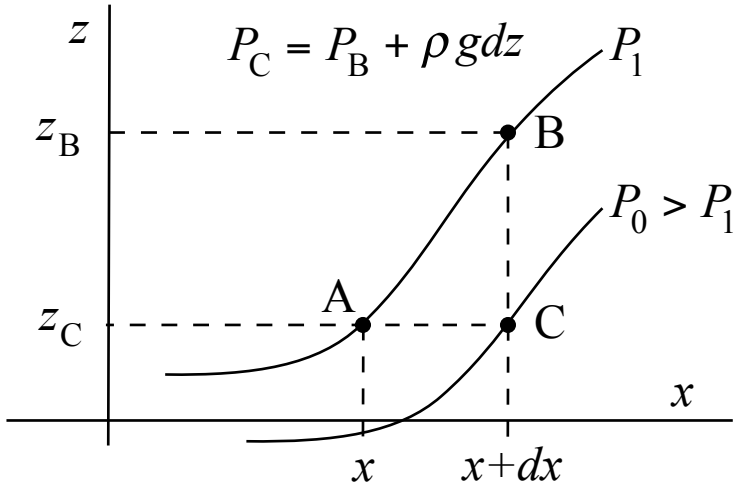
The pressure difference between the two points A and C, where $P_A = P_B$, that are at the same height is to first order:

$$\frac{dP}{dx} = \rho g \frac{dz}{dx}$$

Comparing these two results yields

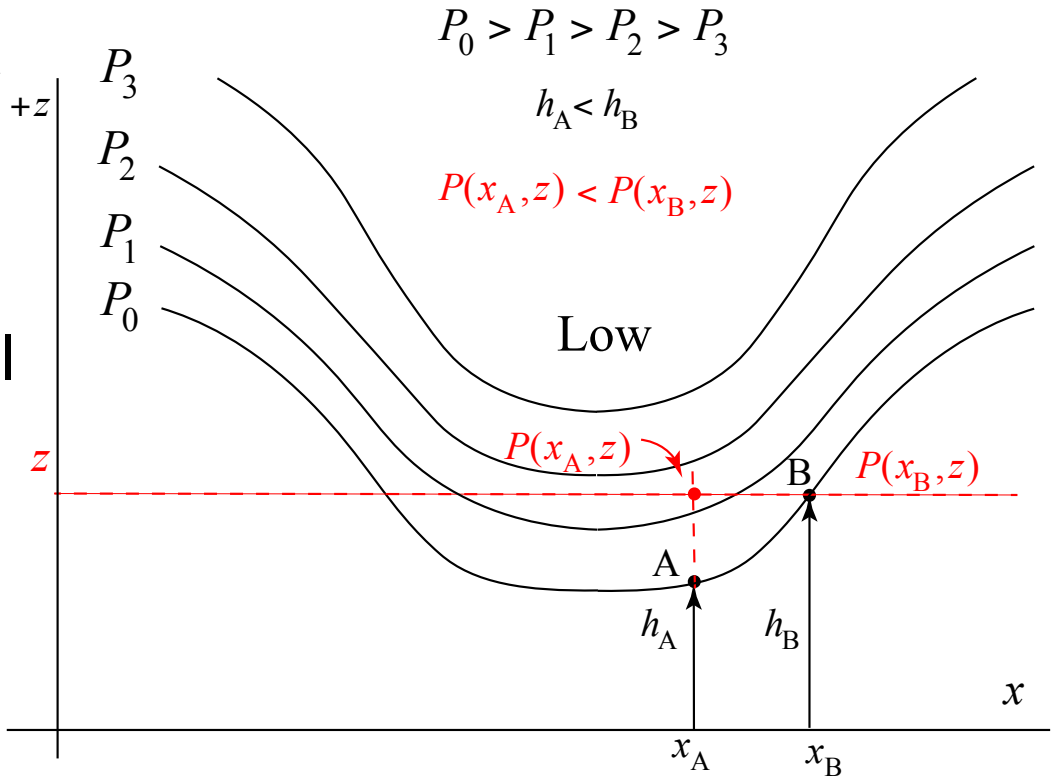
Force per unit mass $$f_x = -\frac{1}{\rho}\frac{dP}{dx} = -g\frac{dz}{dx}$$





Motion on Pressure Contours

In meteorology, a map of a constant pressure surface with contours on the map representing the height above the ground, is useful for visualizing the flow or air. Air will flow from higher heights to lower heights; i.e. downhill on the height surface.

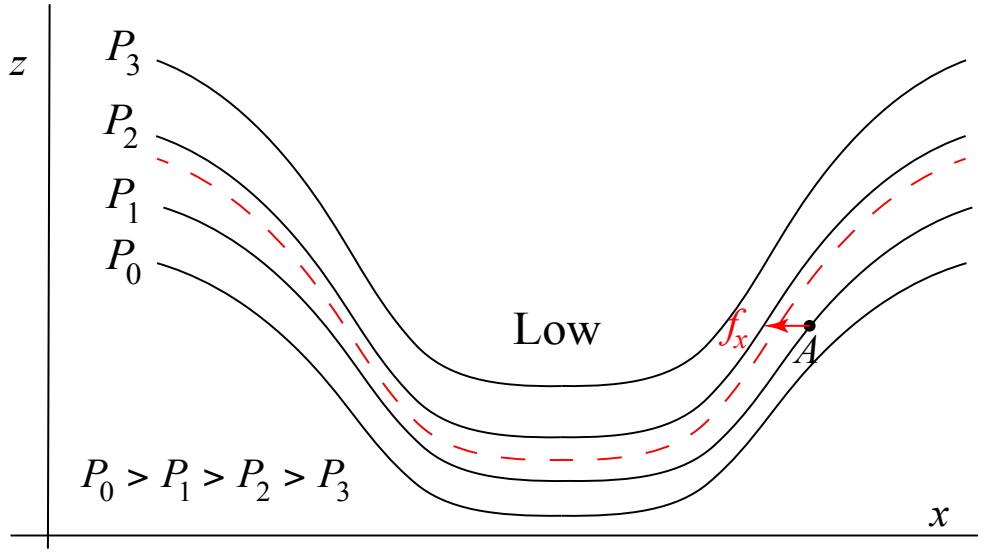


The flow on the constant pressure surface P_0 is from B to A. In the figure the point A is at lower height than the point B. Then on any level surface z = constant

$$P(x_A,z) < P(x_B,z)$$

The flow on the constant pressure surface P_0 is from B to A.

Pressure Contours and Flow



The pressure gradient force per unit mass at point A is

$$f_x = -\frac{1}{\rho}\frac{dP}{dx} = -g\frac{dh}{dx}$$

In the figure above at the point A, dP/dx > 0, dh/dx > 0, so the force is in the negative x-direction .

The force tries to push fluid from the high pressure to low pressure regions or, equivalently, from regions where the pressure surface is higher to regions where it's lower.

Height field

Open up http://cmpo5.mit.edu/8.01

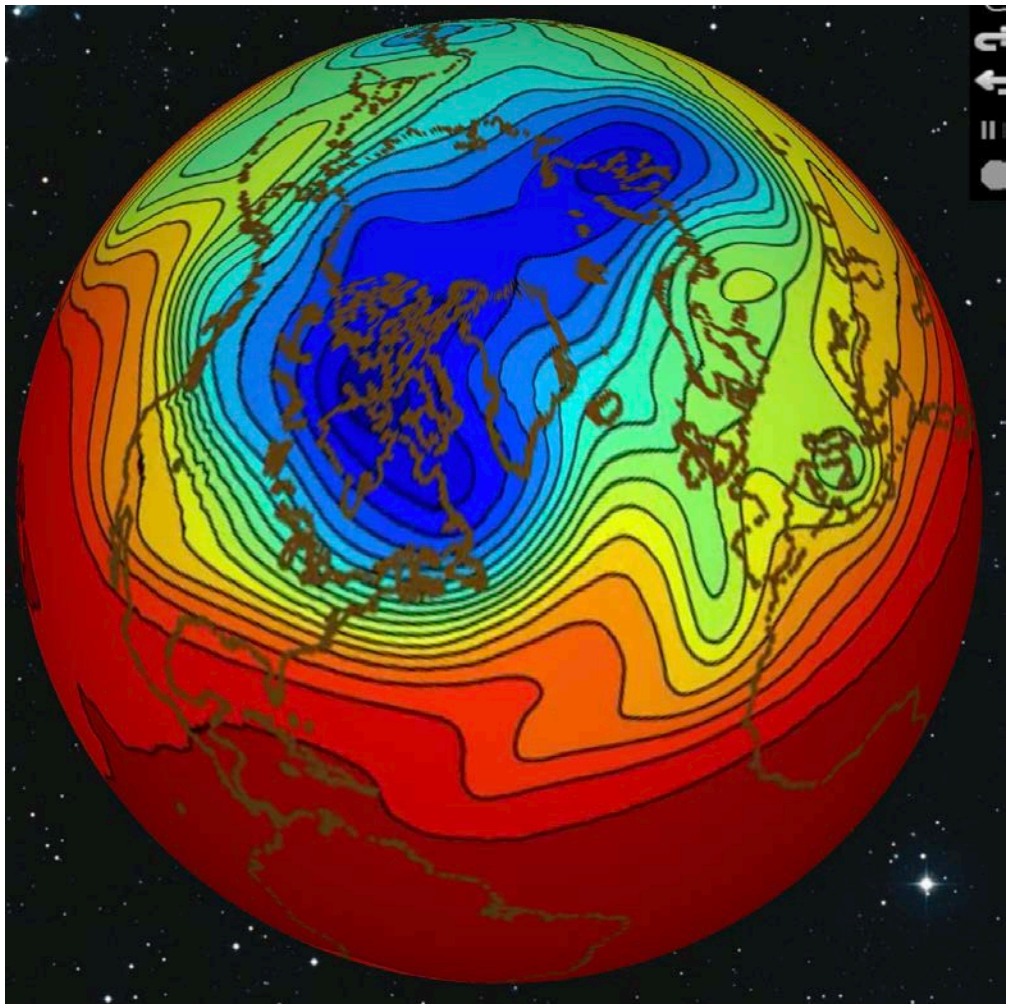
and go to Pressure and height and click on Height field

http://cmpo5.mit.edu/esg1/index.html?8.01/cor.html;r=425

Poles: (blue) lower values of height correspond to low pressure

Tropics (red): higher values of height correspond to high pressure

Mid latitudes: strong height gradients correspond to strong pressure gradients



Height field

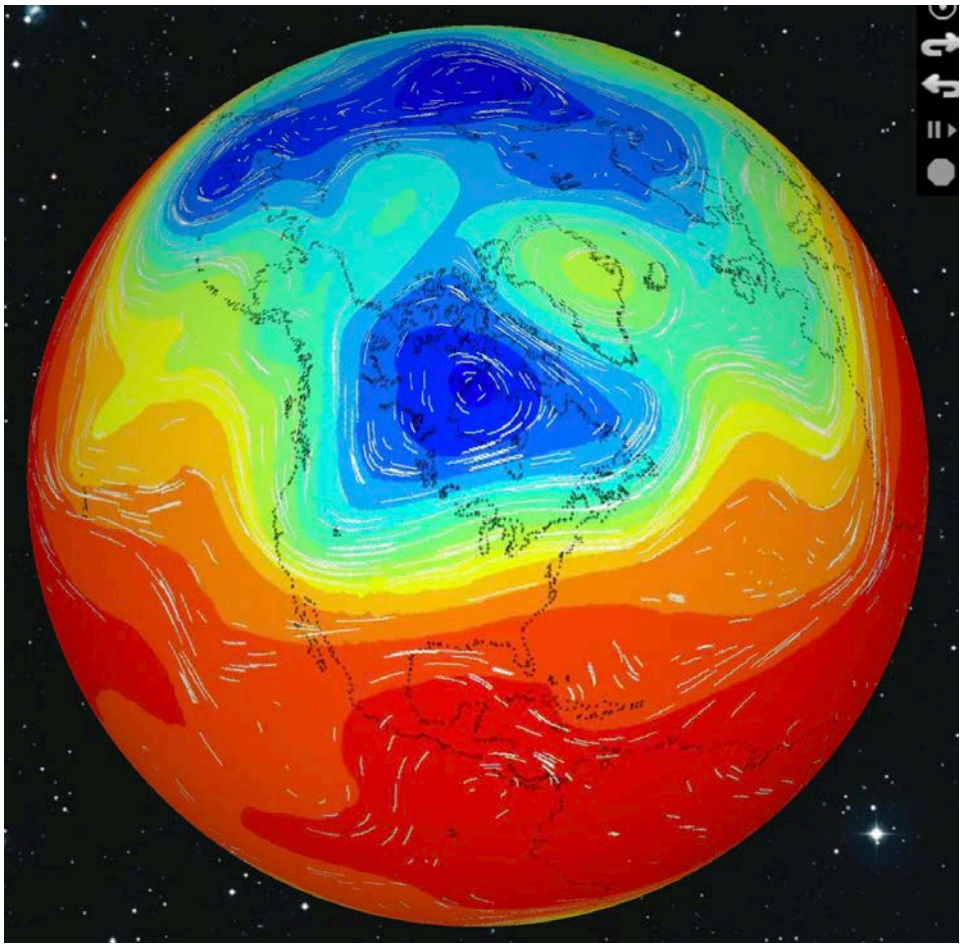
Open up http://cmpo5.mit.edu/8.01

and go to Pressure and height and click on

6hr tracks and Height of 2.5×10^4 Pascal surface

Note eastward flow in our latitudes, the clockwise circulation around the upward bulge in the surface that develops off the west coast of the US, and around the lows near Antarctica, and counterclockwise flow around the lows in the Arctic.

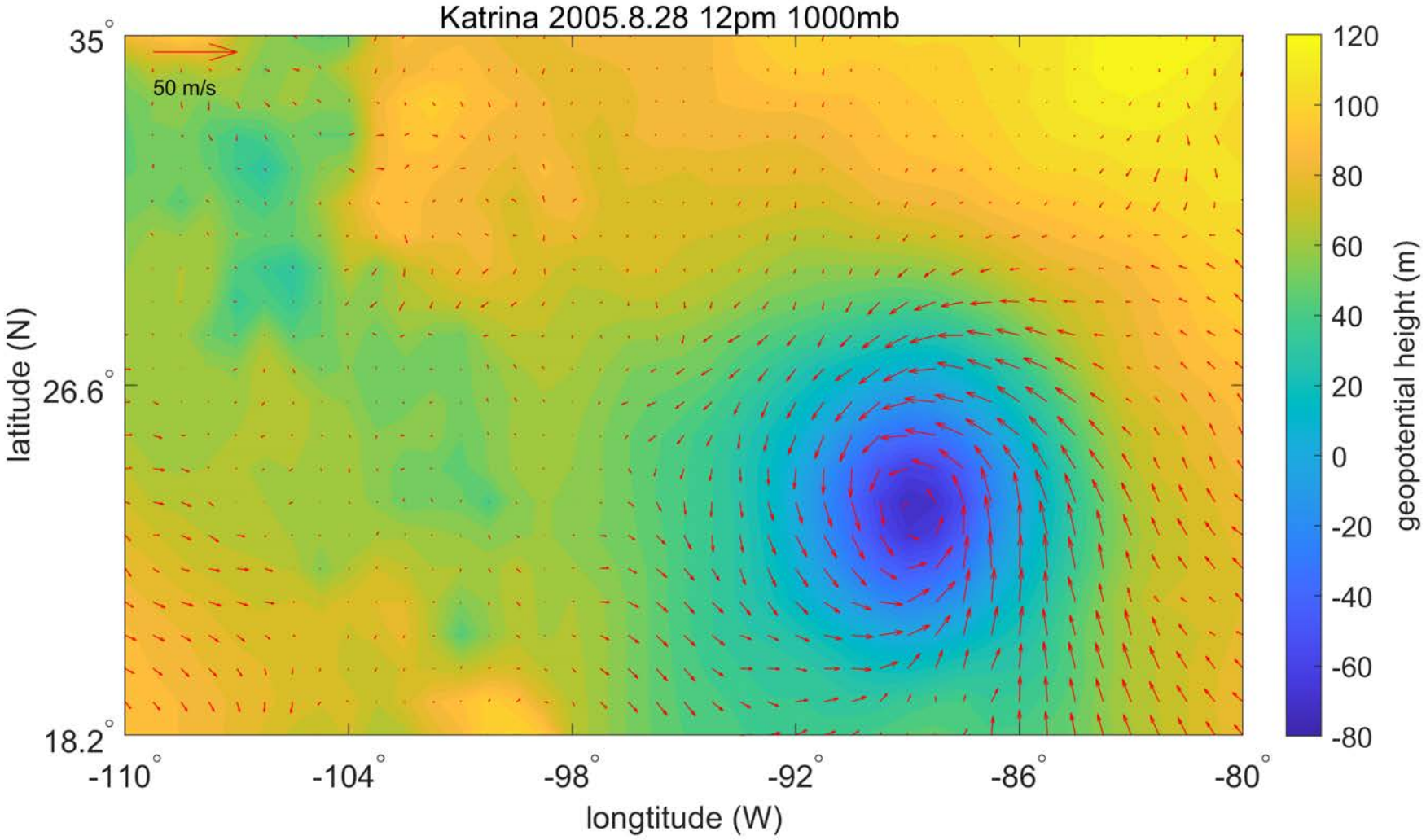
Click on 6hr tracks and Height - climatology to see average pressure and height for the month of November.



You can rotate the globe

11/28/2020-12/13/2020

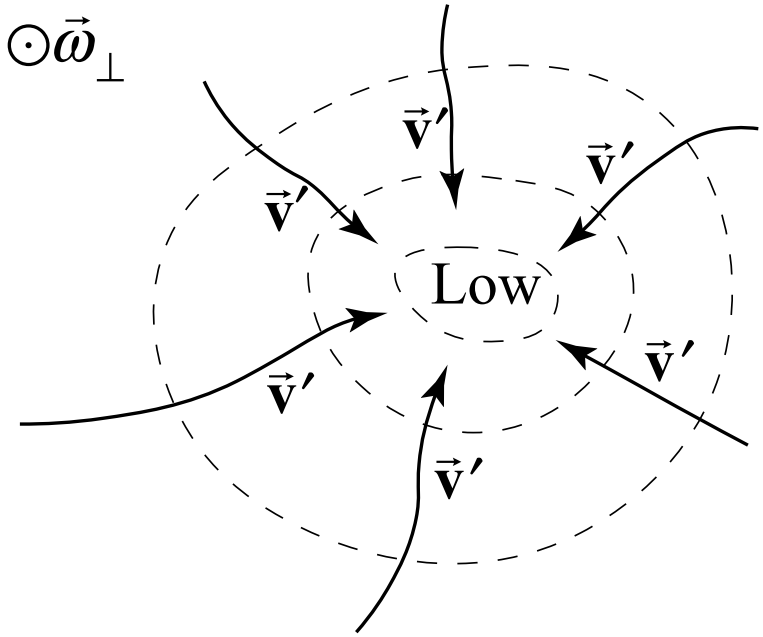
Hurricane Katrina



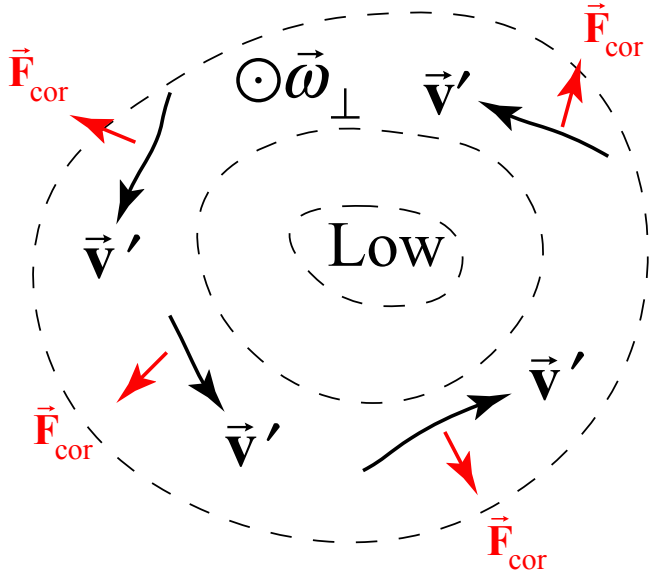
Circulation of winds around Hurricane Katrina Low

Wind Circulation Northern Hemisphere

Simple Model: At a constant height, the pressure gradient in the atmosphere can be balanced by the Coriolis force producing a circulation of air about a region of low pressure. Without any Coriolis force, the air would flow towards a region of low pressure. The Coriolis force exerts a force perpendicular to the flow, which directs the particles to move along lines of constant pressure (isobars). Black arrows indicate velocities.



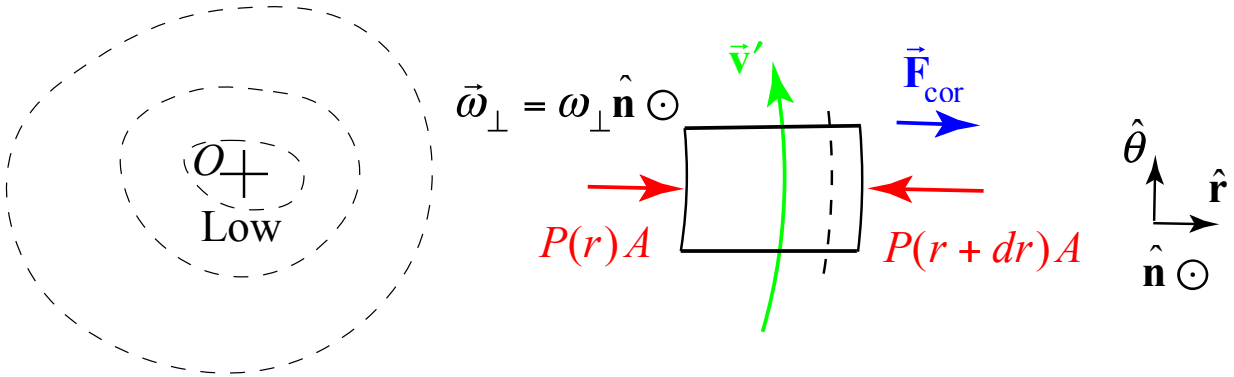
No Coriolis force



Coriolis force

If you stand with your back to the wind, low pressure is on your left

Wind Circulation:



a) The radial component of the pressure gradient force is directed radially inward towards the region of low pressure

$$F_r = -(dP/dr)\Delta V = -(dP/dr)Adr = -(dP)A$$

b) The Coriolis force points radially outward and is given by

$$\vec{\mathbf{F}}_{\text{cor}} = -2\Delta m\vec{\omega}_\perp \times \vec{\mathbf{v}}' = -2\Delta m\omega\sin(\lambda)\hat{\mathbf{k}} \times v'\hat{\theta} = 2\Delta m\omega\sin(\lambda)v'\hat{\mathbf{r}} = 2\rho Adr\omega\sin(\lambda)v'\hat{\mathbf{r}}$$

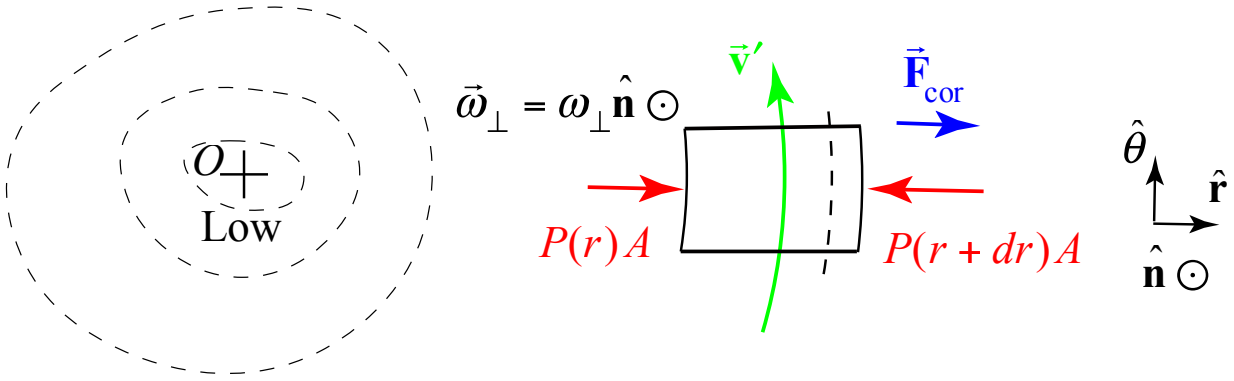
c) Newton's Second Law in the rotating frame is then

$$-dPA + 2\rho Adr\omega\sin(\lambda)v' = -\rho Adrv'^2/r$$

Thus

$$\frac{dP}{dr} = 2\rho\omega\sin(\lambda)v' + \rho v'^2/r$$

Balancing Pressure Forces and Coriolis



Newton's Second Law in the rotating frame is

$$\frac{dP}{dr} = 2\rho\omega\sin(\lambda)v' + \rho v'^2 / r$$

Recall that air masses do not rotate as rigid bodies. Near the center of the low the pressure gradient is large and the wind speeds are strong. Far from the center, the wind speed is small and the term v'^2/r can be neglected. Thus

$$v' = \frac{1}{\rho 2\omega\sin(\lambda)}\frac{dP}{dr}, \qquad \text{(far from center of low)}$$

Hurricane Tracks

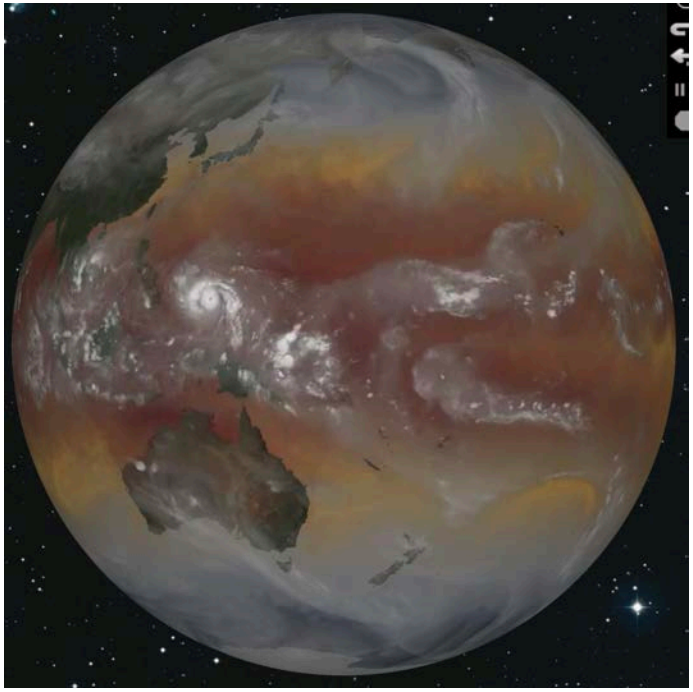
Open up http://cmpo5.mit.edu/8.01

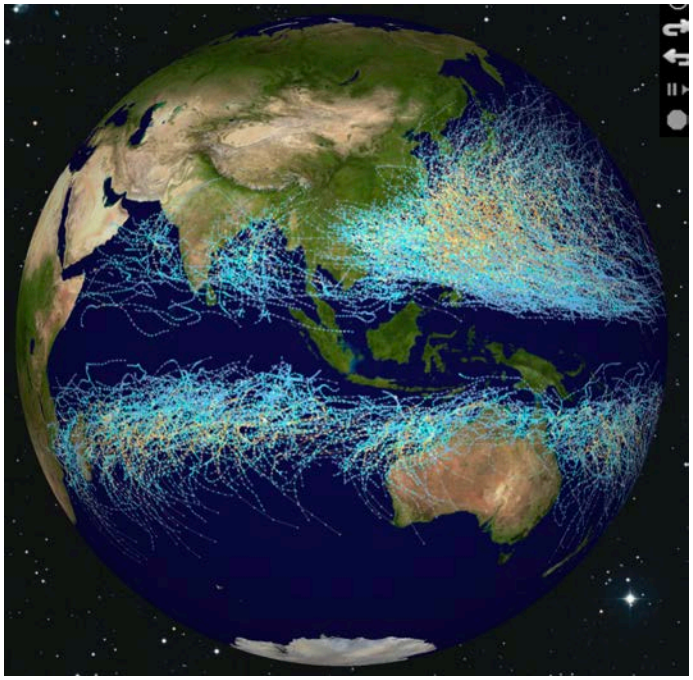
and go to Hurricanes and typhoons
and click on Typhoon Haiyun

These are low pressure systems with strong counter-clockwise circulation (northern hemisphere) so that both the centrifugal force and Coriolis force are directed outward. Typhoon Haiyun was a very destructive example. It had winds of 79 m/s at about 30 km radius and a pressure drop of 90 mb -- almost 10%.

Then click on Hurricane tracks

Tracks from 1980-2005 show clearly the importance of the vertical component of the rotation vector (zero at the equator)





Final Demo:
Ink in Rotating Tank

The pressure gradient force is independent of height, so the water circulates as vertical sheets created dye concentrated patterns in columns.