MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

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XII. AC Circuits - Worked Examples

Example 1: Series RLC Circuit

A sinusoidal voltage $V(t) = (40.0\,\text{V})\sin(100t)$ is applied to a series *RLC* circuit with L = 160 mH, $C = 99.0\,\mu\text{F}$, and $R = 68.0\,\Omega$.

(a) What is the impedance of the circuit?

(b) Let the current at any instant in the circuit be $I(t) = I_0 \sin(\omega t - \phi)$. Find I_0.

(c) What is the phase constant ϕ?

Solution:

(a) The impedance of a series *RLC* circuit is given by

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \tag{1.1}$$

where

$$X_L = \omega L \tag{1.2}$$

and

$$X_C = \frac{1}{\omega C} \tag{1.3}$$

are the inductive reactance and the capacitive reactance, respectively. Since the general expression of the voltage source is $V(t) = V_0 \sin(\omega t)$, where V_0 is the maximum output voltage and ω is the angular frequency, we have $V_0 = 40$ V and $\omega = 100$. Thus, the impedance Z becomes

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{(68)^2 + \left((100)(.16) - \frac{1}{(100)(99\times 10^{-6})}\right)^2} = 109\,\Omega \tag{1.4}$$

(b) With $V_0 = 40.0\,\text{V}$, the amplitude of the current is given by

$$I_0 = \frac{V_0}{Z} = \frac{40.0\,\text{V}}{109\,\Omega} = 0.367\,\text{A} \tag{1.5}$$

(c) The phase angle between the current and the voltage is determined by

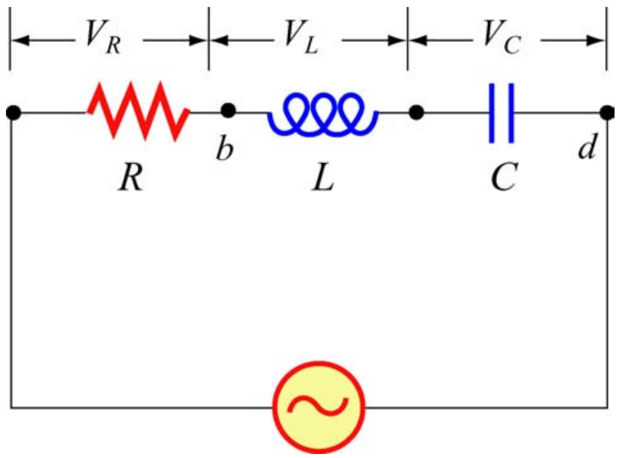
$$\tan\phi = \frac{X_L - X_C}{R} = \frac{\omega L - \dfrac{1}{\omega C}}{R} \tag{1.6}$$

Numerically, we have

$$\phi = \tan^{-1}\left(\frac{(100)(0.160) - \dfrac{1}{(100)(99.0\times10^{-6})}}{68.0}\right) = -51.3° \tag{1.7}$$

Example 2: Series RLC Circuit

Suppose an AC generator with $V(t) = (150\,\text{V})\sin(100\pi t)$ is connected to a series RLC circuit where R=40.0 Ω, L=185 mH, and C=65.0 μF.



Calculate the following:

(a) V_{R0}, V_{L0} and V_{C0}, the maximum voltage drops across each circuit element, and

(b) the maximum voltage drop across points b and d shown in the figure.

Solution:

(a) The inductive reactance, capacitive reactance and the impedance of the circuit are given by

$$X_C = \frac{1}{\omega C} = \frac{1}{(100\pi)(65.0\times10^{-6})} = 49.0\,\Omega \tag{2.1}$$

$$X_L = \omega L = (100\pi)(185\times10^{-3}) = 58.1\,\Omega \tag{2.2}$$

and

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(40.0)^2 + (58.1-49.0)^2} = 41.0\,\Omega \tag{2.3}$$

respectively. Therefore, the corresponding maximum current amplitude is

$$I_0 = \frac{V_0}{Z} = \frac{150}{41.0} = 3.66\,\text{A} \tag{2.4}$$

The maximum voltage across the resistance would be just the product of maximum current and the resistance:

$$V_{R0} = I_0 R = (3.66)(40) = 146\,\text{V} \tag{2.5}$$

Similarly, the maximum voltage across the inductor is

$$V_{L0} = I_0 X_L = (3.66)(58.1) = 212\,\text{V} \tag{2.6}$$

and the maximum voltage across the capacitor is

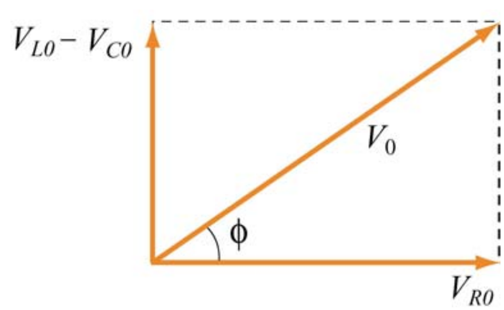
$$V_{C0} = I_0 X_C = (3.66)(49.0) = 179\ \text{V} \tag{2.7}$$

(b) The maximum input voltage V_0 is related to V_{R0}, V_{L0} and V_{C0} by

$$V_0 = \sqrt{V_{R0}{}^2 + (V_{L0} - V_{C0})^2} \tag{2.8}$$

Thus, from *b* to *d*, the maximum voltage would be the difference between V_{L0} and V_{C0}:

$$V_{bd} = V_{L0} - V_{C0} = 212.5 - 179.1 = 33.4\,\text{V} \tag{2.9}$$



Example 3: Resonance

A sinusoidal voltage $V(t) = (100\,\mathrm{V})\sin\omega t$ is applied to a series *RLC* circuit with L = 20.0 mH, C = 100 nF and R = 20.0 Ω. Find the following quantities:

(a) the resonant frequency,

(b) the amplitude of the current at the resonant frequency,

(c) the quality factor Q of the circuit, and

(d) the amplitude of the voltage across the inductor at resonance.

Solution:

(a) The resonant frequency for the circuit is given by

$$f = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} = \frac{1}{2\pi}\sqrt{\frac{1}{(20.0\times10^{-3})(100\times10^{-9})}} = 3560\,\mathrm{Hz} \tag{3.1}$$

(b) At resonance, the current is

$$I_0 = \frac{V_0}{R} = \frac{100}{20.0} = 5.00\,\mathrm{A} \tag{3.2}$$

(c) The quality factor Q of the circuit is given by

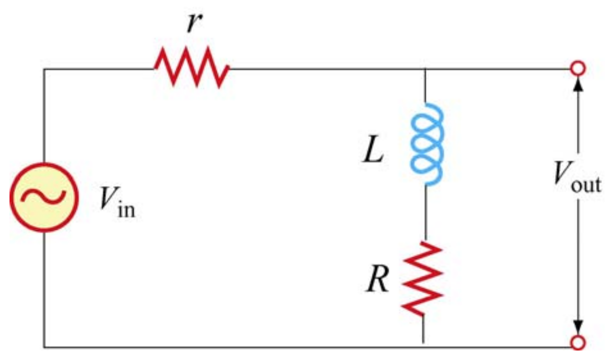
$$Q = \frac{\omega_0 L}{R} = \frac{2\pi(3560)(20.0\times10^{-3})}{(20.0)} = 22.4 \tag{3.3}$$

(d) At resonance, the amplitude of the voltage across the inductor is

$$V_{L0} = I_0 X_L = I_0\omega_0 L = (5.00)(2\pi\times3560)(20.0\times10^{-3}) = 2.24\times10^3\,\mathrm{V} \tag{3.4}$$

Example 4: High-pass RL filter

A high-pass *RL* filter can be represented by the circuit in the figure below, with *r* being the internal resistance of the inductor.



(a) Find $V_{out,0}/V_{in,0}$, the ratio of the maximum output voltage $V_{out,0}$ to the maximum input voltage $V_{in,0}$.

(b) Let *r* =20.0 Ω, *R*=5.0 Ω, and *L*=250 mH. What is the frequency if $\frac{V_{out,0}}{V_{in,0}} = \frac{1}{2}$?

Solution:

(a) The impedance for the input circuit is $Z_{in} = \sqrt{(R+r)^2 + X_L^2}$ where $X_L = \omega L$ and $Z_{out} = \sqrt{R^2 + X_L^2}$ for the output circuit. The maximum current is given by

$$I_0 = \frac{V_{in,0}}{Z_{in}} = \frac{V_0}{\sqrt{(R+r)^2 + X_L^2}} \tag{4.1}$$

Similarly, the maximum output voltage is related to the output impedance by

$$V_{out,0} = I_0 Z_{out} = I_0 \sqrt{R^2 + X_L^2} \tag{4.2}$$

This implies

$$\frac{V_{out,0}}{V_{in,0}} = \frac{\sqrt{R^2 + X_L^2}}{\sqrt{(R+r)^2 + X_L^2}} \tag{4.3}$$

(b) For $\frac{V_{out,0}}{V_{in,0}} = \frac{1}{2}$, we have

$$\frac{R^2 + X_L^2}{(R+r)^2 + X_L^2} = \frac{1}{4} \tag{4.4}$$

Rearranging the terms, we have

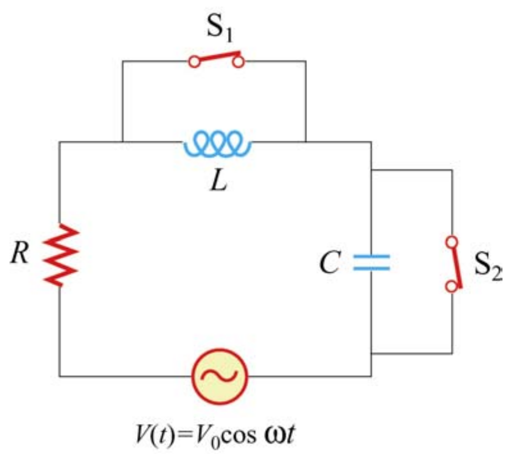
$$X_L = \sqrt{\frac{(r+R)^2 - 4R^2}{3}} \tag{4.5}$$

Since $X_L = \omega L = 2\pi f L$, we have

$$f = \frac{X_L}{2\pi L} = \frac{1}{2\pi(0.250)}\sqrt{\frac{(25.0)^2 - 4(5.00)^2}{3}} = 8.42\,\text{Hz} \tag{4.6}$$

Example 5: RLC Circuit

Consider the circuit shown below, assuming that R, L, V_0 and ω are known. If both switches are closed initially, find the following:



(a) the current as a function of time,

(b) the average power delivered to the circuit,

(c) the current as a function of time after *only* switch 1 is opened.

(d) the capacitance C after switch 2 is *also* opened, with the current and voltage in phase,

(e) the impedance of the circuit when both switches are open,

(f) the maximum energy stored in the capacitor during oscillations,

(g) the maximum energy stored in the inductor during oscillations.

(h) the phase difference between the current and the voltage if the frequency of the voltage source is doubled, and

(i) the frequency that makes the inductive reactance one-half the capacitive reactance.

Solution:

(a) When both switches are closed, the current goes through only the generator and the resistor, so the total impedance of the circuit is *R* and the current is

$$I(t) = \frac{V_0}{R}\cos\omega t \tag{5.1}$$

(b) The average power is given by

$$< P >=< I(t)V(t) >= \frac{V_0^{\ 2}}{R} < \cos^2 \omega t >= \frac{V_0^{\ 2}}{2R} \tag{5.2}$$

(c) If only switch 1 is opened, the current will pass through the generator, the resistor and the inductor. For this *RL* circuit, the impedance becomes

$$Z = \frac{1}{\sqrt{R^2 + X_L^2}} = \frac{1}{\sqrt{R^2 + \omega^2 L^2}} \tag{5.3}$$

and the phase angle ϕ is

$$\phi = \tan^{-1}\left(\frac{\omega L}{R}\right) \tag{5.4}$$

Thus, the current as a function of time is

$$\boxed{I(t) = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}}\cos\left(\omega t + \tan^{-1}\frac{\omega L}{R}\right)} \tag{5.5}$$

(d) If both switches are opened, then this would be a driven *RLC* circuit, with the phase angle ϕ given by

$$\tan\phi = \frac{X_L - X_C}{R} = \frac{\omega L - \dfrac{1}{\omega C}}{R} \tag{5.6}$$

If the current and the voltage are in phase, then $\phi = 0$, implying $\tan\phi = 0$. Let the corresponding angular frequency be ω_0, we then obtain

$$\omega_0 L = \frac{1}{\omega_0 C} \tag{5.7}$$

and the capacitance is

$$C = \frac{1}{\omega_0{}^2 L} \tag{5.8}$$

(e) From (d), we see that when both switches are opened, the circuit is at resonance with $X_L = X_C$. Thus, the impedance of the circuit becomes

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = R \tag{5.9}$$

(f) The energy stored in the capacitor is

$$U_C = \frac{1}{2} C V_C^2 = \frac{1}{2} C I^2 X_C^2 \tag{5.10}$$

It attains maximum when the current is at its maximum I_0:

$$U_{C,\max} = \frac{1}{2} C I_0^2 X_C^2 = \frac{1}{2} C \left(\frac{V_0}{R} \right)^2 \frac{1}{\omega_0{}^2 C^2} = \frac{V_0^2 L}{2R^2} \tag{5.11}$$

where we have used $\omega_0^2 = 1 / LC$.

(g) The maximum energy stored in the inductor is given by

$$U_{L,\max} = \frac{1}{2} L I_0^2 = \frac{L V_0^2}{2R^2} \tag{5.12}$$

(h) If the frequency of the voltage source is doubled, i.e., $\omega = 2\omega_0 = 1/\sqrt{LC}$, then the phase angle is given by

$$\phi = \tan^{-1}\left(\frac{\omega L - 1/\omega C}{R} \right) = \tan^{-1}\left(\frac{\left(2/\sqrt{LC}\right)L - \left(\sqrt{LC}/2C\right)}{R} \right) = \tan^{-1}\left(\frac{3}{2R}\sqrt{\frac{L}{C}} \right) \tag{5.13}$$

(i) If inductive reactance is one-half the capacitive reactance, i.e.,

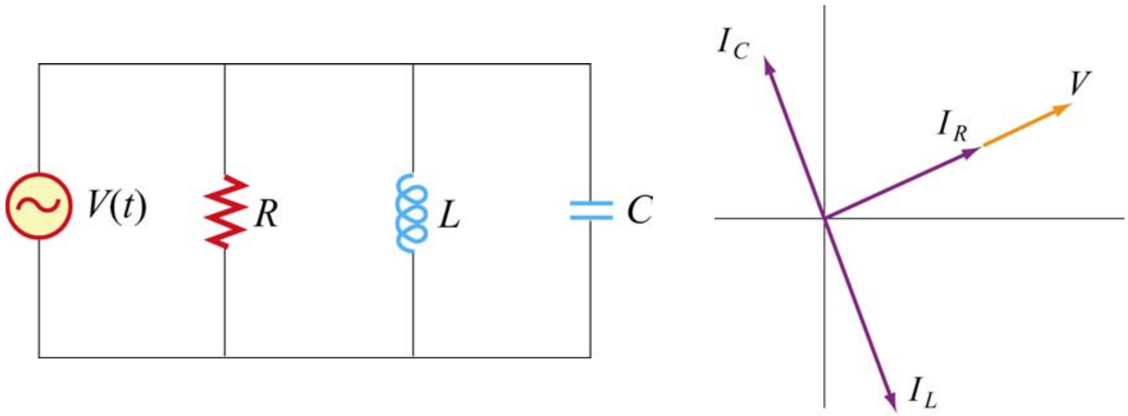
$$\omega L = \frac{1}{2}\left(\frac{1}{\omega C} \right) \tag{5.14}$$

then

$$\omega = \frac{1}{\sqrt{2LC}} = \frac{\omega_0}{\sqrt{2}} \tag{5.15}$$

Example 6: Parallel RLC Circuit

The figures below illustrate a parallel *RLC* circuit and its corresponding phasor diagram.



The instantaneous voltages and rms voltages across the three circuit elements are the same, and each is in phase with the current through the resistor. The currents in C and L lead or lag behind the current in the resistor.

(a) Show that the rms current delivered by the source is

$$I_{rms} = V_{rms}\sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2} \tag{6.1}$$

(b) Find the phase angle ϕ between V_{rms} and I_{rms}.

Solution:

Denote I_R, I_L and I_C as the currents that pass through the resistor, the inductor and the capacitor, respectively. Since the instantaneous voltages and rms voltages across the three circuit elements are the same, we then have

$$I_R = \frac{V_{\text{rms}}}{R} \tag{6.2}$$

$$I_L = \frac{V_{\text{rms}}}{X_L} = \frac{V_{\text{rms}}}{\omega L} \tag{6.3}$$

and

$$I_C = \frac{V_{\text{rms}}}{X_C} = \omega C V_{\text{rms}} \tag{6.4}$$

From the phasor diagram, we see that the rms current is given by

$$I_{\text{rms}} = \sqrt{I_R^2 + \left(I_C - I_L\right)^2} \tag{6.5}$$

or

$$I_{\text{rms}} = \sqrt{\left(\frac{V_{\text{rms}}}{R}\right)^2 + \left(\omega C V_{\text{rms}} - \frac{V_{\text{rms}}}{\omega L}\right)^2} = V_{\text{rms}}\sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2} \tag{6.6}$$

(b) From the phasor diagram, we see that the phase angle can be obtained as

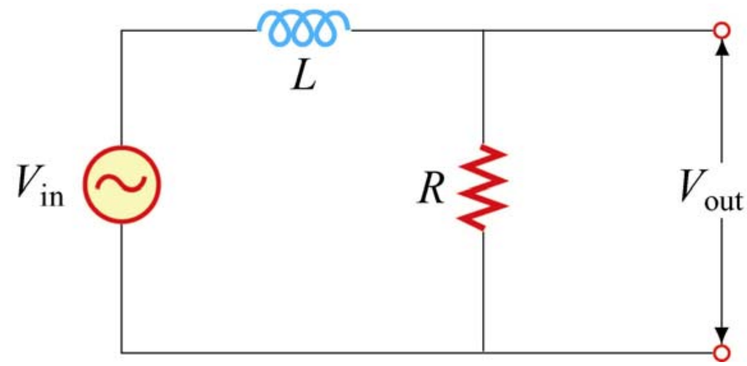
$$\tan\phi = \left(\frac{I_C - I_L}{I_R}\right) = \frac{\dfrac{V_{\text{rms}}}{X_C} - \dfrac{V_{\text{rms}}}{X_L}}{\dfrac{V_{\text{rms}}}{R}} = R\left(\frac{1}{X_C} - \frac{1}{X_L}\right) \tag{6.7}$$

or

$$\phi = \tan^{-1}\left[R\left(\frac{1}{X_C} - \frac{1}{X_L}\right)\right] \tag{6.8}$$

Example 7: RL low-pass filter

The circuit below represents an RL low-pass filter.



Suppose the input voltage is $V(t) = (10.0\,\mathrm{V})\sin 200t$ with L = 500 mH, find

(a) the value of R such that the output voltage lags behind the input voltage by $30.0°$,

(b) the amplitude of the output voltage.

Solution:

(a) The phase relationship between V_L and V_R is given by

$$\tan\phi = \frac{V_L}{V_R} = \frac{IX_L}{IX_R} = \frac{\omega L}{R} \tag{7.1}$$

Thus, we have

$$R = \frac{\omega L}{\tan\phi} = \frac{(200\,\mathrm{s}^{-1})(0.500\,\mathrm{H})}{\tan 30.0°} = 173\,\Omega \tag{7.2}$$

(b) Since

$$\frac{V_{out}}{V_{in}} = \frac{V_R}{V_{in}} = \cos\phi \tag{7.3}$$

we have

$$V_{out} = V_{in}\cos\phi = (10.0\,\mathrm{V})\cos 30.0° = 8.66\ \mathrm{V} \tag{7.4}$$